# Differential Geometry and Mechanics

Fan Feng

Department of Mechanics and Engineering Science, College of Engineering,
Peking University, Beijing 100087, China

## 1 Preliminaries

### 1.1 Vector space

Vector space $\mathcal{V}$ is a set closed under $+$ and $\cdot$. We usually say $\mathcal{V}$ is a vector space over the scalar field $\mathcal{F}$. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{F}$, the following conditions are satisfied:

- Commutativity: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- Associability: $(\mathbf{x}+ \mathbf{y}) + \mathbf{z}+ \mathbf{x}+ (\mathbf{y}+ \mathbf{z})$
- Zero vector: $\mathbf{0} \in \mathcal{V}$, $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$
- Inverse vector: $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
- Scalar multiplication: $\alpha \cdot (\beta \cdot \mathbf{x}) = (\alpha \cdot \beta) \cdot \mathbf{x}$
- Scalar distributivity: $(\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$
- Vector distributivity: $\alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$
- Scalar multiplication identity: $1 \cdot \mathbf{x} = \mathbf{x}$.

### 1.2 Euclidean space

Euclidean space is a vector space equipped with an inner product $< \cdot, \cdot >$, e.g. $\mathbb{R}^3$. An inner product satisfies

- $< \mathbf{x}, \mathbf{y} >=< \mathbf{y}, \mathbf{x} >$
- $< \mathbf{x} + \mathbf{y}, \mathbf{z} >=< \mathbf{x}, \mathbf{z} > + < \mathbf{y}, \mathbf{z} >$
- $< \alpha \cdot \mathbf{x}, \mathbf{y} >= \alpha < \mathbf{x}, \mathbf{y} >$
- $< \mathbf{x}, \mathbf{x} >\geq 0$, "$=$" holds if and only if $\mathbf{x} = \mathbf{0}$.

Example: In $\mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, $< \mathbf{x}, \mathbf{y} >= \sqrt{x_1y_1 + x_2y_2 + x_3y_3}$.

### 1.3 Euclidean affine space

Euclidean affine space is an Euclidean space with no fixed origin $\mathbf{0}$. Let $\mathcal{V}$ be a vector space over $\mathcal{F}$, and let $\mathcal{A}$ be a nonempty set. Now define addition $p + \mathbf{a} \in \mathcal{A}$ for any vector $\mathbf{a} \in \mathcal{V}$ and $p \in \mathcal{A}$ subject to

- $p + \mathbf{0} = \mathrm{p}$
- $(\mathrm{p} + \mathbf{a}) + \mathbf{b} = \mathrm{p} + (\mathbf{a} + \mathbf{b})$
- For any $q \in \mathcal{A}$, there exists an unique $\mathbf{a} \in \mathcal{V}$ such that $q = p + \mathbf{a}$.

  Example: the universe.

### 1.4 Vector product and linear algebra

Two ordered basis $e = \{\hat{e}_i\}$ and $f = \{\hat{f}_i\}$ have the same orientation if the matrix of change of basis has positive determinant, $e \sim f$.

**Dot product.** $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, dot product $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

**Vector product.** $\mathbf{u} \wedge \mathbf{v} \in \mathbb{R}^3$ characterized by $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$, where

$$\det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1}$$

which implies

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{e}_3. \tag{2}$$

Vector product satisfies the following identities:

- $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$
- $\mathbf{u} \wedge \mathbf{v}$ depends linearly on $\mathbf{u}$ and $\mathbf{v}$. $(\alpha \mathbf{u} + \beta \mathbf{w}) \wedge \mathbf{v} = \alpha \mathbf{u} \wedge \mathbf{v} + \beta \mathbf{w} \wedge \mathbf{v}$.
- $\mathbf{u} \wedge \mathbf{v} = 0$ if and only if $\mathbf{u}$ $\mathbf{v}$ are linearly dependent.
- $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{v} = 0$
- $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$
- Cyclic property: $\mathbf{w} = \mathbf{u} \wedge \mathbf{v} \implies \mathbf{v} = \mathbf{w} \wedge \mathbf{u}$ if $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \perp \mathbf{v}$.

