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Γ-convergence for Beginners

Andrea Braides
Professor of Mathematics Università di Roma 'Tor Vergata'

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A mio padre

PREFACE

The object of Γ-convergence is the description of the asymptotic behaviour of families of minimum problems, usually depending on some parameters whose nature may be geometric or constitutive, deriving from a discretization argument, an approximation procedure, etc. Since its introduction by De Giorgi in the early 1970s Γ-convergence has gained an undiscussed role as the most flexible and natural notion of convergence for variational problems, and is now been widely used also outside the field of the Calculus of Variations and of Partial Differential Equations. Its flexibility is due to its being linked to no a priori *ansatz* on the form of minimizers, which is in a sense automatically described by a process of optimization. In this way Γ-convergence is not bound to any prescribed setting, and it can be applied to the study of problems with discontinuities in Computer Vision as well as to the description of the overall properties of nonlinear composites, to the formalization of the passage from discrete systems to continuum theories, to the modelling of thin films or plates, etc., and may be potentially of help in a great variety of situations where a variational limit intervenes or an approximation process is needed.

This books stems from the lecture notes of a course I gave at the SISSA in Trieste in Spring 1999 aimed at all PhD students in Applied Functional Analysis. The idea of the course was to describe all the main features of Γ-convergence to an audience interested in applications but not necessarily wishing to work in that field of the Calculus of Variations, and at the same time to give a simplified introduction to some topics of active research. After a brief presentation of the main abstract properties of Γ-convergence, the lectures were organized as a series of examples in a one-dimensional setting. This choice was aimed at separating those arguments proper of the variational convergence from the technicalities of higher dimensions that render the results at times much more interesting but often are not directly related to the general issues of the convergence process. This structure (with some changes in the order of the chapters) is kept also in the present book, with the addition of some final chapters, which are thought as an introduction to a selected choice of higher-dimensional problems. The scope of this final part of the book is showing how, contrary to what happens for differential equations where passing from Ordinary Differential Equations to Partial Differential Equations and then to systems involves a substation change of viewpoint, the main arguments of Γ-convergence essentially remain unchanged when passing from one-dimensional problems to higher-dimensional ones and from scalar to vector-valued functions. Apart from these chapters 'for the advanced beginner' (which require some notions on Sobolev spaces and whose title is marked by an asterisk) the rest of the book is reasonably self contained, requiring standard notions of Measure Theory and basic Functional Analysis.

I have tried to describe the principles of variational convergence rather than include the sharpest results. Hence, I have frequently chosen proofs that are not the most efficient for the specific result but illustrate most clearly the arguments that can be repeated elsewhere or the technical points that can be generalized to more complex situations. Conversely, I have frequently left minor details as an exercise. All chapters have a final section of comments where some more refined issues are addressed, an outline of the higher-dimensional problems is often sketched, and some bibliographical indication is given. Since this is not thought as a research book on each single subject treated (homogenization, phase transitions, free-discontinuity problems, etc.) I refer to other monographs for complete references on established results. On the contrary, I have chosen to include references to the most recent advances in some problems that may interest the research-oriented reader.

As an advice for the user, it must be mentioned that it is not by chance that no dynamical problem is treated: Γ-convergence is a purely-variational technique aimed at treating minimum problems, and, even though it may give some precious hints in particular situations, in general it is not designed to treat time-dependent cases. Furthermore, also in the 'static case' the generality of Γ-convergence does not allow to obtain the more accurate results of matched asymptotics techniques whenever a very accurate *ansatz* for optimal sequences is available (for example in linear homogenization).

Finally, I wish to thank the many friends that have fruitfully interacted with me during and after the course at SISSA and another course given at the University of Rome 'La Sapienza' in 2001, where part of the material was again presented. I am indebted to Roberto Alicandro, Nadia Ansini, Marco Cicalese, Lorenzo D'Ambrosio, Francesco Del Fra, Gianpietro Del Piero, Maria Stella Gelli and Chiara Leone for accurately reading parts of the manuscript of the book, and to Lev Truskinovsky for his enthusiastic support. The final form of the material much owes to the precious advices and critical comments of Giovanni Alberti, whom I regard as the invisible second author of the book. Thanks to Adriana Garroni for help in the figures and for being there.

Rome
February 2002

Andrea Braides

CONTENTS

INTRODUCTION

Dear Sir or Madam will you read my book?
It took me years to write, will you take a look?
(Lennon and McCartney, *Paperback Writer*)

Cosa ne può importare alla casalinga di Treviso,
al bracciante lucano, al pastore abruzzese?
(Nanni Moretti, *Sogni d'oro*)

Why a variational convergence?

In many mathematical problems, may they come from the world of Physics, industrial applications or abstract mathematical questions, some parameter appears (small or large, of geometric or constitutive origin, coming from an approximation process or a discretization argument, at times more than a single parameter) which makes those problems increasingly complex or more and more degenerate. Nevertheless, as this parameter varies, it is often possible to foresee some 'limit' behaviour, and 'guess' that we may substitute the complex, degenerate problems we started with, with a new one, simpler and with a more understandable behaviour, possibly of a completely different type, where the parameters have disappeared, or appear in a more handy way.

Sometimes this type of questions may be studied in a variational framework. In this case, it can be rephrased as the study of the asymptotic behaviour of a family of minimum problems depending on a parameter; in an abstract notation,

$$\min\{F_\varepsilon(u) : \ u \in X_\varepsilon\}. \tag{0.1}$$

The next section provides a number of examples in which F_ε range from singularly-perturbed non-convex problems to highly-oscillating integrals, from discrete energies defined on varying lattices to functionals approximating combined bulk and interfacial energies. The form and the dependence on ε of the solutions in those examples as well as the way they convergence may be very different from case to case.

A way to describe the behaviour of the solutions of (0.1) is provided by substituting such a family by an 'effective problem' (not depending on ε)

$$\min\{F(u) : \ u \in X\}, \tag{0.2}$$

which captures the relevant behaviour of minimizers and for which a solution can be more easily obtained. Γ-convergence is a convergence on functionals which loosely speaking amounts to requiring the convergence of minimizers and of minimum values of problems (0.1) and of their continuous perturbations to those of (0.2) with the same perturbations. In this way the relevant properties of the actual solutions of (0.1) can be approximately described by those of the solution of (0.2). Note that the function space X and the form of the energy F may be very different from those at level ε, so that the way this convergence is defined must be quite flexible.

A fundamental remark is that the effectiveness of Γ-convergence is linked to the possibility of obtaining converging sequences (or subsequences) from minimizers (or almost-minimizers) of (0.1). A preliminary fundamental question is then compactness: the notion of convergence of functions u_ε must be given so that the existence of a limit of minimizers of (0.1) — assuming that they exist — is ensured beforehand. A too strong notion of convergence of functions will result in a useless definition of convergence of energies, simply because minimizers will not converge. The candidate space X for the limit problem is the space where this compactness argument leads.

Once a notion of convergence $u_\varepsilon \to u$ is agreed upon, the way the functional F in the limit problem (0.2) is obtained can be heuristically explained as an optimization between lower and upper bounds. A *lower bound* for F is an energy G such that

$$G(u) \leq F_\varepsilon(u_\varepsilon) + o(1) \tag{0.3}$$

(or in other terms $G(u) \leq \liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon)$) whenever $u_\varepsilon \to u$. The requirement that this hold for all u and u_ε (and not only for $u_\varepsilon = u$ or for minimizers) is a characteristic of Γ-convergence that makes it 'stable under perturbations' and at the same time suggests some structure properties on the candidate G (i.e. lower semicontinuity). Condition (0.3) above implies that

$$\inf\{G(u):\ u \in X\} \leq \lim_{\varepsilon\to0^+} \min\{F_\varepsilon(u):\ u \in X_\varepsilon\}$$

(given the limit exists). The sharpest lower bound is then obtained by optimizing the role of G. The way this is obtained in practice differs greatly from case to case, but always involves some minimization argument: in the case of homogenization the minimization is done in classes of periodic functions, for phase transitions it consists in an optimal profile issue, in the study of non-convex discrete systems it amounts to optimize a 'separation of scales' argument, etc. (see the examples below). A crucial point at this stage is the study of (necessary) conditions for lower semicontinuity, that allows to restrict the class of competing G.

Once it is computed, the optimal G in this procedure suggests an *ansatz* for the form of the minimizing sequences: in the case of homogenization it suggests that minimizers oscillate close to their limit following an energetically-optimal locally-periodic pattern, for phase transitions that sharp phases are approximated by smoothened functions with an optimal profile, in the study of non-convex discrete systems that minimizers are obtained by an optimal two-scale discretization, etc. Using this *ansatz* for each $u \in X$ (and not only for minimizers) we may construct a particular $\overline{u}_\varepsilon \to u$ and define $H(u) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\overline{u}_\varepsilon)$. This H is an *upper bound* for the limit energy, and for such H we have

$$\lim_{\varepsilon\to 0^+} \min\{F_\varepsilon(u):\ u \in X_\varepsilon\} \leq \inf\{H(u):\ u \in X\};$$

that is, to an *ansatz* on approximating sequences there corresponds an estimate 'from above' for the limit of the minimum problems.

The Γ-convergence of F_ε is precisely the requirement that these two bounds coincide, and hence it implies the convergence of (0.1) to (0.2). Having taken care of defining the upper and lower bound energies for all functions and not only for minimizers Γ-convergence enjoys important properties, such as:

— Γ-convergence itself implies the convergence of minimum problems (that for the sake of simplicity was assumed true in the argument above) and the convergence of (sub)sequences of (almost-)minimizers to minimizers of the Γ-limit,

— it is stable under continuous perturbations. This means that our analysis is still valid if we add to all problems any fixed continuous term. In this sense the Γ-limit F provides a 'limit theory' which describes all relevant features of F_ε and not only those related to a specific minimum problem,

— the Γ-limit F is a lower semicontinuous functional. This is a very useful structure property that usually implies existence of minimizers and helps in giving a better description of F through representation results.

Comparing this notion with others used for asymptotic expansion we note that the main issue here is the computation of the lower bound, which uniquely involves minimization and 'optimization' procedures and is totally *ansatz*-free. To this lower bound there corresponds an upper bound where the *ansatz* on minimizers is automatically driven by the lower bound itself. As a result Γ-convergence does not require the computation of minimizers of (0.1) — that indeed may or may not exist — nor the solution of the associated Euler–Lagrange equations, and it is not linked to any structure of X_ε and X.

It must be mentioned that, given the generality of applications of Γ-convergence, whenever a good *ansatz* for minimizers is reached, additional *ad hoc* techniques should be also used to give a more complete characterization of the convergence of minimum problems. This is the case, for example, of periodic (linear) homogenization where asymptotic expansion in locally-periodic functions provide a more complete description of the behaviour of minimizers, and finer issues can be fruitfully addressed by different methods. The same example of homogenization shows that we must be very careful when we start from an *ansatz* that looks completely natural but is not justified by a convergence result: in the vector-valued non-linear case minimizers are in general (locally) almost periodic (i.e. oscillations at all scales must be taken into account). This behaviour is natural from the viewpoint of Γ-convergence but it is easily missed if we start from the wrong assumptions on the (local) periodicity of minimizing sequences.

In the rest of this chapter we provide a series of examples, which serve also as an introduction to the core of the book, and a final section in which we introduce the definition of Γ-convergence as a 'natural' extension of the so-called direct methods of the Calculus of Variations.

Parade of examples

In this section we include a number of examples, in which we show how a notion of variational convergence must be sensible, as it must include cases where the limit problem is set on a space X completely different from all X_ε, and even when X is the same it may be very different from pointwise convergence. Furthermore, by describing the approximate forms of minimizers in these examples, which will be obtained as a final result in the Γ-convergence process and exhibit a variety of structures, we want to highlight how the convergence must not rely on any a priori *ansatz* on the asymptotic form of minimizers, and it should in a sense itself suggest the precise meaning of this asymptotic question, as this could not be supplied by problems (0.1). These examples will be dealt with in detail in the next chapters.

Example 0.1 (gradient theory of phase transitions). The simplest example that shows a dramatic change of type in the passage to the limit, is perhaps that of the *gradient theory of phase transitions* for a homogeneous isothermal fluid contained in a bounded region Ω. If we denote the concentration of the fluid with a function $u : \Omega \to [0,1]$, then the equilibrium configurations are described as minimizing a suitable energy depending on u under a mass constraint:

$$\min\Big\{E(u) : \ u : \Omega \to [0,1], \ \int_\Omega u\,dx = C\Big\}, \tag{0.4}$$

where the energy is of the form

$$E(u) = \int_\Omega W(u)\,dx. \tag{0.5}$$

The energy density $W : (0, +\infty) \to \mathbf{R}$ is a non-convex function given by the Van der Waals Cahn Hilliard theory, whose graph is of the form as in Fig. 0.1.

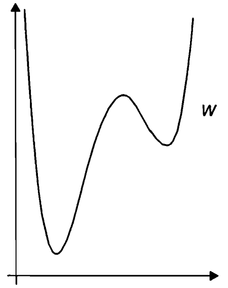

FIG. 0.1. The van der Waals energy density

In order to make problem (0.4) clearer and understand the properties of minimizers, we may add an affine change of variable to W; that is, consider $W(u) + c_1 u + c_2$ in place of W. Note that this change in the energy density does not affect the minimum problem (0.4) since it amounts to add the fixed quantity

$$\int_\Omega (c_1 u + c_2)\, dx = c_1 C + c_2 |\Omega|$$

to $E(u)$. At this point, we may choose c_1 and c_2 so that the new energy density, which we still denote by W, is non-negative and has precisely two zeros at points α and β, as in Fig. 0.2.

It is clear now that, if this is allowed by the mass constraint, minimizers of (0.4) will be simply given by (all!) functions u which take only the values α and β and still satisfy the constraint $\int_\Omega u\, dx = C$. For such a function the two regions $\{u = \alpha\}$ and $\{u = \beta\}$ are called the two *phases* of the fluid and form a partition of Ω. Note that minimizing problem (0.4) does not provide *any* information about the interface between the two phases, which may be irregular or even dense in Ω. This is not what is observed in those equilibrium phenomena: among these minimizers some special configuration are preferred, instead, and precisely those with minimal interface between the phases. This *minimal-interface criterion* is interpreted as a consequence of higher-order terms: in order to prevent the appearance of irregular interfaces, we add a term containing the derivative of u as a *singular perturbation*, which may be interpreted as giving a (small) *surface tension* between the phases. The new problem, in which we see the appearance of a small positive parameter ε, takes the form

$$\min\Big\{\int_\Omega (W(u) + \varepsilon^2 |Du|^2)\, dx : \int_\Omega u\, dx = C\Big\} \tag{0.6}$$

(the power ε^2 comes from dimensional considerations), where now some more regularity on u is required. The solutions to this problem indeed have the form

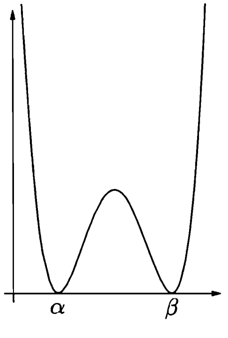

FIG. 0.2. The energy density after the affine translation

$$u_\varepsilon(x) \approx u(x) + u_1\Big(\frac{\operatorname{dist}(x,S)}{\varepsilon}\Big),$$

where $u : \Omega \to \{\alpha, \beta\}$ is a phase-transition function with minimal interface S in Ω, and $u_1 : \mathbf{R} \to \mathbf{R}$ is a function with limit 0 at infinity, which gives the optimal profile between the phases at $\varepsilon > 0$. Fig. 0.3 picture a minimizer u_ε corresponding to a minimal u with a minimal (linear) interface between the phases.

This is a natural *ansatz* and is proved rigorously by a Γ-convergence arguments. We can picture this behaviour in the one-dimensional case, where, then, u is simply a function with a single discontinuity point. In Fig. 0.4 are represented functions u_ε for various values of ε.

The behaviour of u_ε cannot be read out directly by examining small-energy functions for problem (0.6), but may be more easily deduced if that problem is rewritten as

$$\min\Big\{\int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon|Du|^2\Big)\,dx : \int_\Omega u\,dx = C\Big\}. \tag{0.7}$$

In this way it may be seen that the contributions of the two terms in the integral have the same order as ε tends to 0 for minimizing sequences; the qualitative

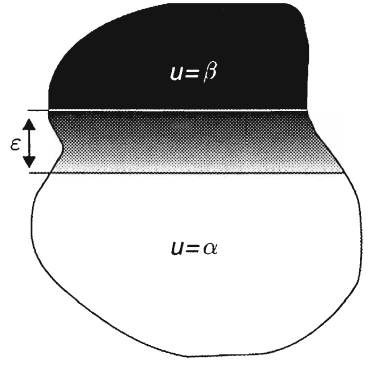

FIG. 0.3. Approximate phase transition with a minimal interface

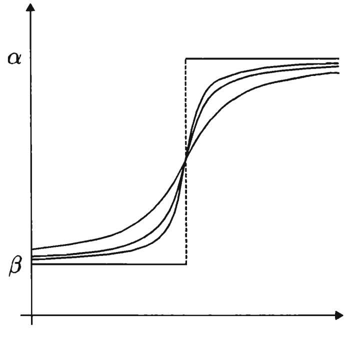

FIG. 0.4. Behaviour of approximate phase transitions

effect of the first term is that u tends to get closer to α or β, while the effect of the second term is to penalize unnecessary interfaces. It can (and it will) be seen that problem (0.7) is well approximated as ε gets small by a *minimal interface problem*:

$$\min\Big\{\operatorname{Per}(\{u=\alpha\},\Omega) : u:\Omega\to\{\alpha,\beta\},\ \int_\Omega u\,dx = C\Big\}, \tag{0.8}$$

where $\operatorname{Per}(A,\Omega)$ denotes the (suitably defined) perimeter of A in Ω. In this case we have a complete change of type in the problems: in particular, while problem (0.7) involves only (sufficiently) smooth functions, its limit counterpart (0.8) gets into play only discontinuous functions. The treatment by Γ-convergence of this example will be done in Chapters 6 and 14.

Example 0.2 (homogenization of variational problems). Another class of problems, which can be (partly) set in this framework are some types of *homogenization problems*. 'Homogenization' is a general term which underlines the asymptotic description of problems with increasingly oscillating solutions. In its simplest form it regards the description of static phenomena involving the study of minimum points of some energy functional whose energy density is periodic on a very small scale (see Fig. 0.5). The simplest case is related to the stationary heat equation in a (locally isotropic) composite medium of $\mathbf{R}^n$ of thermal conductivity $a(x/\varepsilon)$ occupying a region Ω. The function a is periodic (say, of period one) in each coordinate direction, so that the integrand above is periodic of period ε. To fix ideas we may assume that a takes only two values (say, α and β). In this case the medium we have in mind is a *composite* of two materials whose 'microscopic pattern' is described by the function a. If f is a source term and we impose a boundary condition (for simplicity homogeneous) the temperature u_ε will satisfy

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u}{\partial x_j}\Big) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

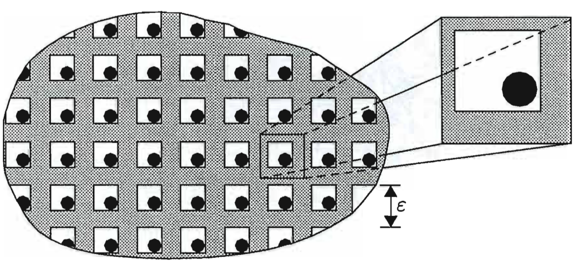

FIG. 0.5. A 'composite medium' and its microscopic pattern

or, equivalently, can be characterized as the minimizer of

$$\min\Big\{E_\varepsilon(u) - 2\int_\Omega fu\,dx : u = 0 \text{ on } \partial\Omega\Big\}, \tag{0.9}$$

where

$$E_\varepsilon(u) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|Du|^2\,dx. \tag{0.10}$$

If the dimensions of the set Ω are very large with respect to ε we may expect that the overall 'macroscopic' behaviour of the medium described above is 'very similar' to a (now, possibly anisotropic) homogeneous material. Indeed the solutions u_ε of (0.9) 'oscillate' close to the solution of a limit problem as we let ε tend to 0; that is, they have the form, at least locally in Ω, (see Figure 0.6)

$$u_\varepsilon(x) \approx u(x) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big). \tag{0.11}$$

The function u is the solution of a problem of the type

$$\min\Big\{\int_\Omega \sum_{i,j} q_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx - 2\int_\Omega fu\,dx : u = 0 \text{ on } \partial\Omega\Big\}. \tag{0.12}$$

The constant 'homogenized' coefficients q_{ij} do not depend on f and Ω, and can be computed directly from a through some auxiliary minimum problems on sets of periodic functions. It is instructive to look at the one-dimensional case, where $\Omega \subset \mathbf{R}$; in this case the *pointwise* limit of the functionals E_ε exists and is given simply by

$$E(u) = \overline{a}\int_\Omega |u'|^2\,dt, \tag{0.13}$$

where the coefficient $\overline{a}$ is the *mean value* of a, $\overline{a} = \int_0^1 a(s)\,ds$, but the coefficient $\widehat{a}$ ($= q_{11}$ in this simple case) appearing in (0.12) is given by the *harmonic mean* of a:

$$\widehat{a} = \Big(\int_0^1 \frac{1}{a(s)}\,ds\Big)^{-1}. \tag{0.14}$$

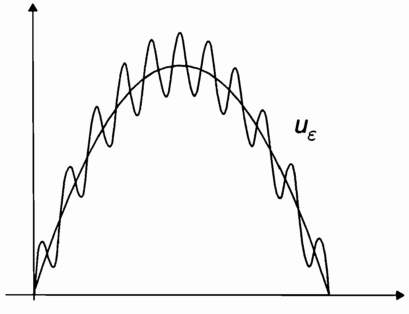

FIG. 0.6. Oscillating solutions and their limit

This observation shows an interesting non-trivial effect of the oscillations in the minimizing sequences, that interact with those of $a(x/\varepsilon)$. By optimizing this interaction we obtain the value of $\widehat{a}$. Some issues of this type of homogenization will be addressed in Chapters 3 and 12.

