HW1: Due March 11, Monday.

1. 10 pts

Let $\alpha: I \to R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

2. 20 pts

(*Straight Lines as Shortest.*) Let $\alpha: I \to R^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p, \alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|\, dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

3. 10 pts

Show that an equation of a plane passing through three noncolinear points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ is given by

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0,$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

4. 20 pts (Helix is the fundamental example of space curves)

Given the parametrized curve (helix)

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \quad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

5. 10 pts (Compute torsion for an arc-length curve)

Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

6. 20 pts (this gives an alternative interpretation of curvature)

Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

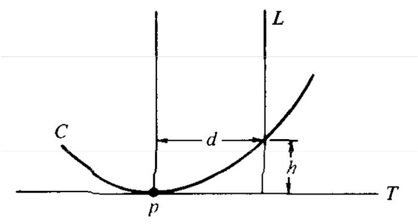

Figure 1-36

7. Bonus 20 pts

If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature k of C at p satisfies $|k| \geq 1/r$.