第16讲　求解常系数非齐次方程特解的算子解法

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3.1 算子解法

- 非齐次线性方程：$P(D)x = f(t)$.

$$P(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

- 形式上：$x(t) = \dfrac{1}{P(D)} f(t)$.

  $\dfrac{1}{P(D)}$称为算子$P(D)$的逆算子，定义如下：

  $\dfrac{1}{P(D)} f(t)$表示一个函数，它用$P(D)$作用后恰好等于$f(t)$。

- 例：$\dfrac{1}{D} f(t) = \displaystyle\int f(t)dt$

  $\dfrac{1}{D^2} f(t) = \displaystyle\int\int f(t)(dt)^2$.

算子解法

算子$\dfrac{1}{P(D)}$的基本性质：

(a) $\dfrac{1}{P(D)}af(t) = a\dfrac{1}{P(D)}f(t)$.

(b) $\dfrac{1}{P(D)}(f_1(t)+f_2(t)) = \dfrac{1}{P(D)}f_1(t) + \dfrac{1}{P(D)}f_2(t)$.

(c) 设$P(D) = P_1(D)P_2(D)$，则

$$\frac{1}{P(D)}f(t) = \frac{1}{P_1(D)}\left(\frac{1}{P_2(D)}f(t)\right) = \frac{1}{P_2(D)}\left(\frac{1}{P_1(D)}f(t)\right).$$

算子解法

几个简单的算子运算公式：

(1) $P(D)e^{\lambda t} = e^{\lambda t}P(\lambda)$.

(2) $P(D^2)\cos\alpha t = \cos\alpha t \cdot P(-\alpha^2)$.

(3) $P(D^2)\sin\alpha t = \sin\alpha t \cdot P(-\alpha^2)$.

(4) $P(D)(e^{\lambda t}v(t)) = e^{\lambda t}P(D+\lambda)v(t)$. **（已证过，归纳法）**

证明： **（2）** $e^{i\alpha t} = \cos\alpha t + i\sin\alpha t, \quad e^{-i\alpha t} = \cos\alpha t - i\sin\alpha t.$

$$\cos\alpha t = \frac{e^{i\alpha t}+e^{-i\alpha t}}{2}, \quad \sin\alpha t = \frac{e^{i\alpha t}-e^{-i\alpha t}}{2i}.$$

$$\begin{aligned} P(D^2)\cos\alpha t &= \frac{1}{2}e^{i\alpha t}P((i\alpha)^2) + \frac{1}{2}e^{-i\alpha t}P((-i\alpha)^2) \\ &= \frac{1}{2}P(-\alpha^2)\left[e^{i\alpha t}+e^{-i\alpha t}\right] = P(-\alpha^2)\cos\alpha t. \end{aligned}$$

算子解法

利用上面的算子运算公式易证明如下的逆算子运算公式：

(i) $\dfrac{1}{P(D)}e^{\lambda t} = \dfrac{1}{P(\lambda)}e^{\lambda t}, \quad (P(\lambda) \neq 0).$

(ii) $\dfrac{1}{P(D^2)}\cos\alpha t = \dfrac{1}{P(-\alpha^2)}\cos\alpha t, \quad (P(-\alpha^2) \neq 0).$

(iii) $\dfrac{1}{P(D^2)}\sin\alpha t = \dfrac{1}{P(-\alpha^2)}\sin\alpha t, \quad (P(-\alpha^2) \neq 0).$

(iv) $\dfrac{1}{P(D)}(e^{\lambda t}v(t)) = e^{\lambda t}\dfrac{1}{P(D+\lambda)}v(t).$

证明（1）： $P(D)\left[\dfrac{1}{P(\lambda)}e^{\lambda t}\right] = \dfrac{1}{P(\lambda)} \cdot P(\lambda)e^{\lambda t} = e^{\lambda t}.$

(2): $P(D^2)\left[\dfrac{1}{P(-\alpha^2)}\cos\alpha t\right] = \dfrac{1}{P(-\alpha^2)} \cdot P(-\alpha^2)\cos\alpha t = \cos\alpha t.$