Another 'homogenization' problem intervenes as a question concerning the convergence of distances. A Riemannian distance is characterized (in local coordinates) by minimum problems of the form

$$\min\Big\{\int_0^1 \sum\nolimits_{i,j} a_{ij}(u)u_i'u_j'\,dx : u(0)=u_0,\ u(1)=u_1\Big\}, \tag{0.15}$$

where $u:[0,1]\to\mathbf{R}^n$ vary among all (regular) curves joining u_0 and u_1. In some problems (e.g. when dealing with families of *viscosity solutions*) it is necessary to characterize the limit of *oscillating* Riemannian metrics of the form

$$\min\Big\{\int_0^1 \sum\nolimits_{i,j} a_{ij}\Big(\frac{u}{\varepsilon}\Big)u_i'u_j'\,dx : u(0)=u_0,\ u(1)=u_1\Big\}. \tag{0.16}$$

In this case we may still characterize the limit of these problems, but it can be seen that it is not related to a Riemannian metric anymore; that is, such problems behave as $\varepsilon\to 0$ as

$$\min\Big\{\int_0^1 \psi(u')\,dx : u(0)=u_0,\ u(1)=u_1\Big\}, \tag{0.17}$$

but in general ψ is not a quadratic form. To understand this behaviour it is instructive to consider the case when $n=2$ and $a_{ij}(u)=a(u)\delta_{ij}$ (δ_{ij} denotes Kronecker's delta), and a models a 'chessboard structure' with two values α, β with $\sqrt{2}\,\alpha<\beta$. With this condition, it is 'not convenient' for the competing curves u in (0.16) to cross the β region, and with this constraint in mind it is easy to find the exact form of ψ and to check that it is not a quadratic form (see Chapter 3). The solutions to (0.16) are pictured in Fig. 0.7.

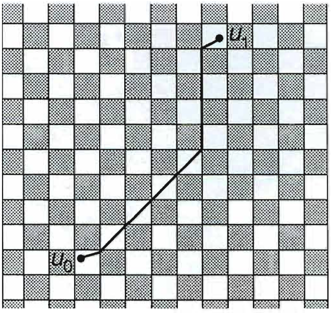

FIG. 0.7. Curves of minimal distance on the 'chessboard' are not line segments

The choice of the scaling (i.e. the dependence on ε) is not always as obvious as above. Another 'classical' example of problems in a periodic setting is that of Dirichlet problems in *perforated domains*. In this case the problem we encounter is of the form

$$\min\Big\{\int_{\Omega_\varepsilon} |Du|^2\,dx - 2\int_{\Omega_\varepsilon} fu\,dx : u = 0 \text{ on } \partial\Omega_\varepsilon\Big\}, \tag{0.18}$$

where Ω_ε is a 'perforation' of a fixed bounded open set $\Omega \subset \mathbf{R}^n$. The simplest case is when Ω_ε is obtained by removing from Ω a periodic array of closed balls of equal radius $\delta = \delta(\varepsilon)$ with centres placed on a regular lattice of spacing ε; that is, of the form

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{\mathbf{i}\in\mathbf{Z}^n} \overline{B(\varepsilon\mathbf{i}, \delta(\varepsilon))} \tag{0.19}$$

(see Fig. 0.8). In terms of the corresponding stationary heat equation, the condition $u = 0$ can be interpreted as the presence of evenly distributed small particles at a fixed temperature (it is suggestive to think of ice mixed with water) in the interior of Ω.

The behaviour as ε gets smaller is trivial when $n = 1$ (the solutions simply tend to 0 since they are equi-continuous and vanish on a set which tends to be dense), but gives rise to an interesting phenomenon when $n \geq 2$ and $\delta = \delta(\varepsilon)$ is appropriately chosen. Let $n \geq 3$ for the sake of simplicity; in this case the interesting case is when

$$\delta(\varepsilon) \approx \varepsilon^{n/(n-2)}, \tag{0.20}$$

all other cases giving trivial results: either the effect of the perforation is negligible, and the boundary condition in the interior of Ω disappears, or it is too strong, and it forces the solutions to tend to zero. The case (0.20) is the intermediate situation where the effect of the perforation is of the same order as that of the Dirichlet energy and it penalizes the distance of the solution from 0 in a very precise manner. The overall effect as ε tends to 0 is that u_ε 'behave approximately' as the solution u of the problem

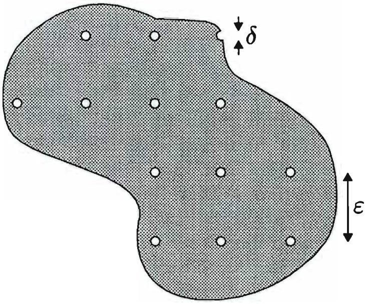

FIG. 0.8. A 'perforated' domain

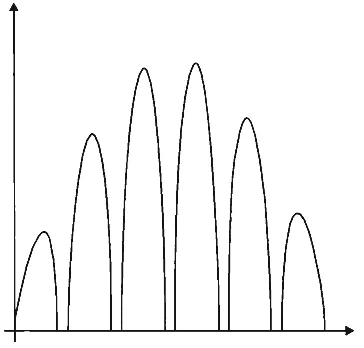

FIG. 0.9. Behaviour of oscillating solutions ('cross section')

$$\min\Big\{\int_\Omega (|Du|^2 + C|u|^2)\,dx - 2\int_\Omega fu\,dx : u \in H_0^1(\Omega)\Big\}, \tag{0.21}$$

meaning that (at least in the interior of Ω)

$$u_\varepsilon(x) \approx u(x)\Big(1 - \sum_{\mathbf{i}} u_1\Big(\frac{x-\varepsilon\mathbf{i}}{\varepsilon^{n/n-2}}\Big)\Big), \tag{0.22}$$

where u_1 is a 'capacitary potential' decreasing to 0 at infinity and with $u_1 = 1$ on the unit ball, and the constant C is computed explicitly and does not depend on f. Figure 0.9 pictures the behaviour of the solutions on a one-dimensional section passing through the perforation.

Note that even though we remain in the same functional space the form of the limit energy is different from the approximating ones and it has an additional 'strange term coming from nowhere' (as baptized by Cioranescu and Murat). An explanation in terms of Γ-convergence is given in Chapter 13.

Example 0.3 (dimension reduction). Other problems where a small parameter ε appears are asymptotic theories of elastic plates, shells, films and rods.

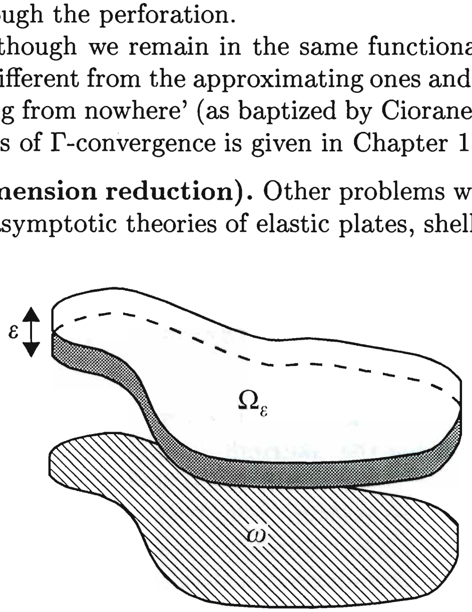

FIG. 0.10. A 'thin' domain

In this case the goal is a rigorous derivation of a lower (one- or two-)dimensional theory (for elastic plates, shells, etc.) from the corresponding three-dimensional one. The starting point (e.g. for thin films) is then to consider energies of the form

$$E_\varepsilon(u) = \int_{\Omega_\varepsilon} f(Du)\,dy, \tag{0.23}$$

where $u : \Omega \to \mathbf{R}^3$ and the domain, in the simplest case of flat films, is of the form

$$\Omega_\varepsilon = \{(y', y_3) : y' \in \omega,\ 0 < y_3 < \varepsilon\} \tag{0.24}$$

and ω is a fixed bounded open subset of $\mathbf{R}^2$ (see Figure 0.10). The simplest type of problems related to such energies are of the form

$$\min\{E_\varepsilon(u) : u = \phi \text{ on } (\partial\omega) \times (0, \varepsilon)\}, \tag{0.25}$$

where $\phi = \phi(y')$ and the boundary conditions are given only on the 'vertical' boundary. After scaling (dividing by ε) the energy E_ε and the change of variables $x' = y'$, $\varepsilon x_3 = y_3$, we have the equivalent energies

$$F_\varepsilon(u) = \int_{\omega\times(0,1)} f\Big(D_1 u, D_2 u, \frac{1}{\varepsilon} D_3 u\Big)\,dy. \tag{0.26}$$

We now have a family of scaled energies, which are defined on a common space of functions, but which tend to degenerate with respect to the derivative in the third direction as ε tends to 0. Problems (0.25) can be rewritten as

$$\min\{F_\varepsilon(u) : u = \phi \text{ on } (\partial\omega) \times (0, 1)\}. \tag{0.27}$$

If u_ε are solutions to such problems, in view of (0.26), one expects that $D_3 u_\varepsilon$ tend to 0 and hence the limit actually to be independent of the third variable. Indeed we have that

$$u_\varepsilon(x) \approx u(x') + \varepsilon x_3 b(x'), \tag{0.28}$$

where u minimizes a two-dimensional problem

$$\min\Big\{\int_\omega \tilde f(D_1 u, D_2 u)\,dx' : u = \phi \text{ on } \partial\omega\Big\}. \tag{0.29}$$

The function $\tilde f$ is independent of the boundary datum ϕ and it is obtained, heuristically, by minimizing the contribution of the function b in (0.26). In this case both the problems at fixed ε and at the limit have the same form, but on domains of different dimension. Minimizing sequences do not necessarily develop oscillations, but the limit lower dimensional theory may not be derived in a trivial way from the full three-dimensional one.

An outline of the approach by Γ-convergence to dimension reduction is given in Chapter 14.

Example 0.4 (approximation of free-discontinuity problems). The terminology 'free-discontinuity problems' (as opposed to free-*boundary* problems) denotes a class of problems in the Calculus of Variations where the unknown is a pair (u, K) with K varying in a class of (sufficiently smooth) closed hypersurfaces contained in a fixed open set $\Omega \subset \mathbf{R}^n$ and $u : \Omega \setminus K \to \mathbf{R}^m$ belonging to a class of (sufficiently smooth) functions. Such problems usually consist in minimizing an energy with competing *volume* and *surface* energies. The main examples in this framework are variational theories in Image Reconstructions and Fracture Mechanics. In the first case $n = 2, m = 1$ and the so-called Mumford Shah functional is taken into account

$$E(u,K) = \int_{\Omega\setminus K} |Du|^2\,dx + c_1\,\mathrm{length}(K) + c_2 \int_{\Omega\setminus K} |u-g|^2\,dx. \tag{0.30}$$

Here, the function g is interpreted as the input picture taken from a camera, u is the 'restored' image, and K is the relevant contour of the objects in the picture; c_1 and c_2 are contrast parameters. Note that the problem is meaningful also adding the constraint $Du = 0$ outside K, in which case we have a *minimal partitioning* problem. In the case of *fractured hyperelastic media* $m = n = 3$ and the volume and surface energies taken into account are very similar (with the area of K in place of the length), Ω is interpreted as the reference configuration of an elastic body, K is the crack surface, and u represents the elastic deformation in the unfractured part of the body.

Functionals arising in free-discontinuity problems present some drawbacks; for example, numerical difficulties arise in the detection of the unknown discontinuity surface. To bypass these drawbacks, a considerable effort has been spent to provide variational approximations, in particular of the Mumford Shah functional E defined above, with differentiable energies defined on smooth functions. An approximation was given by Ambrosio and Tortorelli, who followed the idea of the gradient theory of phase transitions introducing an approximation with an auxiliary variable v. A family of approximating functionals is the following:

$$\begin{aligned} G_\varepsilon(u,v) = &\int_\Omega v^2|Du|^2\,dx + \frac{1}{2}\int_\Omega \Big(\varepsilon|Dv|^2 + \frac{1}{\varepsilon}(1-v)^2\Big)\,dx \\ &+c_2 \int_\Omega |u-g|^2\,dx\,, \end{aligned} \tag{0.31}$$

defined on regular functions u and v with $0 \le v \le 1$. Heuristically, the new variable v in the limit as $\varepsilon \to 0$ approaches $1 - \chi_K$ and introduces a penalty on the length of K in the same way as a phase transition. Since the functionals G_ε are elliptic, even though non-convex, numerical methods can be applied to them. It is interesting to note that the functionals G_ε may have also an interpretation in terms of Fracture Mechanics, as v can be thought as a pointwise *damage parameter.*

Free-discontinuity problems and their approximations are dealt with in Chapters 7 and 8.

Example 0.5 (continuous limits of difference schemes). Another interesting problem is that of the description of variational limits of discrete problems (for the sake of brevity in a one-dimensional setting). Given $n \in \mathbf{N}$ and points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*, which plays the role of the small parameter ε) we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big).$$

If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and u_i represents the displacement of the i-th point, then ψ_n^j can be thought as the energy density of the interaction of points with distance $j\lambda_n$ (j lattice spacings) in the reference lattice. Note that the only assumption we make is that ψ_n^j depends on $\{u_i\}$ through the differences $u_{i+j}-u_i$, but we find it more convenient to highlight its dependence on 'discrete difference quotients'. For a quite general class of energies it is possible to describe the behaviour of solutions of problems of the form

$$\min\Big\{E_n(\{u_i\}) - \sum_{i=0}^{n} \lambda_n u_i f_i : \ u_0 = U_0, \ u_n = U_L\Big\}$$

(and similar), and to show that these problems have a limit continuous counterpart. Their solutions then can be though of (non-trivial) discretizations of the corresponding solution on the continuum. Here $\{f_i\}$ represent the external forces and U_0, U_L are the boundary conditions at the endpoints of the interval $(0, L)$. More general statement and different problems can also be treated. Under some growth conditions, minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min\Big\{\int_0^L (\psi(u') - fu)\,dt : \ u(0) = U_0, \ u(L) = U_L\Big\}.$$

The energy densities ψ can be explicitly identified by a series of operations on the functions ψ_n^j. The case when only *nearest-neighbour interactions* are taken into account,

$$E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big),$$

is particularly simple. In this case, the limit energy density is given by the limit of the convex envelopes of the functions $\psi_n(z)$, which exists up to subsequences. The description of the limit energy gets much more complex when not only nearest-neighbour interactions come into play and involves arguments of homogenization type which highlight that the overall behaviour of a system of interacting points will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities' (see Chapters 4 and 11).

A maieutic approach to Γ-convergence. Direct methods

The scope of this section is to show that we may 'naturally' derive the definition of Γ-convergence for functionals from the requirements that

— it implies the convergence of minimizers and minimum values (under suitable assumptions),

— it is stable under continuous perturbations, and

— it is given in *local* terms (i.e. we can also speak of convergence 'at one point').

The starting point will be the examination of the so-called *direct methods of the Calculus of Variations*. For the sake of simplicity, from here onwards all our problems will be set on metric spaces, so that the topology is described by just using sequences. The idea is very simple: in order to prove the existence of a minimizer of a problem of the form

$$\min\{F(u):\ u \in X\}, \tag{0.32}$$

we examine the behaviour of a *minimizing sequence*; that is, a sequence $(\overline{u}_j)$ such that

$$\lim_j F(\overline{u}_j) = \inf\{F(u):\ u \in X\}, \tag{0.33}$$

which clearly always exists. Such a sequence, in general might lead nowhere. The first thing to check is then that we may find a *converging* minimizing sequence. This property may be at times checked by hand, but it is often more convenient to check that an *arbitrary* minimizing sequence lies in a *compact* subset K of X (i.e. since X is metric, that for any sequence (u_j) in K we can extract a subsequence (u_{j_k}) converging to some $u \in K$). This property is clearly stronger than requiring that there exists *one* converging minimizing sequence, but its verification often may rely on a number of characterizations of compact sets in different spaces. In its turn this compactness requirement can be directly made on the functional F by asking that it be *coercive*; that is, that for all t its sub-level sets $\{F < t\} = \{u \in X : F(u) < t\}$ are *pre-compact* (this means that for fixed t there exists a compact set K_t containing $\{F < t\}$, or, equivalently, in terms of sequences, that for all sequences (u_j) with $\sup_j F(u_j) < +\infty$ there exists a converging subsequence). Again, this is an even stronger requirement, but it may be derived directly from the form of the functional F and not from special properties of minimizing sequences. Once some compactness properties of a minimizing sequence are established, we may extract a (minimizing) subsequence, that we still denote by $(\overline{u}_j)$, converging to some $\overline{u}$.

At this stage, the point $\overline{u}$ is a candidate to be a minimizer of F; we have to prove that indeed

$$F(\overline{u}) = \inf\{F(u):\ u \in X\}. \tag{0.34}$$

One inequality is trivial, since $\overline{u}$ can be used as a test function in (0.34) to obtain an *upper inequality*: $\inf\{F(u):\ u \in X\} \le F(\overline{u})$.

To obtain a *lower inequality* we have to link the value at $\overline{u}$ to those computed at $\overline{u}_j$, to obtain the right inequality

$$F(\overline{u}) \leq \lim_j F(\overline{u}_j) = \inf\{F(u):\ u \in X\}. \tag{0.35}$$

Since we do not want to rely on special properties of $\overline{u}$ or of the approximating sequence $(\overline{u}_j)$, but instead we would like to isolate properties of the functional F, we require that *for all* $u \in X$ and *for all* sequences (u_j) tending to u we have the inequality

$$F(u) \leq \liminf_j F(u_j). \tag{0.36}$$

This property is called the *lower semicontinuity* of F. It is much stronger than requiring (0.35), but it may be interpreted as a structure condition on F and often derived from general considerations.

At this point we have not only proven that F admits a minimum, but we have also found a minimizer $\overline{u}$ by following a minimizing sequence. We may condensate the reasoning above in the following formula

$$\text{coerciveness} \quad + \quad \text{lower semicontinuity} \quad \Rightarrow \quad \text{existence of minimizers}, \tag{0.37}$$

which summarizes the direct methods of the Calculus of Variations. It is worth noticing that the coerciveness of F is easier to verify if we have *many* converging sequences, while the lower semicontinuity of F is more easily satisfied if we have *few* converging sequences. These two opposite requirements will result in a balanced choice of the metric on X, which is in general not given a priori, but in a sense forms a part of the problem.

We now turn our attention to the problem of describing the behaviour of a family of minimum problems depending on a parameter. In order to simplify the notation we deal with the case of a sequence of problems

$$\inf\{F_j(u):\ u \in X_j\} \tag{0.38}$$

depending on a discrete parameter $j \in \mathbf{N}$; the case of a family depending on a continuous parameter ε introduces only a little extra complexity in the notation. As j increases we would like these problems to be approximated by a 'limit theory' described by a problem of the form

$$\min\{F(u):\ u \in X\}. \tag{0.39}$$

In order to make this notion of 'convergence' precise we try to follow closely the direct approach outlined above. In this case we start by examining a minimizing sequence for the family F_j; that is, a sequence $(\overline{u}_j)$ such that

$$\lim_j \Big(F_j(\overline{u}_j) - \inf\{F_j(u):\ u \in X_j\}\Big) = 0, \tag{0.40}$$

and try to follow this sequence.

In many problems the space X_j indeed varies with j, so that now we have to face a preliminary problem of defining the convergence of a sequence of functions which belong to different spaces. This is usually done by choosing X large enough so that it contains the domain of the candidate limit and all X_j. We can always consider all functionals F_j as defined on this space X by identifying them with the functionals

$$\widetilde{F}_j(u) = \begin{cases} F_j(u) & \text{if } u \in X_j \\ +\infty & \text{if } u \in X \setminus X_j. \end{cases} \tag{0.41}$$

This type of identification is customary in dealing with minimum problems and is very useful to include constraints directly in the functional. We may therefore suppose that all $X_j = X$. If one is not used to dealing with functionals which take the value $+\infty$, one may regard this as a technical tool; if the limit functional is not finite on the whole X it will always be possible to restrict it to its domain $\operatorname{dom} F = \{u \in X : F(u) < +\infty\}$.