### 1.5 Continuity

**Definition 1.1.** *Let $\mathcal{X}$ be a metric space. A sequence of points $x_1, x_2 \ldots$ in $\mathcal{X}$ converges if there is a point $x_\infty \in \mathcal{X}$ such that $|x_\infty - x_n|_\mathcal{X} \to 0$ as $n \to \infty$. That is, for every $\epsilon > 0$, $\exists N$ such that for all $n \geq N$, we have $|x_\infty - x_n|_\mathcal{X} < \epsilon$.*

**Definition 1.2.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be matrix spaces. A map $f : \mathcal{X} \to \mathcal{Y}$ is called continuous if, for any convergent sequence $x_n \to x_\infty$ in $\mathcal{X}$, we have $f(x_n) \to f(x_\infty)$ in $\mathcal{Y}$. Equivalently, $f : \mathcal{X} \to \mathcal{Y}$ is continuous if, for any $x \in \mathcal{X}$ and $\epsilon > 0$, $\exists\ \delta > 0$ such that $|x - y|_\mathcal{X} < \delta$ implies that $|f(x) - f(y)|_\mathcal{Y} < \epsilon$.*

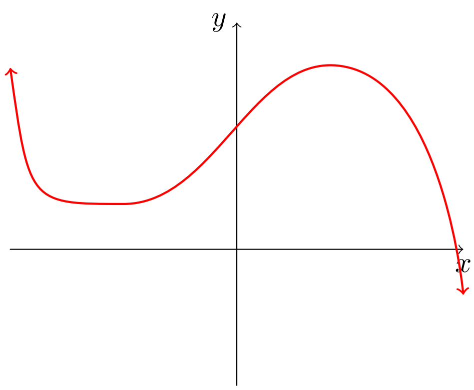
### 1.6 Derivatives

**Definition 1.3.** *Lipschitz condition: a function $f$ between metric spaces is called Lipschitz if $\exists$ a constant $L$ s.t. $|f(x) - f(y)| \leq L|x - y|$ for all values $x, y$ in the domain of definition of $f$.*

**Definition 1.4.** *Let $f : [a, b] \to \mathbb{R}$ be a Lipschitz function. Then the derivative $f'$ of $f$ is a bounded measurable function defined almost everywhere in $[a, b]$, and the following identity $f(b) - f(a) = \int_a^b f'(x)\mathrm{d}x$ holds if the integral is understood in the sense of Lebesgue.*

## 2 Curves

### 2.1 Parameterized curves




We need a parameterization in $\mathbb{R}^3$ for a 1-dimensional differentiable (or smooth) object.

**Definition 2.1.** *A parameterized differentiable curve is a differentiable map $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.*

$t \in I$, $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$

- $x(t), y(t), z(t)$ are differentiable.
- t: parameter of the curve.
- $\boldsymbol{\alpha}'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$: tangent vector.
- $\boldsymbol{\alpha}(I) \subset \mathbb{R}^3$ is called the trace of $\boldsymbol{\alpha}$.

**Example 1.** $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt), t \in \mathbb{R}$.
**Example 2.** $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$ and $\boldsymbol{\beta}(t) = (\cos 2t, \sin 2t)$ have the same trace.

### 2.2 Arc length

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a parameterized differentiable curve.

**Definition 2.2.** $\boldsymbol{\alpha}(t)$ *is regular if* $\boldsymbol{\alpha}'(t) \neq 0$ *for all* $t \in I$.

**Arc length.** Arc length $s(t) = \int_{t_0}^t |\boldsymbol{\alpha}'(t)|\mathrm{d}t$ where $|\boldsymbol{\alpha}| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$.

- $\mathrm{d}s/\mathrm{d}t = 1 = |\boldsymbol{\alpha}'(t)|$: arc-length parameterization
- $\mathrm{d}s/\mathrm{d}t \neq 1$ not arc-length parameterization. We may reparameterize the curve using the arc length: $s(t) = \int_{t_0}^t |\boldsymbol{\alpha}'(t)|\mathrm{d}t \implies t(s)$ and then $\boldsymbol{\alpha}(t(s))$ is the arc-length parameterized curve.

**Example.** Reparameterize $\boldsymbol{\alpha}(t) = (2t, \frac{4}{3}t^{3/2}, \frac{1}{2}t^2), t \in (0, 4)$ using the arc length.

### 2.3 Local theory of curves parameterized by arc length

**Definition 2.3.** *Let* $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ *be an arc length parameterized curve.* $|\boldsymbol{\alpha}''| = \kappa(s)$ *is called the curvature of* $\boldsymbol{\alpha}$ *at* $s$*.*

**Note.** Curvature for a generally parameterized curve is given by

$$\kappa(t) = \frac{|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)|}{|\boldsymbol{\alpha}''(t)|^3} \tag{3}$$

**Example.**

- Straight line $\boldsymbol{\alpha}(s) = s\mathbf{u} + \mathbf{v} \implies \kappa(s) = 0$
- Circle $\boldsymbol{\alpha} = (\cos(s), \sin(s)) \implies \kappa(s) = 1$
- $s \to -s$, $\kappa(s)$ remains invariant.

**Frenet frame.** $\boldsymbol{\alpha}''(s) = \kappa(s)\mathbf{n}(s)$ recalling $\boldsymbol{\alpha}''(s) \perp \boldsymbol{\alpha}'(s)$, i.e., the first derivative of a unit vector field is perpendicular to itself.