算子解法

证明（4）：

$$P(D)\left[e^{\lambda t}\frac{1}{P(D+\lambda)}v(t)\right]=e^{\lambda t}P(D+\lambda)\left[\frac{1}{P(D+\lambda)}v(t)\right]=e^{\lambda t}v(t).$$

(5) 设$f_k(t)=b_0+b_1t+\cdots+b_kt^k,\quad P(0)=a_n\neq 0$，则

$$\frac{1}{P(D)}f_k(t)=Q_k(D)f_k(t),$$

其中$Q_k(D)=c_0+c_1D+\cdots+c_kD^k$是$\frac{1}{P(D)}$在$D=0$附近泰勒展式的前$k+1$项。

证明思路：$\dfrac{1}{P(t)}=c_0+c_1t+\cdots+c_kt^k+c_{k+1}t^{k+1}+\cdots$

$$\frac{1}{P(D)}=Q_k(D)+c_{k+1}D^{k+1}+\cdots$$

算子解法

类型（I）：$P(D)x = f_k(t)$，$f_k(t)$为k次多项式。

(1) $P(0) \neq 0$ （$\lambda = 0$不是特征根），特解为

$$x_0(t) = \frac{1}{P(D)} f_k(t) = (c_0 + c_1 D + \cdots + c_k D^k) f_k(t).$$

(2) $P(0) = 0$，且$\lambda = 0$是k重特征根，即$P(D) = D^k Q(D)$，而$Q(0) \neq 0$。

$$x_0(t) = \frac{1}{D^k} \left[\frac{1}{Q(D)} f_k(t) \right].$$

算子解法

例3.1 $x'' + x = t^2 - t + 2$.

$$x_0(t) = (1 - D^2)(t^2 - t + 2) = t^2 - t.$$

例3.2 $x'' - x = t$.

$$x_0(t) = -\frac{1}{2}\frac{1}{D+1}t + \frac{1}{2}\frac{1}{D-1}t = -t.$$

例3.3 $x'' + x' = 1 + t^2$.

$$x_0(t) = \frac{1}{D^2 + D}(1 + t^2) = \frac{1}{D}\frac{1}{1+D}(1 + t^2) = \frac{1}{3}t^3 - t^2 + 3t.$$

例3.4 $(D - \alpha)^m x = f(t)$，m**为正整数。**

$$x_0(t) = \frac{1}{(D-\alpha)^m}f(t) = e^{\alpha t}\frac{1}{D^m}e^{-\alpha t}f(t)$$

算子解法

类型（II）：$P(D)x = e^{\lambda t} f_k(t)$，$f_k(t)$为$k$次多项式。

$$x_0(t) = \frac{1}{P(D)} e^{\lambda t} f_k(t) = e^{\lambda t} \frac{1}{P(D+\lambda)} f_k(t)$$

例3.5 $x'' - x' + 6x = 2e^{4t}$.

$$x_0(t) = \frac{1}{9} e^{4t}.$$

例3.6 $x'' - 2x' + x = 5te^{t}$.

$$x_0(t) = \frac{5}{6} t^3 e^{t}.$$

例3.7 $P(D)x = e^{\lambda_0 t}$, $P(D) = (D-\lambda_0)^r Q(D)$, $Q(\lambda_0) \neq 0$.

$$x_0(t) = \frac{1}{(D-\lambda_0)^r}\frac{1}{Q(D)}(e^{\lambda_0 t}) = \frac{1}{(D-\lambda_0)^r}\frac{1}{Q(\lambda_0)}(e^{\lambda_0 t}\cdot 1)$$

$$= \frac{1}{Q(\lambda_0)}e^{\lambda_0 t}\frac{1}{D^r}1 = \frac{e^{\lambda_0 t}}{Q(\lambda_0)}\cdot\frac{t^r}{r!}.$$

算子解法

类型（III）：

$$P(D)x = \cos\alpha t \cdot f_k(t) \quad (1)$$

$$P(D)x = \sin\alpha t \cdot f_k(t) \quad (2)$$

其中$P(D)$和$f_k(t)$的系数都是实的。

- 考虑辅助方程：$P(D)x = e^{i\alpha t} f_k(t)$.