As in the case of a minimizing sequence for a single problem, it is necessary to find a converging minimizing (sub)sequence. In general it will be possible to find a minimizing sequence lying in a compact set of X as before, or prove that the functionals themselves satisfy an *equi-coerciveness* property: for all t there exists a compact K_t such that for all j we have $\{F_j < t\} \subset K_t$.

If a compactness property as above is satisfied, then we may suppose that the whole sequence $(\overline{u}_j)$ converges to some $\overline{u}$ (this is a technical point that will be made clear in the next section). The function $\overline{u}$ is a good candidate as a minimizer.

First, we want to obtain an *upper bound* for the limit behaviour of the sequence of minima, of the form

$$\limsup_j \inf\{F_j(u) : u \in X\} \le \inf\{F(u) : u \in X\} \le F(\overline{u}). \tag{0.42}$$

The second inequality is trivially true; the first inequality means that for all $u \in X$ we have

$$\limsup_j \inf\{F_j(v) : v \in X\} \le F(u). \tag{0.43}$$

This is a requirement of global type; we can 'localize' it in the neighbourhood of the point u by requiring a stronger condition: that for all $\delta > 0$ we have

$$\limsup_j \inf\{F_j(v) : d(u,v) < \delta\} \le F(u). \tag{0.44}$$

By the arbitrariness of δ we can rephrase this condition as a condition on sequences converging to u as:

(*limsup inequality*) for all $u \in X$ there exists a sequence (u_j) converging to u such that

$$\limsup_j F_j(u_j) \le F(u). \tag{0.45}$$

This condition can be considered as a local version of (0.42); it clearly implies all conditions above and (0.42) in particular.

Next, we want to obtain a *lower bound* for the limit behaviour of the sequence of minima of the form

$$F(\overline{u}) \leq \liminf_j F_j(\overline{u}_j). \tag{0.46}$$

As we do not want to rely on particular properties of minimizers we regard $\overline{u}$ as an arbitrary point in X and $(\overline{u}_j)$ as any converging sequence; hence, condition (0.46) can be deduced from the more general requirement:

(*liminf inequality*) for all $u \in X$ and for all sequences (u_j) converging to u we have

$$F(u) \leq \liminf_j F_j(u_j). \tag{0.47}$$

This condition is the analog of the lower semicontinuity hypothesis in the case of a single functional.

From the considerations above, if we can find a functional F such that the liminf and limsup inequalities are satisfied and if we have a converging sequence of minimizers, from (0.46) and (0.42) we deduce the chain of inequalities

$$\begin{aligned} \limsup_j \inf\{F_j(u): \; u \in X\} &\leq \inf\{F(u): \; u \in X\} \\ &\leq F(\overline{u}) \leq \liminf_j F_j(\overline{u}_j) \\ &= \liminf_j \inf\{F_j(u): \; u \in X\}. \end{aligned} \tag{0.48}$$

As the last term is clearly not greater than the first, all inequalities are indeed equalities; that is, we deduce that

(i) (*existence*) the limit problem $\min\{F(u): \; u \in X\}$ admits a solution,

(ii) (*convergence of minimum values*) the sequence of infima $\inf\{F_j(u): \; u \in X\}$ converges to this minimum value,

(iii) (*convergence of minimizers*) up to subsequences, the minimizing sequence for (F_j) converges to a minimizer of F on X.

Therefore, if we define the Γ-convergence of (F_j) to F as the requirement that the limsup and the liminf inequalities above both hold, then we may summarize the considerations above in the formula

$$\text{equi-coerciveness} + \Gamma\text{-convergence} \;\Rightarrow\; \text{convergence of minimum problems.} \tag{0.49}$$

As in the case of the application of the direct methods, a crucial role will be played by the type of metric we choose on X. In this case, again, it will be a matter of balance between the convenience of a stronger notion of convergence, that will make the liminf inequality easier to verify, and a weaker one, which would be more convenient both to satisfy an equi-coerciveness condition and to find sequences satisfying the limsup inequality.

1

Γ-CONVERGENCE BY NUMBERS

This chapter is devoted to the general properties of Γ-convergence, which will be easily deduced from its definition. Then main issues in this chapter are

— Γ-limits are stable under continuous perturbations. This means that once a Γ-limit is computed we do not have to redo all computations if 'lower-order terms' are added. Conversely, we can always remove such terms to simplify calculations;

— Under suitable conditions Γ-convergence implies convergence of minimum values and minimizers. Note that some minimizers of the Γ-limit may not be limit of minimizers, so that Γ-convergence may be interpreted as a choice criterion;

— The computation of Γ-limits can be separated into computing lower and upper bounds; the first involving lower-semicontinuity inequalities, the second the construction of suitable approximating sequences of functions. In order to better handle these operations Γ-lower and upper limits are introduced;

— The natural setting of Γ-convergence are lower semicontinuous functions. In particular Γ-upper and lower limits are lower semicontinuous functions, and the operation of Γ-limit does not change if functionals are replaced by their lower semicontinuous envelopes (which, in turn, are usually easier to handle);

— The choice of the convergence with respect to which computing the Γ-limit is essential. Since the arguments of Γ-convergence rely on compactness issues, it is usually more convenient to use weaker topologies, which explains why spaces of 'weakly-differentiable functions' are preferred.

The theoretical effort that we will spend in this chapter will be rewarded in the following ones, where the abstract properties will be helpful to simplify computations.

1.1 Some preliminaries

We spend a few words fixing the notation of lower and upper limits, and of lower semicontinuous functions.

1.1.1 *Lower and upper limits*

In all what follows, unless otherwise specified X will be a *metric space* equipped with the distance d.

Definition 1.1 *Let* $f : X \to [-\infty, +\infty]$. *We define the* lower limit (lim inf *for short*) *of* f *at* x *as*

$$\liminf_{y\to x} f(y) = \inf\{\liminf_j f(x_j) : x_j \in X,\ x_j \to x\}$$

$$= \inf\{\lim_j f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_j f(x_j)\},$$

and the upper limit (lim sup *for short*) *of* f *at* x *as*

$$\limsup_{y\to x} f(y) = \sup\{\limsup_j f(x_j) : x_j \in X,\ x_j \to x\}$$
$$= \sup\{\lim_j f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_j f(x_j)\}.$$

As remarked in the Introduction, the lower limit is linked to our minimum problems much more than the upper limit. The first notion will be preferred in our statements, but many results will obviously hold for the limsup, with the due changes. Definition 1.1 can also be given if f is not defined in the whole X (in this case the x_j must be taken in the domain of f); in particular, we can have $X = \mathbf{N}$ and $x = +\infty$ and recover the usual definition of lim inf and lim sup for sequences.

Note that in Definition 1.1, contrary to what is usually done for the definition of limit, we do take into account the value of f at x; in particular, by taking $x_j = x$ we always get $\liminf_{y\to x} f(y) \leq f(x)$. Moreover, it can easily be checked that

$$\liminf_{y\to x}(-f(y)) = -\limsup_{y\to x}(f(y)), \tag{1.1}$$

$$\liminf_{y\to x}(f(y) + g(y)) \geq \liminf_{y\to x} f(y) + \liminf_{y\to x} g(y) \tag{1.2}$$

and

$$\liminf_{y\to x}(f(y) + g(y)) \leq \limsup_{y\to x} f(y) + \liminf_{y\to x} g(y). \tag{1.3}$$

A way to interpret these limit operations is that they give the sharpest upper and lower bounds for the behaviour of f close to x; that is, for all $\varepsilon > 0$ we will have

$$\liminf_{y\to x} f(y) - \varepsilon < f(x') < \limsup_{y\to x} f(y) + \varepsilon,$$

provided that $d(x, x') < \delta = \delta(\varepsilon)$. With this observation in mind it can be easily checked that we have the equivalent topological definitions:

$$\liminf_{y\to x} f(y) = \sup_{U\in\mathcal{N}(x)} \inf_{y\in U} f(y), \qquad \limsup_{y\to x} f(y) = \inf_{U\in\mathcal{N}(x)} \sup_{y\in U} f(y), \tag{1.4}$$

where we have used the notation $\mathcal{N}(x)$ for the family of all open sets containing a point $x \in X$.

In this last characterization we have used 'elementary operators' of the type inf/sup: those of the type 'evaluation on a set U', which map a point function f (i.e. a function defined on points of X) into a set function (i.e. a function defined on subsets of X) defined by

$$U \mapsto \inf_{y\in U} f(y), \qquad U \mapsto \sup_{y\in U} f(y),$$

and those of the type 'evaluations on neighbourhoods of a point x' which map a set function α into a point function defined by

$$x \mapsto \inf_{U \in \mathcal{N}(x)} \alpha(U), \qquad x \mapsto \sup_{U \in \mathcal{N}(x)} \alpha(U).$$

A trivial but useful observation is that $U \mapsto \inf_{y \in U} f(y)$ is a *decreasing set function*; that is, that $\inf_{y \in U} f(y) \leq \inf_{y \in V} f(y)$ if $V \subset U$. An immediate consequence is that in the definition of lower limit we can substitute $\mathcal{N}(x)$ by a suitable family of balls of center x. For example

$$\begin{aligned}\liminf_{y \to x} f(y) &= \sup_{\delta > 0} \inf_{d(x,y) < \delta} f(y) = \sup_{k \in \mathbf{N}} \inf_{d(x,y) < 1/k} f(y) \\ &= \lim_{\delta \to 0} \inf_{d(x,y) < \delta} f(y) = \lim_k \inf_{d(x,y) < 1/k} f(y).\end{aligned}$$

A similar observation holds for the sup operator and the limsup.

1.1.2 *Lower semicontinuity*

The 'natural' framework in which we set our problems will be that of lower semicontinuous functions. This is somewhat justified by the direct approach explained in the Introduction, and will be further examined in Section 1.7.

Definition 1.2 *A function $f : X \to \overline{\mathbf{R}}$ will be said to be (sequentially)* lower semicontinuous (l.s.c. *for short*) *at $x \in X$, if for every sequence (x_j) converging to x we have*

$$f(x) \leq \liminf_j f(x_j), \tag{1.5}$$

or, in other words, $f(x) = \min\{\liminf_j f(x_j) : x_j \to x\}$. We will say that f is lower semicontinuous (on X) if it is l.s.c. at all $x \in X$.

Remark 1.3 The following conditions are equivalent:

(i) f is lower semicontinuous;

(ii) we have $f(x) = \liminf_{y \to x} f(y)$ for all $x \in X$;

(iii) for all $t \in \mathbf{R}$ the sublevel set $\{f \leq t\}$ is closed.

Indeed, the equivalence of (i) and (ii) is given by (1.4). Note that (i) implies that if $f(x_j) \leq t$ and $x_j \to x$ then $f(x) \leq t$, while if there exist x and $x_j \to x$ such that $f(x) > t > \liminf_j f(x_j)$ then (iii) is violated for such a t.

Remark 1.4 (i) If f and g are l.s.c. at x, then so is $f + g$ by (1.2).

(ii) Let $\{f_i : i \in I\}$ be a family of l.s.c. functions (I an arbitrary set of indices, not necessarily countable). Then the function defined by $f(x) = \sup_i f_i(x)$ is l.s.c. In fact, for fixed $x \in X$ and $x_j \to x$, we have

$$f_i(x) \leq \liminf_j f_i(x_j) \leq \liminf_j f(x_j).$$

By taking the supremum for $i \in I$ we obtain $f(x) \leq \liminf_j f(x_j)$. In particular, the supremum of a family of *continuous* functions is l.s.c.

(iii) If $f = \chi_E$ is the characteristic function of the set E, then f is l.s.c. if and only if E is open, by Remark 1.3(iv).

(iv) A function $f : X \to \overline{\mathbf{R}}$ is called *upper semicontinuous* if $-f$ is l.s.c. All the results of this section have an obvious counterpart for upper semicontinuous functions. In particular $f = \chi_E$ is upper semicontinuous if and only if E is closed.

1.2 Γ-convergence

We can now recall the definition of Γ-convergence, and make some first remarks.

Definition 1.5 (Γ-convergence). *We say that a sequence $f_j : X \to \overline{\mathbf{R}}$* Γ-converges *in X to $f_\infty : X \to \overline{\mathbf{R}}$ if for all $x \in X$ we have*

(i) (lim inf inequality) *for every sequence (x_j) converging to x*

$$f_\infty(x) \le \liminf_j f_j(x_j); \tag{1.6}$$

(ii) (lim sup inequality) *there exists a sequence (x_j) converging to x such that*

$$f_\infty(x) \ge \limsup_j f_j(x_j). \tag{1.7}$$

The function f_∞ is called the Γ-limit *of (f_j), and we write $f_\infty = \Gamma\text{-}\lim_j f_j$.*

Pointwise definition *The definition above can be also given at a fixed point $x \in X$: we say that (f_j) Γ-converges* at x *to the value $f_\infty(x)$ if* (i), (ii) *above hold; in this case we write $f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$. In this notation, f_j Γ-converges to f_∞ if and only if $f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$ at all $x \in X$.*

If we want to highlight the role of the metric, we can add the dependence on the distance d, and write $\Gamma(d)$-$\lim_j$, $\Gamma(d)$-convergence, and so on.

Different ways of writing the limsup inequality Note that if (x_j) satisfies the limsup inequality, then by (1.6) we have

$$f_\infty(x) \le \liminf_j f_j(x_j) \le \limsup_j f_j(x_j) \le f_\infty(x),$$

so that indeed $f_\infty(x) = \lim_j f_j(x_j)$; hence, (ii) can be substituted by

(ii)′ (*existence of a recovery sequence*) there exists a sequence (x_j) converging to x such that

$$f_\infty(x) = \lim_j f_j(x_j). \tag{1.8}$$

On the other hand, sometimes it is more convenient to prove (ii) with a small error and then deduce its validity by an approximation argument; that is, (ii) can be replaced by

(ii)″ (*approximate limsup inequality*) for all $\varepsilon > 0$ there exists a sequence (x_j) converging to x such that

$$f_\infty(x) \ge \limsup_j f_j(x_j) - \varepsilon. \tag{1.9}$$

In the following (and in the literature) all conditions (ii), (ii)′ and (ii)″ are equally referred to as the limsup inequality or as the existence of a recovery sequence.

Note that the liminf inequality (i) can be rewritten as

$$f_\infty(x) \le \inf\{\liminf_j f_j(x_j) : x_j \to x\}.$$

Trivially, we always have

$$\inf\{\liminf_j f_j(x_j) : x_j \to x\} \le \inf\{\limsup_j f_j(x_j) : x_j \to x\},$$

and, if $(\overline{x}_j)$ is a recovery sequence for (ii) we have

$$\inf\{\limsup_j f_j(x_j) : x_j \to x\} \le \limsup_j f_j(\overline{x}_j) \le f_\infty(x),$$

so that (i) and (ii) imply that we have

$$f_\infty(x) = \min\{\liminf_j f_j(x_j) : x_j \to x\} = \min\{\limsup_j f_j(x_j) : x_j \to x\} \quad (1.10)$$

(and actually both minima are obtained as limits along a recovery sequence). It is important to keep in mind this characterization as many properties of the Γ-limit will be easily explained from it.

Remark 1.6 (Γ-convergence as an equality of upper and lower bounds) It is sometimes convenient to state the equality in (1.10) as an equality of *infima*:

$$f_\infty(x) = \inf\{\liminf_j f_j(x_j) : x_j \to x\} = \inf\{\limsup_j f_j(x_j) : x_j \to x\}. \quad (1.11)$$

This equality is indeed equivalent to the definition of Γ-limit; that is, the Γ-limit exists if and only if the two infima in (1.11) are equal. This characterization will be important in that in this way the existence of the Γ-limit (which not always exists) is expressed as the equality of two quantities which are always defined, and which can (and will) be studied separately. The first quantity can be thought as a *lower bound* for the Γ-limit, the second as an *upper bound.*

By (1.11) we obtain in particular that the Γ-limit, if it exists, is unique.

Remark 1.7 (stability under continuous perturbations). An important property of Γ-convergence is its stability under continuous perturbations: if (f_j) Γ-converges to f_∞ and $g : X \to [-\infty, +\infty]$ is a d-continuous function then (f_j+g) Γ-converges to $f_\infty + g$. This is an immediate consequence of the definition, since if (i) holds then for all $x \in X$ and $x_j \to x$ we get

$$f_\infty(x) + g(x) \le \liminf_j f_j(x_j) + \lim_j g(x_j) = \liminf_j (f_j(x_j) + g(x_j)),$$

while if (ii)′ above holds then we get

$$f_\infty(x) + g(x) = \lim_j f_j(x_j) + \lim_j g(x_j) = \lim_j (f_j(x_j) + g(x_j)),$$

and (x_j) is a recovery sequence also for $f_\infty + g$.

Remark 1.8 (Γ-limit of a constant sequence). Consider the simplest case $f_j = f$ for all $j \in \mathbf{N}$. In this case it will be easily seen that (f_j) Γ-converges. By the liminf inequality, the limit f_∞ must satisfy

$$f_\infty(x) \leq \liminf_j f(x_j)$$

for all x and $x_j \to x$. If f is *not* lower semicontinuous then there exists $\overline{x}$ and a sequence $\overline{x}_j \to \overline{x}$ such that

$$\liminf_j f(\overline{x}_j) < f(\overline{x});$$

hence, in particular $f_\infty(\overline{x}) \neq f(\overline{x})$. This shows that Γ-convergence *does not satisfy* the requirement that a constant sequence $f_j = f$ converges to f (if f is not lower semicontinuous). We will see however that this holds true in the family of lower semicontinuous functions (see Remark 1.10).

Remark 1.9 (dependence on the metric). The choice of the metric on X is clearly a fundamental step in problems involving Γ-limits. In general, even when two distances d and d' are confrontable; that is,

$$\lim_j d'(x_j, x) = 0 \qquad \Longrightarrow \qquad \lim_j d(x_j, x) = 0, \tag{1.12}$$

the existence of the Γ-limit in one metric does not imply the existence of the Γ-limit in the second (see the examples in the Section 1.3). However, in this situation, if both Γ-limits exist then we have

$$\Gamma(d)\text{-}\lim_j f_j \leq \Gamma(d')\text{-}\lim_j f_j.$$

This is clear, for example, from the characterization (1.10) since the set of converging sequences for d is larger that that for d'.

Remark 1.10 (comparison with pointwise and uniform limits). As a very particular case, we can consider the metric d' of the *discrete topology* (where the only converging sequences are constant sequences). In this case the Γ-limit coincides with the pointwise limit (if it exists). If d is any other metric then (1.12) holds trivially, so that we obtain

$$\Gamma(d)\text{-}\lim_j f_j \leq \lim_j f_j$$

as a particular case of the previous remark.

If f_j converge *uniformly* to a f on an open set U (in particular if $f_j = f$) and f is l.s.c. then we have also that f_j Γ-converge to f. Indeed, the limsup inequality is obtained by the constant sequence, while the liminf inequality is immediately verified once we remark that if $x_j \to x \in U$ then $x_j \in U$ for j large enough, so that $\liminf_j f_j(x_j) = \lim_j (f_j(x_j) - f(x_j)) + \liminf_j f(x_j) \geq f(x)$.

1.3 Some examples on the real line

In this section we will compute some simple Γ-limits of functions defined on the real line (equipped with the usual euclidean distance), and we will also make some comparisons with the pointwise convergence (which can be thought of as a Γ-limit with respect to the discrete metric, as explained in Remark 1.10).

The computations in these examples will be quite straightforward, but nevertheless will allow us to highlight the different roles of the limsup and liminf inequalities. The first inequality is more constructive, as it amounts to finding the optimal approximating sequence for a fixed target point x, while the second one is more technical, and amounts to proving that the bound given by the recovery sequence is indeed optimal.

Example 1.11 We have seen that a constant sequence $f_j = f$ Γ-converges to f if and only if f is lower semicontinuous; hence, if f is not l.s.c. the pointwise limit and the Γ-limit are different. Now we construct an example where these two limits differ even if the pointwise limit is lower semicontinuous. Take $f_j(t) = f_1(jt)$, where $f_1(t) = \sqrt{2}\, te^{-(2t^2-1)/2}$ or

$$f_1(t) = \begin{cases} \pm 1 & \text{if } t = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_j \to 0$ pointwise, but $\Gamma\text{-}\lim_j f_j = f$, where

$$f(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ -1 & \text{if } t = 0. \end{cases}$$

Indeed, the sequence f_j converges locally uniformly (and hence also Γ-converges) to 0 in $\mathbf{R} \setminus \{0\}$, while clearly the optimal sequence for $x = 0$ is $x_j = -1/j$, for which $f_j(x_j) = -1$. In this case the pointwise and Γ-limits both exist and are different at one point.

