HW3: Due April 22, Monday.

1. (20 points)

3. (Surface of revolution) We have a lot of examples(sphere, cylinder, hyperboloid, torus, etc) are obtained by rotating a regular connected plane curve C about an axis in the plane which does not intersect the curve. Usually, we take the xz plane as the plane of the curve C and the z-axis as the rotation axis. We parameterize the curve C by

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

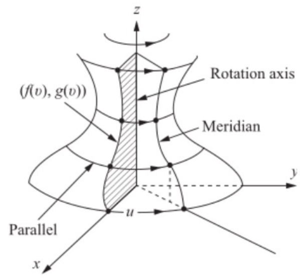
and denote u by the rotation angle about the z-aixs. Then the map

$$\sigma(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

is a parameterizations from the open set $\{0 < u < 2\pi, a < v < b\}$ in $\mathbb{R}^2$ to the surface S. Verify that the first and second fundamental forms are

$$I = f(v)^2 du^2 + dv^2, \quad \text{and} \quad (-1)II = f g_v du^2 + (f_v g_{vv} - f_{vv} g_v) dv^2.$$

A picture(from do Carmo's book) is given below:



2. (20 points)

Derive the formula of Gaussian curvature for orthogonal parameterization.

$$K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right]$$

3. (20 points)

Verify that the parameterization in Problem 1 is an orthogonal parameterization. Compute the Gaussian curvature of a surface of revolution.

4. (30 points)

Derive the Liouville's formula for orthogonal coordinates.

$$\kappa_g = \frac{d\theta}{ds} + \frac{1}{2\sqrt{EG}}\left[G_u \frac{du}{ds} - E_v \frac{du}{ds}\right]$$

5. (10 points)

Derive the formula (Eq. 4.5 in Ref1) for the Gaussian curvature of lines (interfaces) in liquid crystal elastomer sheets.

Ref1: Feng, F., Duffy, D., Warner, M., & Biggins, J. S. (2022). Interfacial metric mechanics: stitching patterns of shape change in active sheets. Proceedings of the Royal Society A, 478(2262), 20220230.

Ref2: Duffy, D., & Biggins, J. S. (2020). Defective nematogenesis: Gauss curvature in programmable shape-responsive sheets with topological defects. Soft Matter, 16(48), 10935-10945.