- $\mathbf{t}(s) = \boldsymbol{\alpha}'(s)$: unit tangent vector.
- $\mathbf{n}(s)$: unit normal vector
- $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$: binormal vector, which is normal to the *osculating plane* spanned by $\mathbf{t}$ and $\mathbf{n}$.

$\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ forms an orthonormal basis, i.e., Frenet frame associated with the curve.

**Derivative of the Frenet frame.**
$\mathbf{b}'(s) = \mathbf{t}'(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) = \mathbf{t}(s) \wedge \mathbf{n}'(s)$

- $\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s)$. $\tau(s)$ : torsion. $\tau(s) = 0$ for planar curves.
- **Note.** W. Chen's book defines $\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s)$, but do Carmo's book defines $\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$. We use the former one.

$\mathbf{n}'(s) = \mathbf{b}'(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge \mathbf{t}'(s) = \tau\mathbf{b} - \kappa\mathbf{t}$.
**Frenet equation.**

$$\begin{cases} \mathbf{t}' = \kappa\mathbf{n} \\ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' = -\tau\mathbf{n} \end{cases} \tag{4}$$

and equivalently,

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{5}$$

### 2.4 Fundamental theorem of the local theory of curves

**Theorem 2.1.** *Given differentiable functions* $\kappa(s) > 0$ *and* $\tau(s)$*,* $s \in I$*,* $\exists$ *a regular parameterized curve* $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ *such that* $s$ *is the arc length,* $\kappa(s)$ *is the curvature and* $\tau(s)$ *is the torsion of* $\boldsymbol{\alpha}$*. Any other curve* $\tilde{\boldsymbol{\alpha}}$ *satisfying the same condition differs from* $\boldsymbol{\alpha}$ *by a rigid motion* $\tilde{\boldsymbol{\alpha}} = \mathbf{R}\boldsymbol{\alpha} + \mathbf{c}$*,* $\mathbf{R} inSO(3)$ *and* $\mathbf{c} \in \mathbb{R}^3$*.*

*Proof.* Sketch of proof: 1. Existence and uniqueness of solutions of ODEs. 2. Frenet frame. □

### 2.5 Planar curves

In $\mathbb{R}^2$, $\boldsymbol{\alpha}(s) = (x(s), y(s))$.

- unit tangent $\mathbf{t}(s) = (x'(s), y'(s))$.
- $\mathbf{t}'(s) = \kappa_r(s)\mathbf{n}(s)$. $\kappa_r(s)$ : relative curvature. Here $\mathbf{n}(s)$ is defined by rotating $\mathbf{t}(s)$ counterclockwise with 90°, i.e., $\mathbf{n}(s) = (-y'(s), x'(s))$.
- $\kappa_r(s) = \mathbf{t}'(s) \cdot \mathbf{n}(s) = -x''(s)y'(s) + y''(s)x'(s)$
- Frenet equation:

$$\begin{cases} \boldsymbol{\alpha}'(s) = \mathbf{t}(s) \\ \mathbf{t}'(s) = \kappa_r(s)\mathbf{b}(s) \\ \mathbf{b}'(s) = -\kappa_r(s)\mathbf{t}(s) \end{cases} \tag{6}$$

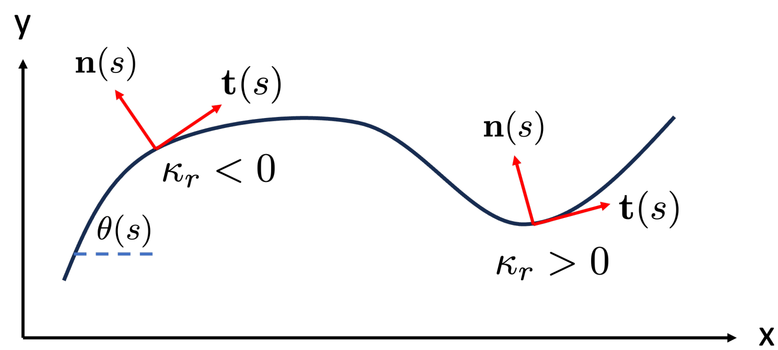



Figure 1: Planar curves.

$\theta(s)$ parameterization: $\theta(s)$ is the angle from $x$ to $\mathbf{t}(s)$.