Example 1.12 Take $f_j(t) = -f_1(jt)$, where f_1 is as in the previous example. Clearly, the Γ-limit remains unchanged. This shows that in general

$$\Gamma\text{-}\lim_j(-f_j) \neq -\Gamma\text{-}\lim_j f_j,$$

$$\Gamma\text{-}\lim_j(f_j + g_j) \neq \Gamma\text{-}\lim_j f_j + \Gamma\text{-}\lim_j g_j$$

(taking in the example $g_j = -f_j$) even if all functions are continuous.

Example 1.13 The pointwise and Γ-limits may exist and be different at *every* point. Take $f_j = g_j$, where

$$g_j(t) = \begin{cases} 0 & \text{if } t \notin \mathbf{Q} \text{ or } t = \frac{k}{n}, \text{ with } k \in \mathbf{Z} \text{ and } n \in \{1, \ldots, j\}, \\ -1 & \text{otherwise.} \end{cases}$$

We then have $f_j \to 0$ pointwise, but $\Gamma\text{-}\lim_j f_j = -1$ The liminf inequality is trivial, and the limsup inequality is easily obtained by remarking that $\{g_j = -1\}$ is dense for all $j \in \mathbf{N}$.

Example 1.14 There may be no pointwise converging subsequence of (f_j) but the $\Gamma\text{-}\lim_j f_j$ may exist all the same. Take, for example, $f_j(t) = -\cos(jt)$. In this case $\Gamma\text{-}\lim_j f_j = -1$. Again, the liminf inequality is trivial, while the limsup inequality is easily obtained by taking, for example, $x_j = [jx/2\pi]2\pi/j$ ($[t]$ the integer part of t).

Example 1.15 The sequence f_j may be converging pointwise, but may not Γ-converge. Take for example $f_j = (-1)^j g_j$ with g_j the function of Example 1.13. In this case $f_j \to 0$ pointwise, but the $\Gamma\text{-}\lim_j f_j$ does not exist at any point.

1.4 The many definitions of Γ-convergence

In this section we state different equivalent definitions of Γ-convergence, which will be useful in different contexts. Some of the different ways to state the limsup inequality have been already pinpointed above. Before this overview we recall the definition of compact set.

Definition 1.16 *By a* compact *subset of X we mean a sequentially compact set $K \subset X$; that is, such that all sequences in K admit a subsequence converging to some point in K. In formula,*

$$\forall (x_j) \subset K \ \exists x \in K, \ \exists (x_{j_k}) : \ x_{j_k} \to x.$$

A set $K \subset X$ is called precompact *if its closure is compact; that is, all sequences in K admit a converging subsequence (but its limit may also be outside K). In formula,*

$$\forall (x_j) \subset K \ \exists (x_{j_k}) : \ x_{j_k} \text{ converges in } X.$$

Topological definitions Sometimes, it is useful to have the definition of Γ-limit directly expressed in terms of the topology of X, and not only through the convergence of sequences. In this case it is easily seen that we can rewrite the equality in (1.11)

$$f_\infty(x) = \sup_{U \in \mathcal{N}(x)} \liminf_j \inf_{y \in U} f_j(y) = \sup_{U \in \mathcal{N}(x)} \limsup_j \inf_{y \in U} f_j(y). \tag{1.13}$$

This definition makes sense in any topological space and in the case of arbitrary topological spaces (in particular if X is not metric) is taken as *the* definition of Γ-convergence (while Definition 1.5 above is called *sequential* Γ-convergence). However, we will always be able to stick to metric spaces. A suggestive observation is that equivalently to (1.13), we may also write

$$f_\infty(x) = \sup_{U \in \mathcal{N}(x)} \sup_{k \in \mathbf{N}} \inf_{j \geq k} \inf_{y \in U} f_j(y) = \sup_{U \in \mathcal{N}(x)} \inf_{k \in \mathbf{N}} \sup_{j \geq k} \inf_{y \in U} f_j(y); \tag{1.14}$$

in this way, Γ-limits may be interpreted as compositions of the 'elementary operators' of the type inf/sup.

Note that in (1.13) we can substitute $\mathcal{N}(x)$ by a suitable family of neighbourhoods generating the topology of X; for example, in the metric case a family of open balls with center in x. For example we can require equivalently that

$$\begin{aligned} f_\infty(x) &= \sup_{n\in\mathbf{N}} \liminf_j \inf\{f_j(y) : d(y,x) < 1/n\} \\ &= \sup_{n\in\mathbf{N}} \limsup_j \inf\{f_j(y) : d(y,x) < 1/n\} \end{aligned}$$

or

$$\begin{aligned} f_\infty(x) &= \sup_{\delta>0} \liminf_j \inf\{f_j(y) : d(y,x) < \delta\} \\ &= \sup_{\delta>0} \limsup_j \inf\{f_j(y) : d(y,x) < \delta\}. \end{aligned}$$

A definition in terms of the convergence of minima Γ-convergence is designed so that it implies the convergence of 'compact' minimum problems. In turn, starting from the topological definition above, a definition of Γ-convergence can be expressed in terms of the asymptotic behaviour of minimum problems (localized on open or compact sets): from the second equality in (1.13) we have

$$\inf_U f_\infty \geq \limsup_j \inf_U f_j \tag{1.15}$$

for all open sets U, while requiring that

$$\inf_K f_\infty \leq \sup\{\liminf_j \inf_U f_j :\ U \supset K,\ U \text{ open}\} \tag{1.16}$$

for all compact sets K implies the first equality in (1.13) by choosing $K = \{x\}$.

Finally, starting from (1.15), back to the case of metric spaces, we can substitute the problems on balls by unconstrained problems, where we penalize the distance from the point x. For example, if all f_j are non-negative, we have that an equivalent definition is that for some $p > 0$

$$\begin{aligned} f_\infty(x) &= \sup_{\lambda\geq 0} \liminf_j \inf_{y\in X} \{f_j(y) + \lambda d(x,y)^p\} \\ &= \sup_{\lambda\geq 0} \limsup_j \inf_{y\in X} \{f_j(y) + \lambda d(x,y)^p\} \end{aligned} \tag{1.17}$$

for all $x \in X$. Note that in this case the Γ-convergence is determined by looking at a family of particular problems, which sometimes can be solved easily.

We can explicitly state the equivalence of all the definitions above in the following theorem.

Theorem 1.17 *Let $f_j, f_\infty : X \to [-\infty, +\infty]$. Then the following conditions are equivalent:*

(i) $f_\infty = \Gamma\text{-}\lim_j f_j$ *in* X *as in Definition* 1.5*;*

(ii) *for every* $x \in X$ (1.10) *holds;*

(iii) *the liminf inequality in Definition* 1.5(i) *and the* approximate limsup inequality (ii)″ *hold;*

(iv) *for every* $x \in X$ (1.11) *holds;*

(v) *for every* $x \in X$ (1.13) *holds;*

(vi) *inequality* (1.15) *holds for all open sets* U *and inequality* (1.16) *holds for all compact sets* K.

Furthermore, if $f_j(x) \geq -c(1 + d(x, x_0)^p)$ *for some* $p > 0$ *and* $x_0 \in X$*, then each of the conditions above is equivalent to*

(vii) *we have* (1.17) *for all* $x \in X$.

The proof of the equivalence of (i)–(vi) is essentially contained in the considerations made hitherto and details are left to the reader; point (vii) will be analysed in Proposition 1.27.

Note that the asymmetry of Definition 1.5 is reflected in the different roles of the sup and inf operators in the equivalent conditions above. Of course, this comes from the fact that Γ-convergence is designed to study *minimum* problems (and not maximum problems!).

1.5 Convergence of minima

We first observe that some requirements on the behaviour of sequences of the form $(f_j(x_j))$ give some information on the behaviour of minimum problems.

Proposition 1.18 *Let* $f_j, f_\infty : X \to [-\infty, +\infty]$ *be functions. Then we have*

(i) *if Definition* 1.5(i) *is satisfied for all* $x \in X$ *and* $K \subset X$ *is a compact set then we have*

$$\inf_K f_\infty \leq \liminf_j \inf_K f_j;$$

(ii) *if Definition* 1.5(ii) *is satisfied for all* $x \in X$ *and* $U \subset X$ *is an open set then we have*

$$\inf_U f_\infty \geq \limsup_j \inf_U f_j.$$

Proof (i) Let $(\tilde{x}_j)$ be such that $\liminf_j \inf_K f_j = \liminf_j f_j(\tilde{x}_j)$. After extracting a subsequence we obtain $(\tilde{x}_{j_k})$ such that

$$\lim_k f_{j_k}(\tilde{x}_{j_k}) = \liminf_j \inf_K f_j,$$

and $x_{j_k} \to \overline{x} \in K$. If $x_j = \begin{cases} \tilde{x}_{j_k} & \text{if } j = j_k \\ \overline{x} & \text{if } j \neq j_k \text{ for all } k, \end{cases}$ then

$$\begin{aligned} \inf_K f_\infty &\leq f_\infty(\overline{x}) \leq \liminf_j f_j(x_j) \\ &\leq \liminf_k f_{j_k}(x_{j_k}) = \lim_k f_{j_k}(\tilde{x}_{j_k}) = \liminf_j \inf_K f_j, \end{aligned} \tag{1.18}$$

as required.

(ii) With fixed $\delta > 0$, let $x \in U$ be such that $f_\infty(x) \le \inf_U f_\infty + \delta$. Then, if (x_j) is a recovery sequence for x we have

$$\inf_U f_\infty + \delta \ge f_\infty(x) \ge \limsup_j f_j(x_j) \ge \limsup_j \inf_U f_j, \tag{1.19}$$

and the thesis follows by the arbitrariness of δ. □

Definition 1.19 (coerciveness conditions). *A function $f : X \to \overline{\mathbf{R}}$ is* coercive *if for all $t \in \mathbf{R}$ the set $\{f \le t\}$ is precompact. A function $f : X \to \overline{\mathbf{R}}$ is* mildly coercive *if there exists a non-empty compact set $K \subset X$ such that $\inf_X f = \inf_K f$. A sequence (f_j) is* equi-mildly coercive *if there exists a non-empty compact set $K \subset X$ such that $\inf_X f_j = \inf_K f_j$ for all j.*

Remark 1.20 If f is coercive then it is mildly coercive. In fact, if f is not identically $+\infty$ (in which case we take K as any compact subset of X), then there exists $t \in \mathbf{R}$ such that $\{f \le t\}$ is not empty, and we take K as the closure of $\{f \le t\}$ in X. An example of a non-coercive, mildly coercive function is given by any periodic function $f : \mathbf{R}^n \to \mathbf{R}$. An intermediate condition between coerciveness and mild coerciveness is the requirement that there exists $t \in \mathbf{R}$ such that $\{f \le t\}$ is not empty and precompact.

We immediately obtain the required convergence result as follows.

Theorem 1.21 *Let (X, d) be a metric space, let (f_j) be a sequence of equi-mildly coercive functions on X, and let $f_\infty = \Gamma\text{-}\lim_j f_j$; then*

$$\exists \min_X f_\infty = \lim_j \inf_X f_j. \tag{1.20}$$

Moreover, if (x_j) is a precompact sequence such that $\lim_j f_j(x_j) = \lim_j \inf_X f_j$, then every limit of a subsequence of (x_j) is a minimum point for f_∞.

Proof The proof follows immediately from Proposition 1.18. In fact, if $\overline{x}$ is as in the proof of Proposition 1.18(i) (in particular we can take $\overline{x} = \lim_k x_{j_k}$ if (x_{j_k}) is a converging subsequence such that $\lim_k f_j(x_{j_k}) = \lim_j \inf_X f_j$) then by (1.18) and (1.19) with $U = X$, and by the equi-mild coerciveness we get

$$\begin{aligned}\inf_X f_\infty \le \inf_K f_\infty \le f_\infty(\overline{x}) &\le \liminf_j \inf_K f_j \\ &= \liminf_j \inf_X f_j \le \limsup_j \inf_X f_j \le \inf_X f_\infty.\end{aligned}$$

As the first and last terms coincide, we easily get the thesis. □

Remark 1.22 (Γ-convergence as a choice criterion). If in the theorem above all functions f_j admit a minimizer x_j then, up to subsequences, x_j converge to a minimum point of f_∞. The converse is clearly not true: we may have minimizers of f_∞ which are not limits of minimizers of f_j. A trivial example is $f_j(t) = \frac{1}{j}t^2$ on the real line. This situation is not exceptional; on the contrary, we may often view some functional as a Γ-limit of some particular perturbations, and single out from its minima those chosen as limits of minimizers.

Remark 1.23 (convergence of local minima). Unfortunately, Γ-convergence does not imply the convergence of *local minimizers*. Choose, for example, $f_j(t) = t^2 + \sin(jt)$ or $f_j(t) = e^t + \sin(jt)$ (with Γ-limit $t^2 - 1$ and $e^t - 1$, respectively). In order to have convergence of local minimizers we have to assume some uniform strict minimality in order to use Proposition 1.18.

1.6 Upper and lower Γ-limits

Condition (iv) in Theorem 1.17 justifies the following definition.

Definition 1.24 *Let $f_j : X \to \overline{\mathbf{R}}$ and let $x \in X$. The quantity*

$$\Gamma\text{-}\liminf_j f_j(x) = \inf\{\liminf_j f_j(x_j) : \; x_j \to x\} \tag{1.21}$$

is called the Γ-lower limit *of the sequence (f_j) at x. The quantity*

$$\Gamma\text{-}\limsup_j f_j(x) = \inf\{\limsup_j f_j(x_j) : \; x_j \to x\} \tag{1.22}$$

is called the Γ-upper limit *of the sequence (f_j) at x. If we have the equality*

$$\Gamma\text{-}\liminf_j f_j(x) = \lambda = \Gamma\text{-}\limsup_j f_j(x) \tag{1.23}$$

for some $\lambda \in [-\infty, +\infty]$, then we write

$$\lambda = \Gamma\text{-}\lim_j f_j(x), \tag{1.24}$$

and we say that λ is the Γ-limit *of the sequence (f_j)* at x. *Again, if we need to highlight the dependence on the metric d we may add it in the notation.*

Remark 1.25 Clearly, the Γ-lower limit and the Γ-upper limit exist at every point $x \in X$. Definition 1.24 is in agreement with Definition 1.5, and we can say that a sequence (f_j) Γ-converges to f_∞ if and only if for fixed $x \in X$ the Γ-limit exists and we have $\lambda = f_\infty(x)$ in (1.24).

Remark 1.26 It can be easily checked, as we did for the Γ-limit, that we have

$$\begin{aligned}\Gamma\text{-}\liminf_j f_j(x) &= \min\{\liminf_j f_j(x_j) : x_j \to x\} \\ &= \sup_{U \in \mathcal{N}(x)} \liminf_j \inf_{y \in U} f_j(y), \end{aligned} \tag{1.25}$$

$$\begin{aligned}\Gamma\text{-}\limsup_j f_j(x) &= \min\{\limsup_j f_j(x_j) : x_j \to x\} \\ &= \sup_{U \in \mathcal{N}(x)} \limsup_j \inf_{y \in U} f_j(y). \end{aligned} \tag{1.26}$$

The reader is encouraged to fill the details of the proof of this statement.

We also have the following characterization of upper and lower Γ-limits, which proves in particular the last statement of Theorem 1.17.

Proposition 1.27 *Let $f_j : X \to [-\infty, +\infty]$ be a sequence of functions satisfying $f_j(x) \geq -c(1 + d(x, x_0)^p)$ for some $p > 0$ and $x_0 \in X$; then we have*

$$\Gamma\text{-}\liminf_j f_j(x) = \sup_{\lambda \geq 0} \liminf_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\}, \tag{1.27}$$

$$\Gamma\text{-}\limsup_j f_j(x) = \sup_{\lambda \geq 0} \limsup_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\} \tag{1.28}$$

for all $x \in X$.

Proof We only prove (1.27); the proof of (1.28) being the same.
We first assume $f_j \geq 0$ for all j. With fixed $x \in X$, let

$$r = \Gamma\text{-}\liminf_j f_j(x) = \sup_{\varepsilon \to 0} \liminf_j \inf_{d(x,y)<\varepsilon} f_j(y),$$
$$s = \sup_{\lambda \geq 0} \liminf_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\}.$$

Let $t < r$. There exists $\varepsilon > 0$ such that $t < \liminf_j \inf_{d(x,y)<\varepsilon} f_j(y)$; hence, there exists $k \in \mathbf{N}$ such that $t < \inf_{d(x,y)<\varepsilon} f_j(y)$ for all $j \geq k$. For all $\lambda > t\varepsilon^{-p}$ we get

$$t < \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\}$$

for all $j \geq k$. Hence

$$t < \liminf_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\} \leq s.$$

By the arbitrariness of $t < r$ we get $r \leq s$.
We now prove that $s \leq r$. If $\lambda \geq 0$ and $\varepsilon > 0$, then

$$\inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\} \leq \inf_{d(x,y)<\varepsilon} \{f_j(y) + \lambda d(x,y)^p\} \leq \inf_{d(x,y)<\varepsilon} f_j(y) + \lambda \varepsilon^p$$

and

$$\liminf_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\} \leq \liminf_j \inf_{d(x,y)<\varepsilon} f_j(y) + \lambda \varepsilon^p \leq r + \lambda \varepsilon^p.$$

Letting $\varepsilon \to 0$ we obtain

$$\liminf_j \inf_{y \in X} \{f_j(y) + \lambda d(x,y)^p\} \leq r,$$

and finally, taking the supremum for $\lambda \geq 0$, $s \leq r$.

In the general case, with fixed $\overline{x}$, note that there exists $C > 0$ such that $c(1 + d(x, x_0)^p) \leq C(1 + d(\overline{x}, x)^p)$ for all $x \in X$. We can apply the thesis of the theorem to the sequence of non-negative functions $g_j(x) = f_j(x) + C(1 + d(\overline{x}, x)^p)$ at the point $\overline{x}$ to obtain

$$
\begin{aligned}
C + \Gamma\text{-}\liminf_j f_j(\overline{x}) &= \Gamma\text{-}\liminf_j g_j(\overline{x}) \\
&= \sup_{\lambda \geq 0} \liminf_j \inf_{y \in X} \{ f_j(y) + (C+\lambda) d(\overline{x}, y)^p + C \} \\
&= \sup_{\eta \geq 0} \liminf_j \inf_{y \in X} \{ f_j(y) + \eta d(\overline{x}, y)^p \} + C,
\end{aligned}
$$

that is, the thesis. □

1.7 The importance of being lower semicontinuous

As already remarked, the notion of Γ-convergence does not have the property that a constant sequence $f_j = f$ converges to f. However, this property is true for lower semicontinuous functions. Moreover, on the family of lower semicontinuous functions, Γ-convergence enjoys more interesting and useful properties.

1.7.1 *Lower semicontinuity of Γ-limits*

Proposition 1.28 *The Γ-upper and lower limits of a sequence (f_j) are lower semicontinuous functions.*

Proof We just perform the proof for the $\Gamma\text{-}\liminf_j f_j$

$$
\begin{aligned}
\liminf_{y \to x} \big(\Gamma\text{-}\liminf_j f_j(y)\big) &= \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} \sup_{V \in \mathcal{N}(y)} \liminf_j \inf_{z \in V} f_j(z) \\
&= \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} \sup_{U \supset V \in \mathcal{N}(y)} \liminf_j \inf_{z \in V} f_j(z) \\
&\geq \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} \sup_{U \supset V \in \mathcal{N}(y)} \liminf_j \inf_{z \in U} f_j(z) \\
&= \sup_{U \in \mathcal{N}(x)} \liminf_j \inf_{z \in U} f_j(z) = \Gamma\text{-}\liminf_j f_j(x),
\end{aligned}
$$

as required. □

Remark 1.29 (proof of the limsup inequality by density). The lower semicontinuity of the Γ-limsup in the proposition above is sometimes useful for the estimate of Γ-limits as follows. Let d' be a distance on X inducing a topology which is not weaker than that induced by d; that is, $d'(x_j, x) \to 0$ implies $d(x_j, x) \to 0$. Suppose D is a dense subset of X for d' and that we have $\Gamma\text{-}\limsup_j f_j(x) \leq f(x)$ on D, where f is a function which is continuos with respect to d. Then we have $\Gamma\text{-}\limsup_j f_j \leq f$ on X. In fact, if $d'(x_j, x) \to 0$ and $x_j \in D$ then

$$
\Gamma\text{-}\limsup_j f_j(x) \leq \liminf_k \big(\Gamma\text{-}\limsup_j f_j(x_k)\big) \leq \liminf_k f(x_k) = f(x).
$$

1.7.2 *The lower-semicontinuous envelope. Relaxation*

If f is not lower semicontinuous, it is sometimes useful to compute the lower semicontinuous envelope of f. This operation is also known as *relaxation*.