  设其特解$x_0(t) = x_1(t) + ix_2(t)$，则

$$P(D)x_1(t) + iP(D)x_2(t) = \cos\alpha t \cdot f_k(t) + i\sin\alpha t \cdot f_k(t).$$

$$P(D)x_1(t) = \cos\alpha t \cdot f_k(t), \quad P(D)x_2(t) = \sin\alpha t \cdot f_k(t).$$

- 辅助方程特解$x_0(t)$的实部和虚部分别是方程**(1)**和**(2)**的特解。

算子解法

例3.8 $x'' - 3x' + 2x = \cos 2t$.

特解： $x_0(t) = -\dfrac{1}{20}\cos 2t - \dfrac{3}{20}\sin 2t$.

注3.1 $P(D^2)x = \cos \alpha t$. （或$P(D^2)x = \sin \alpha t$）

(a) 若$P(-\alpha^2) \neq 0$，则$x_0(t) = \dfrac{1}{P(-\alpha^2)}\cos \alpha t$.

(b) 若$P(-\alpha^2) = 0$，且$-\alpha^2$是$P(\lambda)$的r重根，则

$$P(\lambda) = (\lambda + \alpha^2)^r Q(\lambda),\ Q(-\alpha^2) \neq 0$$

$$P(D^2) = (D^2 + \alpha^2)^r Q(D^2),$$

$$x_0(t) = \frac{1}{(D^2 + \alpha^2)^r}\left(\frac{1}{Q(D^2)}\cos \alpha t\right) = \frac{1}{Q(-\alpha^2)} \cdot \frac{1}{(D^2 + \alpha^2)^r}\cos \alpha t.$$

算子解法

考虑辅助方程：$(D^2+\alpha^2)^r x = e^{i\alpha t}$.

$$
\begin{aligned}
x(t) &= \frac{1}{(D^2+\alpha^2)^r}(e^{i\alpha t}\cdot 1) = e^{i\alpha t}\cdot\frac{1}{D^r(D+2i\alpha)^r}e^{0t} \\
&= e^{i\alpha t}\frac{1}{D^r}\left[\frac{1}{(2i\alpha)^r}\right] = e^{i\alpha t}\frac{1}{(2i\alpha)^r}\frac{1}{D^r}1 \\
&= e^{i\alpha t}\frac{1}{(2i\alpha)^r}\frac{t^r}{r!}.
\end{aligned}
$$

原方程的特解为

$$x_0(t) = \frac{1}{Q(-\alpha^2)}\cdot Re\left\{e^{i\alpha t}\frac{1}{(2i\alpha)^r}\frac{t^r}{r!}\right\}.$$

算子解法

例3.9 $(D^4+1)x = 5\sin 2t$.

$$x_0(t) = \frac{5}{17}\sin 2t.$$

例3.10 $x'' + x = \sin t$.

$$x_0(t) = -\frac{1}{2}t\cos t.$$

例3.11 $x'' - x = \sin t + \cos 2t$.

$$x_0(t) = -\frac{1}{2}\sin t - \frac{1}{5}\cos 2t.$$

消去法

例3.12

$$\begin{cases} x' = 2x + y + e^{2t} + 3t - 1, \\ y' = x + 2y + 2e^{2t} + 9t - 6. \end{cases}$$

写成算子形式，消去y，有：

$$(D^2 - 4D + 3)x = 2e^{2t} + 3t - 1$$

$$x = C_1 e^t + C_2 e^{3t} - 2e^{2t} + t + 1$$

消去法

例3.13

$$\begin{cases} \dfrac{dx_1}{dt} - \dfrac{dx_2}{dt} + x_1 = -t \\ \dfrac{d^2x_1}{dt^2} - \dfrac{dx_2}{dt} + 3x_1 - x_2 = e^{2t} \end{cases}$$

写成算子形式，消去x_2，有：

$$(D^3 - D^2 + D - 1)x_1 = 2e^{2t} + t + 1$$

$$x_1(t) = c_1 e^t + c_2 \cos t + c_3 \sin t + \frac{2}{5}e^{2t} - t - 2$$

如何求x_2？

作业

作业：

P 91: 1((1), (2), (6), (7));

P92: 2((1), (2));

P92: 6((2)).