- $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$, $\mathbf{b}(s) = (-\sin\theta(s), \cos\theta(s)) \implies \kappa_r(s) = \mathrm{d}\theta(s)/\mathrm{d}(s)$
- Fundamental theorem:

$$\begin{cases} \theta(s) = \theta(s_0) + \int_{s_0}^{s} \kappa_r(s)\mathrm{d}s \\ x(s) = x(s_0) + \int_{s_0}^{s} \cos\theta(s)\mathrm{d}s \\ y(s) = y(s_0) + \int_{s_0}^{s} \sin\theta(s)\mathrm{d}s \end{cases} \tag{7}$$

### 2.6 Implicit planar curves

$F(x, y) = 0$

- Implicit function theorem: If $\left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0$, then in a neighborhood of $(x_0, y_0)$ we can write $y = f(x)$ where $f$ is real function.
- Tangent. $F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0$

- Curvature.

$$\kappa = \left| \frac{-F_y^2 F_{xx} + 2F_x F_y F_{xy} - F_x^2 F_{yy}}{(F_x^2 + F_y^2)^{3/2}} \right| \tag{8}$$

- Example. $(x^2 + y^2)^2 = x^2 - y^2$

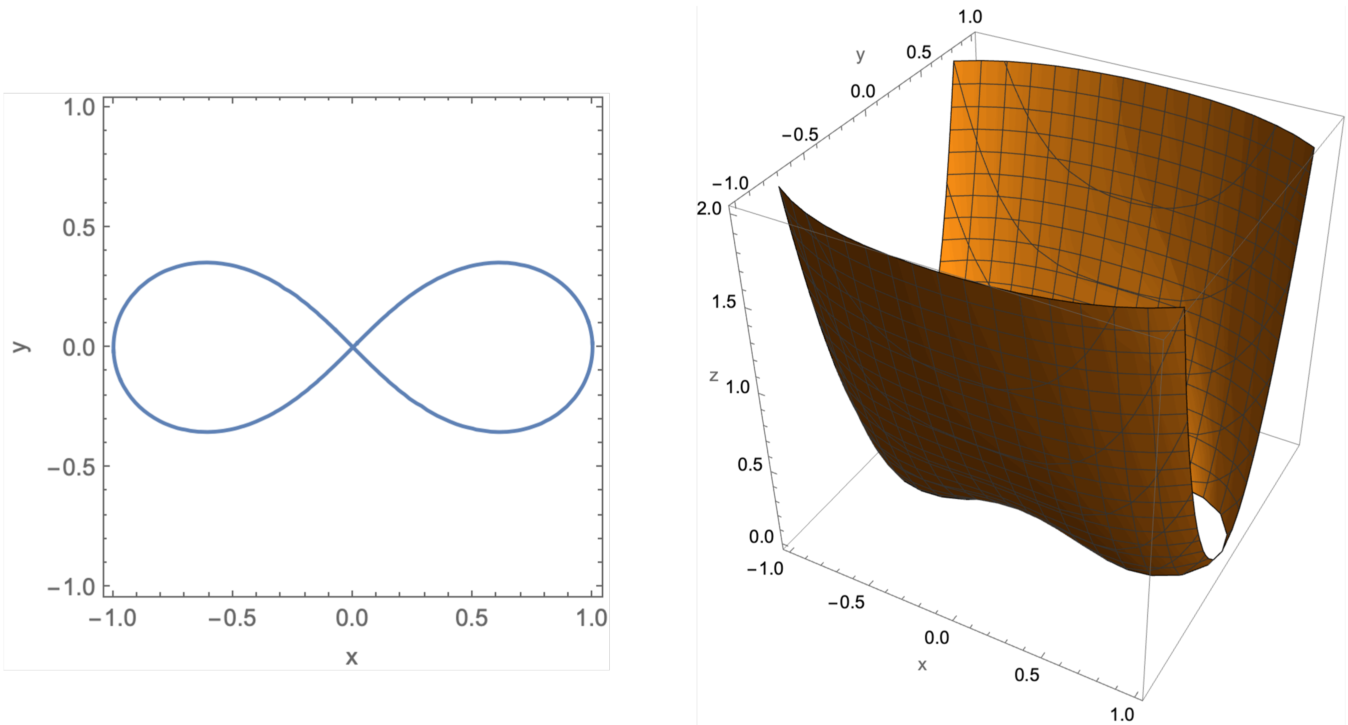



Figure 2: Example.

### 2.7 Some fun global properties (do Carmo's book, Chapter 1-7)

- The isoperimetric inequality: $l^2 - 4\pi A \leq 0$ where $l$ is the length of a simple closed planar curve and $A$ is the area it bounds.
- $\int_0^l \kappa_r(s)\mathrm{d}s = \theta(l) - \theta(0) = 2\pi I$ where $I$ is the rotation index. For simple closed planar curves, $I = \pm 1$.
- The four-vertex theorem. A vertex of a regular plane curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ is a point $t \in [a, b]$ where $\kappa'(t) = 0$. A simple closed convex curve has at least four vertices.
- The Cauchy-Crofton formula. Let $C$ be a regular plane curve with length $l$. The measure of the set of straight lines (counted with multiplicities) which meet $C$ is equal to $2l$.