Definition 1.30 *Let* $f : X \to \overline{\mathbf{R}}$ *be a function. Its* lower-semicontinuous envelope $\mathrm{sc}f$ *is the greatest lower-semicontinuous function not greater than* f, *that is, for every* $x \in X$

$$\mathrm{sc}f(x) = \sup\{g(x) : \ g \ \textit{l.s.c.}, \ g \leq f\}. \tag{1.29}$$

Note that $\mathrm{sc}f$ is l.s.c. by Remark 1.4(ii). Moreover, if $f_1 \leq f_2$ then $\mathrm{sc}f_1 \leq \mathrm{sc}f_2$.

Proposition 1.31 *We have* $\mathrm{sc}f(x) = \Gamma\text{-}\lim_j f(x) = \liminf_{y\to x} f(y)$.

Proof By Proposition 1.28 $f_\infty(x) = \Gamma\text{-}\lim_j f(x) = \liminf_{y\to x} f(y)$ defines a lower-semicontinuous function. Moreover, $f_\infty \leq f$ so that $f_\infty \leq \mathrm{sc}f$.

If $g \leq f$ is l.s.c. then $g = \Gamma\text{-}\lim_j g \leq \Gamma\text{-}\lim_j f = f_\infty$, and the inequality $\mathrm{sc}f \leq f_\infty$ follows by taking the supremum on such g. □

Proposition 1.32 *We have*

$$\Gamma\text{-}\liminf_j f_j = \Gamma\text{-}\liminf_j \mathrm{sc}f_j, \qquad \Gamma\text{-}\limsup_j f_j = \Gamma\text{-}\limsup_j \mathrm{sc}f_j. \tag{1.30}$$

Proof We have

$$\begin{aligned}\Gamma\text{-}\liminf_j \mathrm{sc}f_j(x) &= \sup_{U\in\mathcal{N}(x)} \liminf_j \inf_{y\in U} \sup_{V\in\mathcal{N}(y)} \inf_{z\in V} f_j(z)\\ &= \sup_{U\in\mathcal{N}(x)} \liminf_j \inf_{y\in U} \sup_{V\in\mathcal{N}(y)} \inf_{z\in V\cap U} f_j(z)\\ &\geq \sup_{U\in\mathcal{N}(x)} \liminf_j \inf_{z\in U} f_j(z) = \Gamma\text{-}\liminf_j f_j(x),\end{aligned}$$

while the opposite inequality is trivial. The same proof works for the Γ-upper limit. □

1.7.3 *Approximation of lower-semicontinuous functions*

We address now the problem of the approximation of l.s.c. functions by continuous functions. The following proposition is a particular case of Proposition 1.27.

Proposition 1.33 *Let* $f : X \to \overline{\mathbf{R}}$ *be bounded from below and* $p > 0$; *then for all* $x \in X$

$$\mathrm{sc}f(x) = \sup_{\lambda\geq 0} \inf_{y\in X} \{f(y) + \lambda d(x,y)^p\}.$$

Corollary 1.34 *Every l.s.c. function bounded from below is the supremum of an increasing family of Lipschitz functions.*

Proof Define the *Yosida transform* of f as

$$T_\lambda f(x) = \inf\{f(y) + \lambda d(x,y) : y \in X\} \tag{1.31}$$

From Proposition 1.33 we have, taking $p = 1$, $f(x) = \sup_{\lambda\geq 0} T_\lambda f(x)$. The proof is completed if we show that $T_\lambda f$ is a Lipschitz function. Indeed we have something more:

$$T_\lambda f(x) = \max\{g(x) : g \le f,\ g \text{ is Lipschitz with constant } \lambda\}. \tag{1.32}$$

In fact, with fixed x_1, x_2 and $\varepsilon \in (0,1)$, we can find $y_\varepsilon \in X$ such that $f(y_\varepsilon) + \lambda d(x_1, y_\varepsilon) \le T_\lambda f(x_1) + \varepsilon$. Then

$$T_\lambda f(x_2) \le f(y_\varepsilon) + \lambda d(x_2, y_\varepsilon) \le f(y_\varepsilon) + \lambda(d(x_1, x_2) + d(x_1, y_\varepsilon)),$$

so that $T_\lambda f(x_2) \le T_\lambda f(x_1) + \varepsilon + \lambda d(x_1, x_2)$, and, by a symmetry argument and the arbitrariness of ε, $|T_\lambda f(x_2) - T_\lambda f(x_1)| \le \lambda d(x_1, x_2)$, showing that $T_\lambda f$ is Lipschitz with constant λ. Vice versa, if $g \le f$ is Lipschitz with constant λ then $g = T_\lambda g \le T_\lambda f$. □

1.7.4 *The direct method*

Combined lower semicontinuity and coerciveness ensure the existence of minimum points, as specified by the following version of a well-known theorem.

Theorem 1.35 (Weierstrass' Theorem) *If $f : X \to \overline{\mathbf{R}}$ is mildly coercive, then there exists the minimum value* $\min\{\mathrm{sc}f(x) : x \in X\}$, *and it equals the infimum* $\inf\{f(x) : x \in X\}$. *Moreover, the minimum points for* $\mathrm{sc}f$ *are exactly all the limits of converging sequences (x_j) such that* $\lim_j f(x_j) = \inf_X f$.

Proof The theorem is a particular case of Theorem 1.21. The only thing to notice is that if $\overline{x}$ is a minimum point for $\mathrm{sc}f$, we can find a sequence (x_j) converging to $\overline{x}$ such that $\lim_j f(x_j) = \mathrm{sc}f(\overline{x}) = \inf_X f$. □

Remark 1.36 The previous theorem gives, of course, that if f is l.s.c. and mildly coercive then the problem $\min_X f$ has a solution.

The application of Theorem 1.35, and in particular of the remark above, to prove the existence of solutions of minimum problems is usually referred to as the 'direct method' of the Calculus of Variations.

1.8 More properties of Γ-limits

From the definitions of Γ-convergence we immediately obtain the following properties.

Remark 1.37 If (f_{j_k}) is a subsequence of (f_j) then

$$\Gamma\text{-}\liminf_j f_j \le \Gamma\text{-}\liminf_k f_{j_k}, \qquad \Gamma\text{-}\limsup_k f_{j_k} \le \Gamma\text{-}\limsup_j f_j.$$

In particular, if $f_\infty = \Gamma\text{-}\lim_j f_j$ exists then for every increasing sequence of integers (j_k) $f_\infty = \Gamma\text{-}\lim_k f_{j_k}$.

Remark 1.38 If g is a continuous function then $f_\infty + g = \Gamma\text{-}\lim_j (f_j + g)$; more in general, if $g_j \to g$ uniformly, and g is continuous then $f_\infty + g = \Gamma\text{-}\lim_j (f_j + g_j)$. In particular, if $f_j \to f$ uniformly on an open set U, then

$$\Gamma\text{-}\lim_j f_j = \mathrm{sc}f \tag{1.33}$$

on U.

Remark 1.39 If $f_j \to f$ pointwise then $\Gamma\text{-}\limsup_j f_j \leq f$, and hence, also $\Gamma\text{-}\limsup_j f_j \leq \text{sc}f$.

1.8.1 *Γ-limits of monotone sequences*

We can state some simple but important cases when the Γ-limit does exist, and it is easily computed.

Remark 1.40 (i) (*decreasing sequences*) If $f_{j+1} \leq f_j$ for all $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \text{sc}(\inf_j f_j) = \text{sc}(\lim_j f_j). \tag{1.34}$$

In fact as $f_j \to \inf_k f_k$ pointwise, by Remark 1.39 we have $\Gamma\text{-}\limsup_j f_j \leq \text{sc}(\inf_k f_k)$, while the other inequality is trivially derived from the inequality $\text{sc}(\inf_k f_k) \leq \inf_k f_k \leq f_j$;

(ii) (*increasing sequences*) if $f_j \leq f_{j+1}$ for all $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \sup_j \text{sc}f_j = \lim_j \text{sc}f_j; \tag{1.35}$$

in particular if f_j is l.s.c. for every $j \in \mathbf{N}$, then

$$\Gamma\text{-}\lim_j f_j = \lim_j f_j. \tag{1.36}$$

In fact, since $\text{sc}f_j \to \sup_k \text{sc}f_k$ pointwise,

$$\Gamma\text{-}\limsup_j f_j = \Gamma\text{-}\limsup_j \text{sc}f_j \leq \sup_k \text{sc}f_k$$

by Remark 1.39. On the other hand $\text{sc}f_k \leq f_j$ for all $j \geq k$ so that the converse inequality easily follows.

Remark 1.41 By Remark 1.40(ii), if f_j is a equi-mildly coercive non-decreasing sequence of l.s.c. functions then $\sup_j \min_X f_j = \min_X \sup_j f_j$.

1.8.2 *Compactness of Γ-convergence*

Proposition 1.42 *Let (X, d) be a separable metric space, and for all $j \in \mathbf{N}$ let $f_j : X \to \overline{\mathbf{R}}$ be a function. Then there is a subsequence (f_{j_k}) such that the $\Gamma\text{-}\lim_k f_{j_k}(x)$ exists for all $x \in X$.*

Proof Let (U_k) be a countable base of open sets for the topology of X. Since $\overline{\mathbf{R}}$ is compact, there exists an increasing sequence of integers $(\sigma_j^0)_j$ along which the limit

$$\lim_j \inf_{y \in U_0} f_{\sigma_j^0}(y)$$

exists, and for all $k \geq 1$ we define $(\sigma_j^k)_j$ as any subsequence of $(\sigma_j^{k-1})_j$ along which the limit

$$\lim_j \inf_{y \in U_k} f_{\sigma_j^k}(y)$$

exists. The 'diagonal' sequence $j_k = \sigma_k^k$, being a subsequence of all $(\sigma_j^l)_j$, has the property that the limit

$$\lim_k \inf_{y \in U_l} f_{j_k}(y)$$

exists for all $l \in \mathbf{N}$. In particular we have

$$\liminf_k \inf_{y \in U_l} f_{j_k}(y) = \limsup_k \inf_{y \in U_l} f_{j_k}(y)$$

for all $l \in \mathbf{N}$, and we deduce (1.23) by (1.13). □

Remark 1.43 If (X, d) is not separable, then Proposition 1.42 fails. As an example, we can take $X = \{-1, 1\}^{\mathbf{N}}$ equipped with the discrete topology. X is metrizable, and Γ-convergence on X is equivalent to pointwise convergence. We take the sequence $f_j : X \to \{-1, 1\}$ defined by $f_j(\mathbf{x}) = x_j$ if $\mathbf{x} = (x_0, x_1, \ldots)$. If (f_{j_k}) is a subsequence of (f_j) and we define $\mathbf{x}$ by $x_{j_k} = (-1)^k$, and $x_j = 1$ if $j \notin \{j_k : k \in \mathbf{N}\}$, then the limit $\lim_k f_{j_k}(\mathbf{x})$ does not exist. Hence no subsequence of (f_j) Γ-converges.

1.8.3 Γ-*convergence by subsequences*

Γ-convergence enjoys the following useful property.

Proposition 1.44 (Urysohn property of Γ-convergence). *We have $f_\infty = \Gamma\text{-}\lim_j f_j$ if and only if for every subsequence (f_{j_k}) there exists a further subsequence which Γ-converges to f_∞.*

Proof Clearly if f_j Γ-converges to f_∞, then every subsequence of f_j Γ-converges to the same limit (Remark 1.37(iii)).

For every increasing sequence of integers (j_k) we have

$$\begin{aligned}\Gamma\text{-}\liminf_j f_j &\leq \Gamma\text{-}\liminf_k f_{j_k}\\ &\leq \Gamma\text{-}\limsup_k f_{j_k} \leq \Gamma\text{-}\limsup_j f_j.\end{aligned}$$

Hence if $\Gamma\text{-}\lim_k f_{j_k}(x) = f_\infty(x)$ but the $\Gamma\text{-}\lim_j f_j(x)$ does not exist we have either

$$f_\infty(x) < \Gamma\text{-}\limsup_j f_j(x) \qquad \text{or} \qquad f_\infty(x) > \Gamma\text{-}\liminf_j f_j(x).$$

In the first case we have

$$f_\infty(x) < \sup_{U \in \mathcal{N}(x)} \limsup_j \inf_{y \in U} f_j(y),$$

so that there exists $U \in \mathcal{N}(x)$ such that

$$f_\infty(x) < \limsup_j \inf_{y \in U} f_j(y).$$

This means that a subsequence (f_{j_k}) of (f_j) exists along which

$$f_\infty(x) < \liminf_k \inf_{y \in U} f_{j_k}(y),$$

so that $f_\infty(x) < \Gamma\text{-}\liminf_k f_{j_k}(x)$ leads to a contradiction. In the second case a sequence x_j converging to x exists such that $\liminf_j f_j(x_j) < f_\infty(x)$. We can extract a subsequence (x_{j_k}) of (x_j) such that $\lim_k f_j(x_{j_k}) < f_\infty(x)$. This means that $\Gamma\text{-}\limsup_k f_{j_k}(x) < f_\infty(x)$, thus giving a contradiction. □

1.9 Γ-limits indexed by a continuous parameter

In applications, our energies will often depend on a continuous parameter $\varepsilon > 0$, so that we will have a family of functions $f_\varepsilon : X \to \overline{\mathbf{R}}$. It is necessary then to make precise the definition of Γ-limit in this case, as follows.

Definition 1.45 *We say that f_ε Γ-converges to f_0 if for all sequences (ε_j) converging to* 0 *we have* $\Gamma\text{-}\lim_j f_{\varepsilon_j} = f_0$.

Remark 1.46 It can be easily checked that all the characterizations and properties of the Γ-limits, as well as the definitions of Γ-upper and lower limits, can be still obtained in this case with the due changes. We usually prefer to stick to sequences, as in the proof it is more convenient to extract subsequences.

1.10 Development by Γ-convergence

As already remarked, the process of Γ-limit entails a choice in the minimizers of the limit problem by minimizers of the approximating ones. In the case that this 'choice' is still not unique, we may proceed further to a 'Γ-limit of higher order'.

Theorem 1.47 (development by Γ-convergence). *Let $f_\varepsilon : X \to \overline{\mathbf{R}}$ be a family of d-equi-coercive functions and let* $f^0 = \Gamma(d)\text{-}\lim_{\varepsilon\to 0} f_\varepsilon$. *Let* $m_\varepsilon = \inf f_\varepsilon$ *and* $m^0 = \min f^0$. *Suppose that for some* $\alpha > 0$ *there exists the Γ-limit*

$$f^\alpha = \Gamma(d')\text{-}\lim_{\varepsilon\to 0} \frac{f_\varepsilon - m^0}{\varepsilon^\alpha}, \tag{1.37}$$

and that the sequence $f^\alpha_\varepsilon = (f_\varepsilon - m^0)/\varepsilon^\alpha$ *is d'-equi-coercive for a metric d' which is not weaker than d. Define* $m^\alpha = \min f^\alpha$ *and suppose that* $m^\alpha \neq +\infty$; *then we have that*

$$m_\varepsilon = m^0 + \varepsilon^\alpha m^\alpha + o(\varepsilon^\alpha) \tag{1.38}$$

and from all sequences (x_ε) *such that* $f_\varepsilon(x_\varepsilon) - m_\varepsilon = o(\varepsilon^\alpha)$ *(in particular this holds for minimizers) there exists a subsequence converging in* (X, d') *to a point* x *which minimizes both* f^0 *and* f^1.

Proof The proof is a simple refinement of that of Theorem 1.21. Since we have

$$m^\alpha = \lim_{\varepsilon\to 0} \min f^\alpha_\varepsilon = \lim_{\varepsilon\to 0} \frac{m_\varepsilon - m^0}{\varepsilon^\alpha},$$

we deduce immediately (1.38). Let (x_ε) be such that $f_\varepsilon(x_\varepsilon) = m_\varepsilon + o(\varepsilon^\alpha)$. By the equi-coerciveness of f^α_ε we can assume that x_ε converges to some x in (X, d') and hence also in (X, d), upon extracting a subsequence. By Theorem 1.21 applied to f_ε x is a minimizer of f^0. From (1.37) we get that $\min f^\alpha_\varepsilon = (m_\varepsilon - m^0)/\varepsilon^\alpha = f^\alpha_\varepsilon(x_\varepsilon) + o(1)$, so that we can also apply Theorem 1.21 to f^α_ε and obtain that x is a minimizer of f^1. □

1.11 Exercises

1.1 Let $f_j = f$ if j is even, $f_j = g$ if j is odd. Compute $\Gamma\text{-}\liminf_j f_j$ and $\Gamma\text{-}\limsup_j f_j$.

1.2 Prove that

$$\Gamma\text{-}\liminf_j (f_j + g_j) \geq \Gamma\text{-}\liminf_j f_j + \Gamma\text{-}\liminf_j g_j,$$

$$\Gamma\text{-}\limsup_j (f_j + g_j) \geq \Gamma\text{-}\limsup_j f_j + \Gamma\text{-}\liminf_j g_j.$$

1.3 Show that in general we do not have

$$\Gamma\text{-}\limsup_j (f_j + g_j) \leq \Gamma\text{-}\limsup_j f_j + \Gamma\text{-}\limsup_j g_j,$$

and in general $\Gamma\text{-}\lim_j (f_j + g_j) \neq \Gamma\text{-}\lim_j f_j + \Gamma\text{-}\lim_j g_j$ even if g_j is l.s.c. and independent of j. (Take e.g. $f_j = f$ not l.s.c. such that $g_j = -f$ is l.s.c. The inequality above is $0 \leq \text{sc} f - f$ which is violated at points where $f \neq \text{sc} f$.)

1.4 Let f_j be lower semicontinuous functions and let $f_j \leq f_\infty$ and $f_j \to f_\infty$ pointwise. Prove that $f_\infty = \Gamma\text{-}\lim_j f_j$.

1.5 Let (X, d) be a topological vector space and let $f_\infty = \Gamma\text{-}\lim_j f_j$. Prove that if every f_j is positively homogeneous of degree α (i.e. if $f_j(tx) = t^\alpha f_j(x)$ for all $x \in X$ and $t > 0$), then so is f_∞.

For the definition and some of the properties of convexity, which intervenes in Exercises 1.6 and 1.10, we refer to Appendix A.

1.6 Let X be also a topological vector space; prove that if every f_j is convex then $\Gamma\text{-}\limsup_j f_j$ is convex. Show that this does not hold in general for the $\Gamma\text{-}\liminf_j f_j$.

1.7 Let X be also a topological vector space; prove that if every f_j is a quadratic form then $\Gamma\text{-}\lim_j f_j$ is a quadratic form (use the fact that a quadratic form f is characterized by $f(0) = 0$, $f(tx) = t^2 f(x)$ for all $t \neq 0$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. A proof of this fact can be found in Dal Maso (1993)).

1.8 Prove that f is lower semicontinuous if and only if either of the two following conditions is satisfied:

(a) the *epigraph* of f, $epi(f) = \{(x,t) : x \in X, t \in \mathbf{R}, t \geq f(x)\}$, is closed in $X \times \mathbf{R}$;

(b) for all $x \in X$ and for all $t < f(x)$ there exists $U \in \mathcal{N}(x)$ such that $f(y) > t$ for all $y \in U$.

1.9 Prove that if $\{f_i\}$ is a finite family of l.s.c. functions then $f = \min f_i$ is l.s.c. Show with an example that this is not true in general, if $\{f_i\}$ is infinite.

1.10 Let $X = \mathbf{R}$ (equipped with the Euclidean distance) and $T_\lambda f(x) = \inf\{f(y) + \lambda|x - y| : y \in \mathbf{R}\}$. Show that if f is convex then $T_\lambda f$ is convex. Use this fact, Corollary 1.34 and an approximation argument (e.g. by convolution) to prove that every positive convex l.s.c. function on $\mathbf{R}$ is the pointwise limit of an increasing sequence of convex smooth Lipschitz functions.

Comments on Chapter 1

Γ-convergence was introduced by De Giorgi in the early 1970s and its name mirrors that of *G-convergence* (i.e. convergence of Green functions) for differential operators (see Spagnolo (1968)). Its first definition was stated in De Giorgi and Franzoni (1975), where all the main properties were presented. Γ-convergence is linked to previous notions of convergence such as Mosco's convergence (see Mosco (1969)) or Kuratowski's convergence of sets. Indeed Γ-convergence of a sequence of functions can be viewed as a convergence of their epigraphs (*epi-convergence*), just like semicontinuity can be seen as a property of the epigraph. Note that a symmetric notion of Γ-convergence can be given to treat maximum problems, whose natural setting is that of upper-semicontinuous functions; in the early papers a more complex notation was used to precise which type of notion was used (such as Γ^-, Γ^+, etc.). The main reference for the general properties of Γ-convergence is Dal Maso (1993), where in particular one can find a complete treatment of Γ-convergence in non-metrizable topological spaces (such as those equipped with weak topologies), and conditions under which Γ-convergence itself can be deduced from a metric on the space of lower semicontinuous functions. This property is very useful, for example, when we have to use diagonal arguments, and allows to treat variational problems on spaces of lower semicontinuous functions; a nice illustrative application to some problems in Economics (namely, the *monopolist's problem*) is given by Ghisi and Gobbino (2002). An exposition of the theory of Γ-convergence from the viewpoint of epiconvergence with applications is given in Attouch (1984).

Relaxation of minimum problems (i.e. their solution via their lower semicontinuous envelope) is a problem that is classically treated in convex analysis (see e.g. Ekeland and Temam (1976)). In a nonlinear setting it has been treated by many authors using the techniques developed for Γ-convergence; a presentation of these relaxation problems can be found for example in Buttazzo (1989) and Fonseca and Leoni (2002).

The notion of development by Γ-convergence was introduced by Anzellotti and Baldo (1993). Another definition and applications to the derivation of low-dimensional theories from three-dimensional linear elasticity are given in Anzellotti *et al.* (1994). An approach by Γ-convergence to problems with multiple scales is proposed by Alberti and Müller (2001). An application to multiscale modelling of materials is given by Conti *et al.* (2002a).

2

INTEGRAL PROBLEMS

The first type of problems we deal with are 'classical problems' in the Calculus of Variations, in which the energies under consideration are integral functionals defined on spaces of suitably-integrable or suitably-differentiable functions. We start with these problems since they are likely those the reader is most accustomed to, even though they are not those we can tackle by using the simplest tools.

The main issues in this chapter are the following:

— Strong topologies are not suited for the application of the direct methods to integral functionals. We then have to resort to *weak topologies*. The typical example of weakly (not strongly)-converging sequence are fast-oscillating functions,

— The qualitative notion entailing the lower semicontinuity of integral functionals (in one dimension) is *convexity* of the integrands. Relaxation and Γ-convergence are expressed in terms of convex envelopes of the integrands,

— The class of *integral functionals of p-growth* ($1 < p < \infty$) on Sobolev spaces is stable with respect to Γ-convergence and addition to boundary data. Sequences failing such growth conditions may produce Γ-limits of different type, for example, defined on discontinuous functions, or of 'supremal type'.

2.1 Problems on Lebesgue spaces

Now we face the problem of a natural definition of convergence or 'closeness' to a function u, compatible with energetic considerations. The problem we examine, loosely speaking, is as follows: with given a sequence of functionals F_j defined on some space of functions, we look at the behaviour of sequences of functions (u_j) with small energy $F_j(u_j)$, or, more in general, of 'perturbations with least energy' of a target function u.

The simplest example is that of the relaxation of a non-convex energy on a Lebesgue space. Take, for example, a single integral functional: $F_j = F$, where

$$F(u) = \int_0^1 u^2(u-1)^2\,dt \tag{2.1}$$

is defined on the space of functions $u \in L^4(0,1)$. If (u_j) is a sequence such that $F(u_j) = 0$, or more in general $\lim_j F(u_j) = 0$, we can only infer that the values of u_j will be close in measure to 0 and 1. In general (u_j) will not converge in $L^4(0,1)$. The only information we may obtain directly from the fact that $F(u_j)$

is equi-bounded is a uniform bound on the $L^4(0,1)$ norm of u_j. This is enough to infer that we may define a *weak* limit of (a subsequence of) (u_j), as in the following section.

2.1.1 *Weak convergences*

We start with the definition of weak convergence on Lebesgue spaces, whose basic theory and notations are assumed to be known to the reader.

Definition 2.1 (weak convergence in Lebesgue spaces). *Let* $1 \le p < \infty$ *($p = \infty$, respectively); then we say that a sequence* (u_j) converges weakly *(weakly*, respectively) to* u *in* $L^p(a,b)$*, and we write* $u_j \rightharpoonup u$ *($u_j \overset{*}{\rightharpoonup} u$, respectively) if we have*

$$\lim_j \int_a^b v(u_j - u)\,dt = 0 \tag{2.2}$$

for all $v \in L^{p'}(a,b)$*, where* p' *is the* conjugate exponent *of* p*; that is,* $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 2.2 Let $1 < p \le \infty$. Then the space $\{u \in L^p(a,b) : \|u\|_p \le C\}$ equipped with the topology of the weak (weak* if $p = \infty$) convergence is metrizable. This fact follows immediately from the separability of $L^{p'}(a,b)$: take a dense sequence (φ_i) in the unit sphere $\{v \in L^{p'}(a,b) : \|v\|_{p'} = 1\}$. Then the distance

$$d(u,v) = \sum_i 2^{-i}\Big|\int_a^b \varphi_i(u-v)\,dt\Big| \tag{2.3}$$

gives the desired metric.

Proposition 2.3 *Let* $1 \le p \le \infty$*. If* $u_j \rightharpoonup u$ *in* $L^p(a,b)$ *($u_j \overset{*}{\rightharpoonup} u$ if $p = \infty$) then we have*

(i) $\sup_j \|u_j\|_{L^p(a,b)} < +\infty$;

(ii) $\|u\|_{L^p(a,b)} \le \liminf_j \|u_j\|_{L^p(a,b)}$.

Note that (i) above comes from the Banach Steinhaus Theorem, while it is interesting to note that (ii) is a lower semicontinuity statement, and can be proved arguing as in Remark 1.4(ii) upon noticing that

$$\|u\|_{L^p(a,b)} = \sup\Big\{\int_a^b uv\,dt : v \in L^{p'}(a,b),\ \|v\|_{L^{p'}(a,b)} \le 1\Big\}$$

(the proof of this equality is left as an exercise) so that the norm can be seen as a supremum of a family of continuous functionals (by (2.2)), and hence lower semicontinuous.

We now describe a class of weakly (but not strongly) converging functions with the following fundamental example.

Example 2.4 (weak convergence of highly-oscillating functions). Let $1 \le p \le \infty$ and let u be a 1-periodic function in $L^p_{\rm loc}(\mathbf{R})$. Let $u_\varepsilon(t) = u(\frac{t}{\varepsilon})$ and let $\overline{u} = \int_0^1 u\,dt$ be the *average of* u, which we also regard as a constant function. Then we have $u_\varepsilon \rightharpoonup \overline{u}$ ($u_\varepsilon \overset{*}{\rightharpoonup} \overline{u}$ if $p = \infty$) (meaning that for all $\varepsilon_j \to 0$ we have $u_{\varepsilon_j} \rightharpoonup \overline{u}$ or $u_{\varepsilon_j} \overset{*}{\rightharpoonup} \overline{u}$, respectively).

To check this fact, we first treat the case $p = \infty$. We note preliminarily that given (v_j) in $L^\infty(a,b)$ we have $v_j \overset{*}{\rightharpoonup} v$ if and only if $\sup_j \|v_j\|_\infty < +\infty$ and

$$\int_E v\,dt = \lim_j \int_E v_j\,dt \tag{2.4}$$

for all intervals E of (a,b). One implication is trivial by taking $v = \chi_E$ as test function in (2.2). The converse implication is easily proved by taking (φ_i) a dense set of piecewise-constant functions in (2.3) and noticing that (2.4) implies that $\lim_j \int_a^b \varphi_i(v_j - v)\,dt = 0$ for all i.

Let then $E = (a', b')$ be an interval. In this case the thesis is trivially checked since ($[t]$ denotes the integer part of t)

$$\begin{aligned}\int_{a'}^{b'} u_\varepsilon\,dt &= \int_{a'}^{a'+[(b'-a')/\varepsilon]\varepsilon} u_\varepsilon\,dt + \int_{a'+[(b'-a')/\varepsilon]\varepsilon}^{b'} u_\varepsilon\,dt \\ &= \left[\frac{b'-a'}{\varepsilon}\right]\int_0^\varepsilon u_\varepsilon\,dt + O(\varepsilon) = \left[\frac{b'-a'}{\varepsilon}\right]\varepsilon\,\overline{u} + O(\varepsilon) = |E|\overline{u} + o(1).\end{aligned}$$

If $u \in L^p(a,b)$ then we can consider the 'truncated function' $u^T = (T \wedge u) \vee (-T)$ and the corresponding $u_\varepsilon^T(t) = u^T(\frac{t}{\varepsilon})$ which converge to $\overline{u}_T$. For $v \in L^{p'}(a,b)$ we may write

$$\int_a^b v(u_\varepsilon - \overline{u})\,dt = \int_a^b v(u_\varepsilon - u_\varepsilon^T)\,dt + \int_a^b v(u_\varepsilon^T - \overline{u}_T)\,dt + \int_a^b v(\overline{u}_T - \overline{u})\,dt,$$

and the thesis follows since we have

$$\int_0^1 |u_T - u|^p\,dt \le \int_{\{|u|>T\}} |u|^p\,dt = o(1)$$

as $T \to +\infty$ and we can use the estimates

$$\begin{aligned}|\overline{u}_T - \overline{u}| &\le \int_0^1 |u_T - u|\,dt \le \Big(\int_0^1 |u_T - u|^p\,dt\Big)^{1/p}, \\ \int_a^b v(u_\varepsilon - u_\varepsilon^T)\,dt &\le \|v\|_{p'}\Big(\int_0^1 |u_T - u|^p\,dt\Big)^{1/p},\end{aligned}$$

which are easily derived using Hölder's inequality.

Example 2.5 An important case of the previous example is when we take the 1-periodic piecewise-constant function u defined on $(0,1)$ by

$$u(s) = \begin{cases} z_1 & \text{if } 0 < s < t \\ z_2 & \text{if } t < s < 1; \end{cases}$$

then, $u_\varepsilon \overset{*}{\rightharpoonup} \overline{u} = tz_1 + (1-t)z_2$.

Theorem 2.6 (weak compactness). *Let $p > 1$. Then from each bounded sequence (u_j) in $L^p(a,b)$ we can extract a weakly (weakly* if $p = \infty$) converging subsequence.*

Proof We only sketch the proof: choose (φ_i) as in Remark 2.2. Upon extraction of a subsequence we may suppose that the quantity $L(\varphi_i) = \lim_j \int_a^b \varphi_i u_j \, dt$ is well defined for all i. L is defined by density on the unit sphere of $L^{p'}(a,b)$ and by linearity on the whole $L^{p'}(a,b)$; hence, there exists $u \in L^p(a,b)$ such that $L(v) = \int_a^b uv \, dt$. Hence, we get $\int_a^b \varphi_i u \, dt = \lim_j \int_a^b \varphi_i u_j \, dt$ for all i, so that, by (2.3), we have $u_j \rightharpoonup u$. □

2.1.2 *Weak-coerciveness conditions*

In the sequel we will deal with *integral functionals* on $L^p(a,b)$ of the form

$$F(u) = \int_a^b f(u) \, dt, \tag{2.5}$$

where $f : \mathbf{R} \to [0, +\infty]$ is a Borel function.

From Theorem 2.6 we immediately obtain a coerciveness condition for this type of functionals in terms of a growth condition on the function f.

Proposition 2.7 *Let F be given by* (2.5).

(i) (p-growth from below) *If* $1 < p < \infty$

$$\liminf_{|z|\to+\infty} \frac{f(z)}{|z|^p} > 0 \tag{2.6}$$

then the functional F is weakly coercive on $L^p(a,b)$; that is, from every sequence (u_j) such that $\sup_j F(u_j) < +\infty$ *there exists a subsequence weakly converging in $L^p(a,b)$;*

(ii) (superlinear growth)*If we have*

$$\lim_{|z|\to+\infty} \frac{f(z)}{|z|} = +\infty \tag{2.7}$$

then the functional F is weakly coercive on $L^1(a,b)$.

Proof (i) follows immediately from Theorem 2.6 since $\sup_j F(u_j) < +\infty$ implies that $\sup_j \|u_j\|_p < +\infty$. Statement (ii) is known as the de la Vallée Poussin compactness criterion; the proof is more complex and we do not include it here. □

Remark 2.8 (i) Condition (2.6) can be rephrased as the existence of two positive constants c_1, c_2 such that

$$c_1|z|^p - c_2 \le f(z). \tag{2.8}$$

(ii) Clearly condition (2.6) with $p = 1$ does not ensure a coerciveness condition. Take, for example, $f(z) = |z|$ and $u_j = j\chi_{(0,1/j)}$.

2.2 Weak lower semicontinuity conditions: convexity

Now we look at conditions that ensure the weak lower semicontinuity of integral functionals; that is, the lower semicontinuity inequality along weakly-converging sequences.

Definition 2.9 *We say that a functional* $F : L^p(a,b) \to [0,+\infty]$ *is* (sequentially) weakly lower semicontinuous *in* $L^p(a,b)$ *(weakly* if* $p = \infty$*) if the lower semicontinuity inequality*

$$F(u) \le \liminf_j F(u_j) \tag{2.9}$$

holds for all $u \in L^p(a,b)$ *and* (u_j) *weakly (weakly* if* $p = \infty$*) converging to* u *in* $L^p(a,b)$.

We will show that necessary and sufficient conditions for the weak lower semicontinuity in $L^p(a,b)$ of functionals of the form (2.5) are the combined lower semicontinuity and convexity of f. While lower semicontinuity of f is clearly necessary even for the lower semicontinuity of F with respect to the strong convergence, in order to heuristically explain the necessity of convexity, we note that Jensen's inequality affirms precisely that for convex energies oscillations are never 'energetically convenient'. Hence, its violation allows to construct highly-oscillating (and hence weakly converging) sequences energetically more convenient than their limit. For a review of the relevant facts about convexity we refer to Appendix A.

Remark 2.10 We note that the *strong* lower semicontinuity is an immediate consequence of Fatou's lemma: if f is lower semicontinuous and positive (no convexity needed), and $u_j \to u$ in $L^1(a,b)$ then, upon extracting a subsequence we can suppose that the limit $\lim_j F(u_j)$ exists and moreover that $u_j \to u$ a.e. In particular $f(u(t)) \le \liminf_j f(u_j(t))$ for a.e. t, so that

$$\int_a^b f(u)\,dt \le \int_a^b \liminf_j f(u_j)\,dt \le \liminf_j \int_a^b f(u_j)\,dt.$$

Moreover, if f is continuous (not necessarily positive) and satisfies a growth condition of the form $|f(z)| \le C(1+|z|^p)$ then F is *continuous* on $L^p(a,b)$. In fact, we may apply Fatou's Lemma as above, with either functions $C(1+|z|^p) - f(z)$ and $C(1+|z|^p) + f(z)$ as integrands (being both positive and continuous). We then obtain that both F and $-F$ are lower semicontinuous, so that F is indeed continuous.

Proposition 2.11 (necessity). *Let $F : L^\infty(a,b) \to [0,+\infty]$ be given by* (2.5). *If F is weakly* lower semicontinuous in $L^\infty(a,b)$ then*

(i) *f is lower semicontinuous;* (ii) *f is convex.*

Proof (i) If $z_j \to z$ and $u_j(t) = z_j$, $u(t) = z$, then $u_j \to u$ strongly (and then weakly*) in $L^\infty(a,b)$. Hence, from (2.9) we get

$$(b-a)f(z) = F(u) \le \liminf_j F(u_j) = (b-a)\liminf_j f(z_j);$$

that is, the lower semicontinuity of f at z.

(ii) To prove that

$$f(tz_1 + (1-t)z_2) \le t\,f(z_1) + (1-t)f(z_2) \tag{2.10}$$

for all $z_1, z_2 \in \mathbf{R}$ and $t \in (0,1)$, define

$$v(s) = \begin{cases} z_1 & \text{if } 0 \le s \le t \\ z_2 & \text{if } t \le s \le 1, \end{cases}$$

extended by 1-periodicity to the whole of $\mathbf{R}$.

Let $u(s) = z$ and $u_j(s) = v(js)$ (see Fig. 2.1). Note $\sup_j \|u_j\|_{L^\infty(a,b)} < +\infty$. We can apply Example 2.4 to both sequences (u_j) and $(f(u_j))$, obtaining that $u_j \overset{*}{\rightharpoonup} u$ in $L^\infty(a,b)$ and that $f(u_j)$ weakly* converges to $tf(z_1) + (1-t)f(z_2)$; in particular $\lim_j F(u_j) = (b-a)(tf(z_1) + (1-t)f(z_2))$, so that, by (2.9), we have

$$(b-a)f(z) = F(u) \le \liminf_j F(u_j) = (b-a)(tf(z_1) + (1-t)f(z_2)),$$

and (2.10) is proved. □

Remark 2.12 The proof of Proposition 2.11(i) shows that the lower semicontinuity of f is necessary for the lower semicontinuity of F with respect to the *strong* convergence in $L^1(a,b)$.

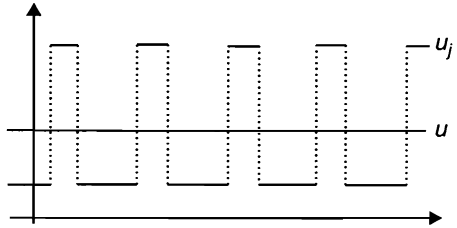

FIG. 2.1. Oscillations near a constant function

Proposition 2.13 (sufficiency). *If $f : \mathbf{R} \to [0, +\infty]$ is a convex lower-semicontinuous function then $F : L^1(a,b) \to [0,+\infty]$ is weakly lower semicontinuous in $L^1(a,b)$.*

Proof Let (f_k) be an increasing sequence of convex Lipschitz C^1 functions such that $f = \sup_k f_k$ (see Exercise 1.10). In this case, by the Monotone Convergence Theorem we have

$$F(u) = \sup_k F_k(u),$$

where $F_k(u) = \int_a^b f_k(u)\,dt$. Hence, it suffices to show that each F_k is weakly lower semicontinuous, since the supremum of a family of lower-semicontinuous functionals is weakly lower semicontinuous.

Let $u_j \rightharpoonup u$ in $L^1(a,b)$. By applying Remark A.1(b) we have $\int_a^b f_k(u)\,dt \le \int_a^b f_k(u_j)\,dt + \int_a^b f_k'(u)(u-u_j)\,dt$; that is,

$$F_k(u) \le F_k(u_j) + \int_a^b f_k'(u)(u-u_j)\,dt.$$

As $f_k'(u) \in L^\infty(a,b)$ and $u_j \rightharpoonup u$ in $L^1(a,b)$ we have

$$\lim_j \int_a^b f_k'(u)(u-u_j)\,dt = 0,$$

so that $F_k(u) \le \liminf_j F_k(u_j)$, as desired. □

Corollary 2.14 *Let $F : L^p(a,b) \to [0,+\infty]$ be of the form* (2.5). *Then, F is weakly lower semicontinuous in $L^p(a,b)$ if and only if f is lower semicontinuous and convex.*

Proof If F is weakly lower semicontinuous in $L^p(a,b)$ then its restriction to $L^\infty(a,b)$ is weakly lower semicontinuous in $L^\infty(a,b)$; hence, f is lower semicontinuous and convex by Proposition 2.11. Conversely, if f is lower semicontinuous and convex then the functional F defined by (2.5) on $L^1(a,b)$ is weakly lower semicontinuous in $L^1(a,b)$; hence its restriction to $L^p(a,b)$ is weakly lower semicontinuous in $L^p(a,b)$. □

Remark 2.15 The same proof can be generalized to show that functionals of the form $F(u) = \int_a^b f(t,u)\,dt$ defined in $L^p(a,b)$ (f a Borel function) are weakly lower semicontinuous in $L^p(a,b)$ if and only if $f(t,\cdot)$ is convex and l.s.c. for a.e. $t \in (a,b)$. If f satisfies a growth condition of the form $|f(t,z)| \le C(1+|z|^p)$ another more abstract proof goes as follows: note that F is a convex functional, and it is continuous with respect to the strong convergence (proceeding as in Remark 2.10). It suffices now to note that a convex functional is weakly continuous if and only if it is strongly continuous, by the Hahn Banach Theorem.

2.3 Relaxation and Γ-convergence in L^p spaces

We only deal with the case $1 < p < \infty$ and of integrands satisfying the condition

$$f(z) \geq c_1|z|^p - c_2 \tag{2.11}$$

for all $z \in \mathbf{R}$, for some $c_1, c_2 > 0$. We will show that the relaxation of an integral functional correspond to the convexification of the corresponding integrand function. To this end we introduce the following notion, whose notation comes from the notation of conjugate functions in Convex Analysis (see Definition 2.34 and Appendix B).

Definition 2.16 *If $f : \mathbf{R} \to [0, +\infty]$ then f^{**} denotes the* convex and lower semicontinuous envelope of f*; that is,*

$$f^{**}(z) = \sup\{g(z) : \ g \textit{ convex and lower semicontinuous}, \ g \leq f\} \tag{2.12}$$

for all $z \in \mathbf{R}$.

Before stating the relaxation result we note some properties of the convex and lower semicontinuous envelope.

Remark 2.17 (a) The supremum in the definition of f^{**} is actually a maximum; that is, f^{**} is convex and lower semicontinuous.

(b) The value $f^{**}(z)$ can be expressed as

$$f^{**}(z) = \inf\bigl\{\liminf_j \bigl(t_j f(z_j^1) + (1-t_j) f(z_j^2)\bigr) : \\ t_j \in (0,1), \ \lim_j \bigl(t_j z_j^1 + (1-t_j) z_j^2\bigr) = z\bigr\}. \tag{2.13}$$

In fact, it is easily seen that the right-hand side of (2.13) is larger than $f^{**}(z)$. On the other hand it can be checked (and is left as an exercise) that it also defines a convex and lower semicontinuous function which is less than f; hence also the converse inequality holds.

(c) If f is lower semicontinuous, then (2.13) can be simplified into

$$f^{**}(z) = \min\{t f(z_1) + (1-t) f(z_2) : t \in [0,1], t z_1 + (1-t) z_2 = z\} \tag{2.14}$$

(just apply the direct methods). Note that this formula implies that if z_1 and z_2 solve the minimum problem then $f^{**}(z_k) = f(z_k)$ for $k = 1, 2$. This shows that f^{**} coincides with g^{**}, where

$$g(z) = \begin{cases} f(z) & \text{if } f^{**}(z) = f(z) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.15}$$

(d) The set $J = \{f^{**} < +\infty\}$ is an interval on the closure of which f^{**} is continuous. In fact, since f^{**} is convex, J is convex, hence an interval. Let $\overline{J} = [x_0, x_1]$. By Remark A.1(d) f^{**} is continuous in (x_0, x_1). By convexity $f^{**}(x_0) \geq$

$\limsup_{t\to x_0^+} f^{**}(t)$, while by lower semicontinuity $f^{**}(x_0) \le \liminf_{t\to x_0^+} f^{**}(t)$, so that f^{**} is continuous at x_0. The same remark applies to x_1.

(e) If $f < +\infty$ then f^{**} coincides with the convex envelope of f, since finite convex functions are continuous. In general this is not true: take for example $f(z) = 0$ if $z \in (0,1)$ and $f(z) = +\infty$ otherwise, or $f(z) = 0$ if $z \in (0,1)$, $f(0) = f(1) = 1$ and $f(z) = +\infty$ otherwise. In both cases f is convex, but not lower semicontinuous. Note that in the second case f is not even lower semicontinuous on $\{f < +\infty\}$.

Theorem 2.18 (relaxation in Lebesgue spaces). *Let* $1 < p < \infty$, *let* $f : \mathbf{R} \to [0, +\infty]$ *be a Borel function satisfying* (2.11), *and let* $F : L^p(a,b) \to [0,+\infty]$ *be given by* (2.5). *Then the weakly-lower semicontinuous envelope of* F *with respect to the* $L^p(a,b)$ *convergence is given by*

$$\mathrm{F}(\mathrm{u}) = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}^{**}(\mathrm{u})\, \mathrm{dt} \tag{2.16}$$

on $L^p(a,b)$.

Proof Let F denote the lower semicontinuous envelope of F with respect to the $L^p(a,b)$ convergence and let G be the functional given by the right-hand side of (2.16). Since G is lower semicontinuous and $G \le F$ we have $G \le \mathrm{F}$. We have to show the converse inequality.

If $J = \{f^{**} < +\infty\}$ is empty or is a point then $f^{**} = f$ and there is nothing to prove. Otherwise, J is a non-degenerate interval. It can easily be seen that for all $u \in L^p(a,b)$ there exist a sequence u_j of piecewise-constant functions such that $u_j \to u$ in $L^p(a,b)$ and $\lim_j G(u_j) = G(u)$. Therefore it is sufficient to show that $\mathrm{F} \le \mathrm{G}$ on piecewise-constant functions, since by the lower semicontinuity of F we have

$$\mathrm{F}(\mathrm{u}) \le \liminf_{\mathrm{j}} \mathrm{F}(\mathrm{u}_{\mathrm{j}}) \le \liminf_{\mathrm{j}} \mathrm{G}(\mathrm{u}_{\mathrm{j}}) = \mathrm{G}(\mathrm{u})$$

for all u (the inequality being trivial if $G(u) = +\infty$).

We begin by treating the case $u(s) = z$ with $f^{**}(z) < +\infty$. In this case, for all $\varepsilon > 0$ let t_ε, z_ε^1 and z_ε^2 be such that

$$t_\varepsilon f(z_\varepsilon^1) + (1 - t_\varepsilon) f(z_\varepsilon^2) \le f^{**}(z) + \varepsilon, \qquad |t_\varepsilon z_\varepsilon^1 + (1-t_\varepsilon) z_\varepsilon^2 - z| < \varepsilon,$$

which exist by (2.13). By repeating the construction in the proof of Proposition 2.11(ii) we obtain a sequence $(u_j^\varepsilon)_j$ of $1/j$-periodic functions converging to the constant function u^ε defined by $u^\varepsilon(s) = (t_\varepsilon z_\varepsilon^1 + (1-t_\varepsilon) z_\varepsilon^2)$ and such that

$$\lim_j F(u_j^\varepsilon) = (t_\varepsilon f(z_\varepsilon^1) + (1-t_\varepsilon) f(z_\varepsilon^2))(b-a) \le G(u^\varepsilon) + \varepsilon(b-a).$$

This shows that $\mathrm{F}(\mathrm{u}^\varepsilon) \le \mathrm{G}(\mathrm{u}^\varepsilon) + \varepsilon(\mathrm{b}-\mathrm{a})$, and also that $\mathrm{F}(\mathrm{u}) \le \mathrm{G}(\mathrm{u})$, by letting $\varepsilon \to 0$ and using the lower semicontinuity of F and the fact that $f^{**}(u^\varepsilon) \le f^{**}(z) + \varepsilon$.

The proof in the case of a piecewise-constant u can be obtained likewise by repeating the construction of the proof of Proposition 2.11(ii) for each interval where the function u is constant. Details are left as an exercise. □

Note that a key point in Theorem 2.18 is proving that F is an integral functional, from which (2.16) is easily deduced. This fact is hidden in the second part of the proof where we use additivity and density properties that are characteristic of integrals. Moreover, note that the growth hypothesis is only used so that all functions can be taken bounded in $L^p(a,b)$.

Example 2.19 As a simple example we can consider the functional at the beginning of the chapter, with $f(u) = u^2(u-1)^2$, for which $f^{**}(u) = ((u^2-u)^+)^2$ ($t^+ = t \vee 0$ is the positive part of t), and in particular $f^{**}(u) = 0$ if $0 \le u \le 1$, which shows that all functions satisfying this constraint can be approximated by functions (u_j) with $\lim_j F(u_j) = 0$.

We now deal with the problem of the Γ-convergence on L^p spaces and show that it can be reduced to a convergence of the integrands via convexification. In the proof of the following theorem we use the fact that we may substitute each F_j with its lower-semicontinuous envelope, and that, in its turn, this can be expressed in terms of convexification.

Theorem 2.20 (Γ-convergence in Lebesgue spaces). *Let* $1 < p < \infty$ *and for all* $j \in \mathbf{N}$ *let* $f_j : \mathbf{R} \to [0,+\infty]$ *be a Borel function. Suppose furthermore that the sequence* (f_j) *satisfies* (2.11) *uniformly and* $\sup_j f_j(0) < +\infty$, *and let* $F_j : L^p(a,b) \to [0,+\infty]$ *be given by*

$$F_j(u) = \int_a^b f_j(u)\, dt. \tag{2.17}$$

Then (F_j) Γ*-converges to some* F *with respect to the weak convergence in* $L^p(a,b)$ *if and only if* (f_j^{**}) Γ*-converges to some* f *in* $\mathbf{R}$. *In this case we have*

$$F(u) = \int_a^b f(u)\, dt. \tag{2.18}$$

If all functions f_j *are locally equi-bounded then* f *is also simply the pointwise limit of* (f_j^{**}).

Proof Let $f = \Gamma\text{-}\lim_j f_j^{**}$. Note that by Proposition 1.32 and Theorem 2.18 $\Gamma\text{-}\liminf_j F_j(u) = \Gamma\text{-}\liminf_j \overline{F}_j(u) = \Gamma\text{-}\liminf_j \int_a^b f_j^{**}(u)\, dt$. Let $u \in L^p(a,b)$, and let $u_j \rightharpoonup u$ in $L^p(a,b)$. With fixed $\lambda > 0$ consider $T_\lambda f_j^{**}$ defined in (1.31) and note that it converges pointwise to $T_\lambda f$, by the property of convergence of minima. By the convexity of all functions and the growth condition this convergence is uniform in $\mathbf{R}$. We then immediately get

$$\liminf_j \int_a^b f_j^{**}(u_j)\,dt \geq \liminf_j \int_a^b T_\lambda f_j^{**}(u_j)\,dt$$
$$= \liminf_j \int_a^b T_\lambda f(u_j)\,dt \geq \int_a^b T_\lambda f(u)\,dt.$$

By taking the supremum in λ we then get the liminf inequality.

As in the proof of Theorem 2.18 it is enough to prove the limsup inequality for $u(t) = z$. In this case, as again $\Gamma\text{-}\limsup_j F_j = \Gamma\text{-}\limsup_j \int_a^b f_j^{**}(u)\,dt$, it is enough to prove that $\Gamma\text{-}\limsup_j \int_a^b f_j^{**}(u)\,dt \leq F(u)$. Then take $u_j(t) = z_j$, where (z_j) is a sequence such that $\lim_j f_j^{**}(z_j) = f(z)$.

Conversely, suppose that (F_j) Γ-converges to some G. By the compactness of Γ-convergence on $\mathbf{R}$ from every subsequence (f_{j_k}) we can extract a further subsequence (not relabelled) such that $(f_{j_k}^{**})$ Γ-converges to some f. From what seen above we then must have $G(u) = \int_a^b f(u)\,dt$. In particular, f does not depend on the subsequence, as desired.

If all f_j are locally equibounded then the convergence of (f_j^{**}) is locally uniform and hence equivalent to the Γ-convergence of (f_j^{**}). □

2.4 Problems on Sobolev spaces

We turn now our attention to problems defined on spaces of weakly-differentiable functions. For a quick review of the relevant notions on Sobolev spaces in dimension one we refer to Appendix A.

2.4.1 *Weak convergence in Sobolev spaces*

Sobolev spaces inherit the notion and terminology of weak and weak* convergences in Lebesgue spaces.

Definition 2.21 (weak convergence in Sobolev spaces). *We say that $u_j \rightharpoonup u$ in $W^{1,p}(a,b)$ ($u_j \overset{*}{\rightharpoonup} u$ if $p = \infty$), if we have $u_j \rightharpoonup u$ and $u_j' \rightharpoonup u'$ in $L^p(a,b)$ ($u_j \overset{*}{\rightharpoonup} u$ and $u_j' \overset{*}{\rightharpoonup} u'$ if $p = \infty$).*

Remark 2.22 Since the inclusion of $W^{1,p}(a,b)$ in $L^p(a,b)$ is compact, $u_j \rightharpoonup u$ in $W^{1,p}(a,b)$ if and only if we have $u_j \to u$ *strongly* in $L^p(a,b)$ and $u_j' \rightharpoonup u'$ weakly in $L^p(a,b)$. Conversely since a sequence which is bounded in $L^1(a,b)$ and with derivatives bounded in $L^p(a,b)$ is also bounded in $W^{1,p}(a,b)$ then $u_j \rightharpoonup u$ in $W^{1,p}(a,b)$ if and only if we have $u_j \to u$ weakly in $L^1(a,b)$ and $u_j' \rightharpoonup u'$ weakly in $L^p(a,b)$. The analogous remark applies to the case $p = \infty$.

Theorem 2.23 (weak compactness). *Let $p > 1$. Then from each bounded sequence (u_j) in $W^{1,p}(a,b)$ we can extract a weakly (weakly* if $p = \infty$) converging subsequence.*

Proof For simplicity of notation suppose $p < \infty$. By Theorem 2.6 applied both to (u_j) and to (u'_j), upon extracting a subsequence we may suppose that $u_j \rightharpoonup u$ and $u'_j \rightharpoonup g$ with $u, g \in L^p(a,b)$. We then have

$$\int_a^b u\varphi'\,dt = \lim_j \int_a^b u_j\varphi'\,dt = -\lim_j \int_a^b u'_j\varphi\,dt = -\int_a^b g\varphi\,dt$$

for all $\varphi \in C_0^1(a,b)$, which shows that u is weakly differentiable and $u' = g$, so that $u_j \rightharpoonup u$ in $W^{1,p}(a,b)$. □

Example 2.24 (weak convergence of oscillating functions). Let $v(t) = zt+u(t)$, where $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R})$ is 1-periodic, and let $v_\varepsilon(t) = \varepsilon\, v(\frac{t}{\varepsilon})$. Then $v_\varepsilon \rightharpoonup zt$ in $W^{1,p}(a,b)$ ($v_\varepsilon \overset{*}{\rightharpoonup} zt$ in $W^{1,\infty}(a,b)$ if $p=\infty$). In particular $v'_\varepsilon \rightharpoonup z$ in $L^p(a,b)$ and $v_\varepsilon \to zt$ in $L^\infty(a,b)$. This example follows immediately from the corresponding one in $L^p(a,b)$.

2.4.2 *Integral functionals on Sobolev spaces. Coerciveness conditions*

We will consider functionals $F : W^{1,p}(a,b) \to [0,+\infty]$. Coerciveness conditions for these functionals will be derived from the following proposition.

Proposition 2.25 *Let $1 < p \le \infty$ and let (u_j) be a sequence in $W^{1,p}(a,b)$ such that* $\sup_j \|u'_j\|_{L^p(a,b)} < +\infty$. *Then*

(i) (closure) *if $u_j \rightharpoonup u$ in $L^p(a,b)$ ($u_j \overset{*}{\rightharpoonup} u$ if $p=\infty$) then $u \in W^{1,p}(a,b)$ and $u_j \rightharpoonup u$ in $W^{1,p}(a,b)$ ($u_j \overset{*}{\rightharpoonup} u$ if $p=\infty$);*

(ii) (compactness) *if* $\sup_j \operatorname{ess-inf}|u_j| < \infty$ *then there exists a subsequence of (u_j) weakly (weakly* if $p=\infty$) converging in $W^{1,p}(a,b)$.*

Proof (for $p < \infty$ only; the case $p = \infty$ is left to the reader)

(i) If $u_j \rightharpoonup u$ then $\sup_j \|u_j\|_{L^p(a,b)} < +\infty$ by Proposition 2.3. Hence, we have $\sup_j \|u_j\|_{W^{1,p}(a,b)} < +\infty$, so that, by Theorem 2.23, we can extract a weakly converging subsequence as desired,

(ii) Let $t_j \in (a,b)$ be such that $\sup_j |u_j(t_j)| < +\infty$. Define $z_j = u_j(t_j)$. Each function $v_j = u_j - z_j$ satisfies the hypotheses of Theorem A.12 (Poincaré's inequality). Hence,

$$\sup_j \|v_j\|_{W^{1,p}(a,b)} \le C \sup_j \|v'_j\|_{L^p(a,b)} = C \sup_j \|u'_j\|_{L^p(a,b)} < +\infty,$$

so that $\sup_j \|u_j\|_{L^p(a,b)} \le \sup_j \Big(\|v_j\|_{L^p(a,b)} + z_j\Big) < +\infty$. This implies again that $\sup_j \|u_j\|_{W^{1,p}(a,b)} < +\infty$, and we may proceed as above. □

Remark 2.26 Let $1 < p < \infty$, and let F satisfy $F(u) \ge c_1 \int_a^b |u'|^p\,dt - c_2$ for some $c_1, c_2 > 0$. Then the following sets are precompact in $L^p(a,b)$ (c denotes an arbitrary constant and $v \in L^{p'}(a,b)$):

$$E_1 = \Big\{u \in W^{1,p}(a,b) :\ F(u) \le c,\ \int_a^b |u|\,dt \le c\Big\}, \tag{2.19}$$

$$E_2 = \Big\{u \in W_0^{1,p}(a,b) : \; F(u) - \int_a^b vu\,dt \le c\Big\}, \tag{2.20}$$

$$E_3 = \Big\{u \in W^{1,p}(a,b) : \; F(u) - \int_a^b vu\,dt \le c, \; \int_a^b u\,dt = 0\Big\}. \tag{2.21}$$

In fact, E_1 is precompact directly by Proposition 2.25(ii). If $u \in E_2$ then we have by Poincaré's inequality (Theorem A.12),

$$\begin{aligned} c \ge F(u) - \int_a^b vu\,dt \; &\ge \; c_1 \int_a^b |u'|^p\,dt - c_2 - \|u\|_{L^p(a,b)}\|v\|_{L^{p'}(a,b)} \\ &\ge \frac{c_1}{C^p}\|u\|^p_{W^{1,p}(a,b)} - c_2 - \|u\|_{L^p(a,b)}\|v\|_{L^{p'}(a,b)}, \end{aligned}$$

which shows that $\|u\|_{W^{1,p}(a,b)} \le c(b-a, c_1, c_2, \|v\|_{L^{p'}(a,b)})$, and the precompactness of E_2 by Proposition 2.25(ii). The same argument shows that E_3 is precompact since the condition $\int_a^b u\,dt = 0$ ensures that u vanishes at some point in (a,b), thus allowing for the application of Theorem A.12.

2.5 Weak lower semicontinuity conditions

We now rephrase the lower semicontinuity conditions obtained for Lebesgue spaces.

Definition 2.27 *As for Lebesgue spaces, we say that* $F : W^{1,p}(a,b) \to [0,+\infty]$ *is* (sequentially) weakly lower semicontinuous *in* $W^{1,p}(a,b)$ *(weakly* if* $p=\infty$*) if the lower semicontinuity inequality*

$$F(u) \le \liminf_j F(u_j) \tag{2.22}$$

holds for all $u \in W^{1,p}(a,b)$ *and* (u_j) *weakly (weakly* if* $p=\infty$*) converging to* u *in* $W^{1,p}(a,b)$.

Remark 2.28 (a) Let $1 < p \le \infty$; by using Theorem A.10 we have that the following conditions are equivalent ('weak' is to be replaced by 'weak*' if $p=\infty$):

(i) F is weakly lower semicontinuous on $W^{1,p}(a,b)$,

(ii) Inequality (2.22) holds whenever $\sup_j \|u_j'\|_{L^p(a,b)} < +\infty$ and $u_j \to u$ strongly in $L^p(a,b)$,

(iii) Inequality (2.22) holds whenever $\sup_j \|u_j'\|_{L^p(a,b)} < +\infty$ and $u_j \rightharpoonup u$ weakly in $L^p(a,b)$.

(b) If $1 < p < \infty$ and $F(u) \ge c_1 \int_a^b |u'|^p\,dt - c_2$ for some $c_1, c_2 > 0$ then (i) is equivalent to the lower semicontinuity of F on $W^{1,p}(a,b)$ with respect to the strong convergence in $L^p(a,b)$.

We will analyse necessary and sufficient conditions for the weak lower semicontinuity in $W^{1,p}(a,b)$ of functionals of the form

$$F(u) = \int_a^b f(u')\,dt, \tag{2.23}$$

where $f : \mathbf{R} \to [0, +\infty]$ is a Borel function. We will show that necessary and sufficient conditions are again the lower semicontinuity and convexity of f. Even though the result can be deduced from that in L^p, we divide the statement in two for future reference.

Proposition 2.29 (necessity). Let $F : W^{1,\infty}(a, b) \to [0, +\infty]$ be given by (2.23). If F is weakly* lower semicontinuous in $W^{1,\infty}(a, b)$ then
(i) f is lower semicontinuous; (ii) f is convex.

Proof (i) If $z_j \to z$ and $u_j(t) = z_j t$, $u(t) = zt$, then $u_j \to u$ strongly (and then weakly*) in $W^{1,\infty}(a, b)$. Hence, from (2.22) we get

$$(b - a)f(z) = F(u) \leq \liminf_j F(u_j) = (b - a)\liminf_j f(z_j),$$

that is, the lower semicontinuity of f.

(ii) Let $z_1, z_2 \in \mathbf{R}$, $t \in (0, 1)$ and $z = tz_1 + (1 - t)z_2$. Define

$$v(s) = \begin{cases} (z_1 - z)s & \text{if } 0 \leq s \leq t \\ (z_1 - z)t + (z_2 - z)(s - t) & \text{if } t \leq s \leq 1, \end{cases}$$

extended by 1-periodicity to the whole of $\mathbf{R}$ (note that $v(1) = t(z_1 - z) + (1 - t)(z_2 - z) = tz_1 + (1 - t)z_2 - z = 0 = v(0)$).

Let $u(s) = zs$ and $u_j(s) = zs + \frac{1}{j}v(js)$ (see Fig. 2.2). Note that u_j' is the $1/j$-periodic piecewise-constant function satisfying

$$u_j'(s) = \begin{cases} z_1 & \text{if } 0 < s < t/j \\ z_2 & \text{if } t/j < s < 1/j; \end{cases}$$

in particular $\sup_j \|u_j\|_{W^{1,\infty}(a,b)} < +\infty$. As $u_j \to u$ in $L^\infty(a, b)$ we have $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(a, b)$. By Example 2.4 we have that the function $f(u_j')$ weakly* converges to $tf(z_1) + (1 - t)f(z_2)$; in particular $\lim_j F(u_j) = (b - a)(tf(z_1) + (1 - t)f(z_2))$, so that, by (2.22), we have

$$(b - a)f(z) = F(u) \leq \liminf_j F(u_j) = (b - a)(tf(z_1) + (1 - t)f(z_2)),$$

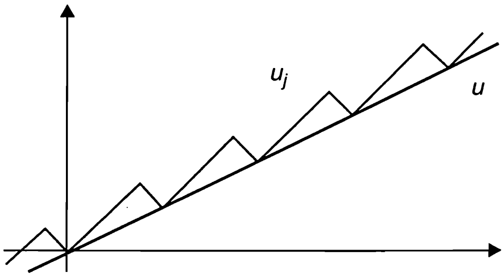

FIG. 2.2. Oscillations near a linear function

and the convexity of f is proved. □

Proposition 2.30 (sufficiency). *If $f : \mathbf{R} \to [0, +\infty]$ is a convex lower semicontinuous function then $F : W^{1,1}(a, b) \to [0, +\infty]$ is weakly lower semicontinuous in $W^{1,1}(a, b)$.*

Proof This is clearly implied by the corresponding result for Lebesgue spaces. □

Corollary 2.31 *Let $F : W^{1,p}(a, b) \to [0, +\infty]$ be of the form (2.23). Then, F is weakly lower semicontinuous in $W^{1,p}(a, b)$ if and only if f is lower semicontinuous and convex.*

If in addition $1 < p < \infty$ and there exist $c_1, c_2 > 0$ such that $f(z) \geq c_1|z|^p - c_2$ for all $z \in \mathbf{R}$ then F is lower semicontinuous on $W^{1,p}(a, b)$ with respect to the $L^p(a, b)$ convergence if and only if f is lower semicontinuous and convex.

Proof If F is weakly lower semicontinuous in $W^{1,p}(a, b)$ then its restriction to $W^{1,\infty}(a, b)$ is weakly lower semicontinuous in $W^{1,\infty}(a, b)$; hence, f is lower semicontinuous and convex by Proposition 2.29. Vice versa, if f is lower semicontinuous and convex then the functional F defined by (2.23) on $W^{1,1}(a, b)$ is weakly lower semicontinuous in $W^{1,1}(a, b)$; hence its restriction to $W^{1,p}(a, b)$ is weakly lower semicontinuous in $W^{1,p}(a, b)$.

The last statement follows by Remark 2.28(b). □

Remark 2.32 (i) The same proof can be generalized to show that functionals of the form $F(u) = \int_a^b f(t, u')\,dt$ defined in $W^{1,p}(a, b)$ (f a Borel function) are weakly lower semicontinuous in $W^{1,p}(a, b)$ if and only if $f(t, \cdot)$ is convex and l.s.c. for a.e. $t \in (a, b)$;

(ii) (relaxation) By applying Theorem 2.18, under the hypotheses on f therein, it can be easily checked that the weak l.s.c. envelope of F given by (2.23) is $\mathrm{F(U)} = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{F}^{**}(\mathrm{U}')\,\mathrm{DT}$.

2.6 Γ-convergence and convex analysis

In the one-dimensional case only, Γ-convergence of integral functionals can be described through a convergence on the integrands, via convex analysis. We begin with an example.

Example 2.33 Consider functionals of the form $\int_a^b \alpha_j(t)|u'|^2\,dt$ with $1 \leq \alpha_j \leq 2$. If such functionals Γ-converge as $j \to +\infty$ then we deduce the convergence of the minimum problems

$$\min\Big\{\int_a^b \big(\alpha_j(t)|u'|^2 - 2gu\big)\,dt : u \in W^{1,2}(a, b)\Big\}$$

for every fixed $g \in L^2(a, b)$ with $\int_a^b g\,dt = 0$ (apply the direct methods to these problems noting that we can restrict to u with $\int_a^b u\,dt = 0$). By computing the Euler–Lagrange equation we obtain that the minimizers u_j satisfy

$$\begin{cases} -(\alpha_j(t)u_j')' = g & \text{on } (a,b) \\ u_j'(a) = u_j'(b) = 0. \end{cases}$$

Let G be defined by $G' = -g$ and $G(a) = 0 (= G(b))$. We then can write $\alpha_j u_j' = G$ for all j. Upon extracting a subsequence, we may suppose that there exists β such that $1/\alpha_j$ converges weakly to $1/\beta$. We then have

$$u_j' = \frac{G}{\alpha_j} \rightharpoonup \frac{G}{\beta},$$

so that, if u is the weak limit of u_j, we deduce that $u' = G/\beta$. In particular, u solves

$$\begin{cases} -(\beta(t)u')' = g & \text{on } (a,b) \\ u'(a) = u'(b) = 0; \end{cases}$$

that is, u is the minimum point of

$$\min\Big\{\int_a^b \Big(\beta(t)|u'|^2 - 2gu\Big)\,dt : u \in W^{1,2}(a,b)\Big\}.$$

This suggests that the convergence of F_j can be stated as a weak convergence of the inverse of its coefficients. In its turn, the latter can be seen as a weak convergence of the conjugate functions of the integrands $a_j(t)|z|^2$ (see further).

Now we state a more precise characterization of Γ-convergence. We will restrict our analysis to a class of integral functionals F which satisfy some growth conditions. With fixed $1 < p < \infty$ and $c_1, c_2, c_3 > 0$, we will consider throughout this section functionals $F : W^{1,p}(a,b) \to [0,+\infty)$ of the form

$$F(u) = \int_a^b f(t, u'(t))\,dt, \tag{2.24}$$

where $f : (a,b) \times \mathbf{R} \to [0,+\infty)$ is a Borel function satisfying the inequalities (*growth conditions of order* p)

$$c_1|z|^p - c_2 \le f(t,z) \le c_3(1 + |z|^p) \tag{2.25}$$

for all $t \in (a,b)$ and $z \in \mathbf{R}$, and such that

$$\text{the function } s \mapsto f(t,s) \text{ is convex for all } t \in (a,b). \tag{2.26}$$

We will denote by $\mathcal{F} = \mathcal{F}(p, c_1, c_2, c_3)$ the class of these functionals.

Our aim is to characterize the Γ-convergence of a sequence of functionals (F_j) belonging to $\mathcal{F}$. To this end we have to recall the following notion of conjugate function.

Definition 2.34 *Let $f : (a,b) \times \mathbf{R} \to \mathbf{R}$ be a function. We define the* conjugate function of f *as*

$$f^*(t,z^*) = \sup\{z^* z - f(t,z) : \ z \in \mathbf{R}\} \tag{2.27}$$

for all $t \in (a,b)$ and $z^* \in \mathbf{R}$.

In order to properly state the characterization theorem, we introduce the following *localized* versions of a functional in $\mathcal{F}$: if $F \in \mathcal{F}$ is given by (2.24) then for every I open sub-interval of (a,b) and $u \in W^{1,p}(I)$ we define

$$F(u,I) = \int_I f(t,u'(t))\,dt. \tag{2.28}$$

With this notation the characterization result reads as follows.

Theorem 2.35 (characterization of Γ-convergence). *Let (F_j) be a sequence in $\mathcal{F}$ with integrand f_j, and $F \in \mathcal{F}$ with integrand f. Then the following statements are equivalent:*

(i) *for all I open subintervals of (a,b), $F(\cdot,I)$ is the $\Gamma(L^p(I))$-$\lim_j F_j(\cdot,I)$ on $W^{1,p}(I)$;*

(ii) *for all $z^* \in \mathbf{R}$, $f^*(\cdot,z^*)$ is the weak*-limit of the sequence $(f_j^*(\cdot,z^*))$.*

Moreover, both conditions are compact. In particular, for every sequence (F_j) in $\mathcal{F}$ there exists a subsequence $\Gamma(L^p(a,b))$-converging to some $F \in \mathcal{F}$ on $W^{1,p}(a,b)$.

The proof of Theorem 2.35 can be found in Appendix B.

Example 2.36 As a particular case, take $f_j(t,z) = a_j(t)|z|^2$ with $0 < c_1 \le a_j \le c_2 < +\infty$. Then

$$f_j^*(t,z^*) = \frac{(z^*)^2}{4a_j(t)}.$$

Hence, $f_j^*(\cdot,z^*)$ converges weakly* if and only if

$$\frac{1}{a_j(t)} \overset{*}{\rightharpoonup} \frac{1}{\beta(t)} \text{ for some } \beta \in L^\infty(a,b),$$

and in this case we get

$$\Gamma\text{-}\lim_j \int_a^b a_j(t)|u'|^2\,dt = \int_a^b \beta(t)|u'|^2\,dt.$$

As a particular case we can take $a_j(t) = \alpha(jt)$ with α 1-periodic. In this case β is constant and

$$\beta = \Big(\int_0^1 \frac{1}{\alpha(t)}\,dt\Big)^{-1},$$

the *harmonic mean* of a. Recall that in contrast $a_j \rightharpoonup \overline{\alpha} = \int_0^1 a(s)\,ds$.

2.7 Addition of boundary data

In minimum problems we often deal with prescribed boundary data. In order to deduce a convergence result with boundary conditions from an unconstrained Γ-limit, it is necessary then to make sure that the addition of these data is in some sense 'compatible' with Γ-convergence. This means that the set of functions satisfying the boundary conditions must be a closed set (to satisfy the liminf inequality), and that for these functions we may find a recovery sequence still satisfying the boundary conditions (to satisfy the limsup inequality). Here we provide a simple proof in the case that all the functionals belong to the same class $\mathcal{F}$ defined in the previous section.

Proposition 2.37 *Let (F_j) be a sequence in $\mathcal{F}$ and let $F = \Gamma(L^p(a,b))\text{-}\lim_j F_j$. Then, for all $u \in W^{1,p}(a,b)$ there exists a sequence u_j such that $u_j - u \in W_0^{1,p}(a,b)$ and converges to 0 weakly in $W^{1,p}(a,b)$, and $F(u) = \lim_j F_j(u_j)$.*

Proof Let $v_j \in W^{1,p}$ be such that $v_j \to u$ in $L^p(a,b)$ and $F(u) = \lim_j F_j(v_j)$. By Proposition 2.25(i) we have that $v_j \rightharpoonup u$ in $W^{1,p}(a,b)$. Let $v \in W_0^{1,p}(a,b)$ be such that $v > 0$ on (a,b) (e.g., $v(t) = \min\{(t-a),(b-t)\}$), and define $u_j(s) = u + \min\{\max\{v_j - u, -v\}, v\}$ (see Fig. 2.3); that is,

$$u_j(s) = \begin{cases} v_j(s) & \text{if } u(s) - v(s) \le v_j(s) \le u(s) + v(s) \\ u(s) - v(s) & \text{if } v_j(s) < u(s) - v(s) \\ u(s) + v(s) & \text{if } v_j(s) > u(s) + v(s). \end{cases}$$

We then have

$$\begin{aligned} F_j(u_j) &= \int_{\{u_j \ne v_j\}} f_j(t, u_j')\,dt + \int_{\{u_j = v_j\}} f_j(t, u_j')\,dt \\ &= \int_{\{u_j \ne v_j, u_j = u - v\}} f_j(t, u' - v')\,dt + \int_{\{u_j \ne v_j, u_j = u+v\}} f_j(t, u' + v')\,dt \\ &\quad + \int_{\{u_j = v_j\}} f_j(t, v_j')\,dt \end{aligned}$$

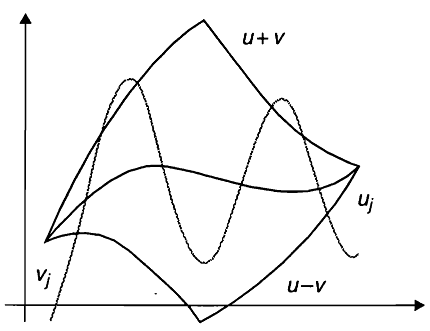

FIG. 2.3. Matching boundary conditions

$$
\begin{aligned}
&\le \int_{\{u_j\neq v_j,u_j=u-v\}} c_3(1+|u'-v'|^p)\,dt \\
&\qquad + \int_{\{u_j\neq v_j,u_j=u+v\}} c_3(1+|u'+v'|^p)\,dt + \int_a^b f_j(t,v_j')\,dt \\
&\le c_3 2^{p-1}\int_{\{u_j\neq v_j\}} (1+|u'|^p+|v'|^p)\,dt + F_j(v_j).
\end{aligned}
$$

Since $\lim_j |\{u_j \neq v_j\}| = 0$ we finally obtain $\limsup_j F_j(u_j) \le \limsup_j F_j(v_j) = F(u)$, and the desired equality. □

Remark 2.38 (compatibility of boundary data). If $u_0 \in W^{1,p}(a,b)$, by the last statement of the previous proposition we deduce that the restriction of F_j to $u_0 + W_0^{1,p}(a,b)$ Γ-converges to the restriction of F to $u_0 + W_0^{1,p}(a,b)$. In particular, we have

$$
\begin{aligned}
&\lim_j \min\Big\{F_j(u) - \int_a^b vu\,dt :\ u(a)=u_0(a),\ u(b)=u_0(b)\Big\} \\
&= \min\Big\{F(u) - \int_a^b vu\,dt :\ u(a)=u_0(a),\ u(b)=u_0(b)\Big\}
\end{aligned}
$$

for all $v \in L^1(a,b)$ by Remark 2.26 and Theorem 1.21.

2.8 Some examples with degenerate growth conditions

We give two examples here below showing how degeneracy of polynomial growth conditions may lead to functionals of different type.

2.8.1 *Degeneracy of lower bounds: discontinuities*

We have noticed that the class of integral functionals of p-growth is closed with respect to Γ-convergence. If this type of growth condition is violated we may end up with an energy of a different form. We here include a simple example where we do not have a uniform growth condition from below.

Example 2.39 Consider the functionals

$$F_j(u) = \int_{-1}^1 a_j(t)|u'|^2\,dt$$

defined on $W^{1,2}(-1,1)$ and extended as $+\infty$ otherwise, where

$$a_j(t) = \begin{cases} 1 & \text{if } |t| > \frac{1}{2j} \\ \frac{1}{j} & \text{otherwise.}\end{cases}$$

We want to compute the Γ-limit with respect to the L^2-convergence. It is clear that if $u_j \to u$ in $L^2(-1,1)$ and $\sup_j F_j(u_j) < +\infty$ then this sequence is weakly compact in $W^{1,2}((-1,-1/k)\cup(1/k,1))$ for all $k > 1$, and

$$\sup_k \|u'\|_{L^2((-1,-1/k)\cup(1/k,1))} \leq \sup_j F_j(u_j) \leq C$$

independently of k, so that indeed $u \in W^{1,2}((-1,1)\setminus\{0\})$. In particular the values $u(0\pm)$ are well defined and we have $\lim_j u_j(\pm 1/2j) = \lim_j u(\pm 1/2j) = u(0\pm)$. For each fixed k we have the estimate

$$\begin{aligned}\liminf_j F_j(u_j) &\geq \liminf_j \int_{-1}^{-1/k} |u_j'|^2\,dt \\ &\quad + \liminf_j \int_{1/k}^{1} |u_j'|^2\,dt + \liminf_j \frac{1}{j}\int_{-1/2j}^{1/2j} |u_j'|^2\,dt \\ &\geq \int_{-1}^{-1/k} |u'|^2\,dt + \int_{1/k}^{1} |u|^2\,dt + \lim_j |u_j(1/2j) - u_j(-1/2j)|^2,\end{aligned}$$

where we have used the simple Jensen's inequality

$$\int_{-1/2j}^{1/2j} |u_j'|^2\,dt \geq \frac{1}{j}\Big(j\int_{-1/2j}^{1/2j} u_j'\,dt\Big)^2 = j|u_j(1/2j) - u_j(-1/2j)|^2.$$

By taking the supremum over all k we finally get that

$$\Gamma\text{-}\liminf_j F_j(u) \geq \int_{(-1,1)\setminus\{0\}} |u'|^2\,dt + |u(0+) - u(0-)|^2$$

if $u \in W^{1,2}((-1,1)\setminus\{0\})$.

Conversely, if $u \in W^{1,2}((-1,1)\setminus\{0\})$ a recovery sequence is readily constructed by taking

$$u_j(t) = \begin{cases} u\Big(t - \frac{t}{2j|t|}\Big) & \text{if } |t| > 1/2j \\ j(u(0+) - u(0-))t + \dfrac{u(0+) + u(0-)}{2} & \text{if } |t| \leq 1/2j \end{cases}$$

to show that $\Gamma\text{-}\limsup_j F_j(u) \leq \int_{(-1,1)\setminus\{0\}} |u'|^2\,dt + |u(0+) - u(0-)|^2$.

Note that not only the Γ-limit's domain is not the same as that of the energies F_j, but the form of the Γ-limit is not an integral (with respect to the Lebesgue measure at least).

2.8.2 *Degeneracy of upper bounds: functionals of the sup norm*

As an example of problems where a polynomial growth condition 'from above' is not uniformly satisfied, we study the behaviour of the minimum problems

$$\min\Big\{\frac{1}{p}\int_0^1 |a(t)u'(t)|^p\,dt : u \in W^{1,p}(0,1),\ u(0) = u_0,\ u(1) = u_1\Big\} \tag{2.29}$$

as $p \to +\infty$. We suppose that a is a bounded measurable function with $\inf a > 0$. To deduce the convergence of (2.29) from a Γ-convergence result, we consider the functionals

$$F_p(u) = \frac{1}{p}\int_0^1 |au'|^p\,dt \tag{2.30}$$

defined on $W^{1,1}(0,1)$ and study their Γ-limit as $p \to +\infty$. In fact, these problems are equicoercive, since, by Hölder's inequality for every fixed $q \geq 1$ we get

$$\int_0^1 |au'|^q\,dt \leq (pF_p(u))^{q/p}, \tag{2.31}$$

so that we deduce that if $\sup\{F_p(u_p) : p \geq 1\} < +\infty$ and the boundary conditions are satisfied, then (u_p) is bounded in $W^{1,q}(0,1)$ and hence precompact in $L^1(0,1)$. As a consequence, the convergence of the minimum problems can be easily deduced from the Γ-convergence of F_p.

Proposition 2.40 *The functionals F_p Γ-converge with respect to the L^1 convergence as $p \to +\infty$ to the functional F_∞ defined on $W^{1,1}(0,1)$ by*

$$F_\infty(u) = \begin{cases} 0 & \text{if } |au'| \leq 1 \text{ a.e. in } (0,1) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.32}$$

Proof The limsup inequality is trivially satisfied choosing the constant recovery sequence. To prove the liminf inequality, upon extraction of subsequences, it is not restrictive to take $u_p \to u$ such that there exists $\lim_{p\to+\infty} F_p(u_p) < +\infty$. Fixed $q > 1$, by (2.31) we may suppose that $u_p \rightharpoonup u$ in $W^{1,q}(0,1)$ and hence that $au_p' \rightharpoonup au'$ weakly in $L^q(0,1)$. By the lower-semicontinuity of the L^q-norm for every fixed $\lambda > 1$ using (2.31) we then have

$$|\{|au'| > \lambda\}|\lambda^q \leq \int_0^1 |au'|^q\,dt \leq \liminf_{p\to+\infty} \int_0^1 |au_p'|^q\,dt \leq 1. \tag{2.33}$$

Letting $q \to +\infty$ we obtain $|\{|au'| > \lambda\}| = 0$ so that $|au'| \leq 1$ a.e. This implies the liminf inequality. □

The convergence of minimum problems (2.29) can be improved by considering the equivalent minimum problems

$$\min\Big\{\Big(\frac{1}{p}\int_0^1 |a(t)u'(t)|^p\,dt\Big)^{1/p} : u(0) = u_0,\ u(1) = u_1\Big\}, \tag{2.34}$$

which is deduced as above from the Γ-convergence of the functionals defined in $W^{1,1}(a,b)$ by

$$G_p(u) = \Big(\frac{1}{p}\int_0^1 |au'|^p\,dt\Big)^{1/p}. \tag{2.35}$$

Proposition 2.41 *The functionals G_p Γ-converge with respect to the L^1 convergence as $p \to +\infty$ to the functional G_∞ defined on $W^{1,1}(0,1)$ by*

$$G(u) = \|au'\|_\infty. \tag{2.36}$$

Proof Since $p^{1/p} \to 1$ the Γ-convergence of G_p is equivalent to that of $p^{1/p}G_p$. Since this sequence converges pointwise increasingly to G_∞ the Γ-convergence is assured by (1.36). □

As a consequence we see that problems (2.34) converge as $p \to +\infty$ to

$$\min\{\|au'\|_\infty : u(0) = u_0,\ u(1) = u_1\}. \tag{2.37}$$

2.9 Exercises

2.1 Let $W : \mathbf{R} \to [0, +\infty)$ be a non-strictly convex lower-semicontinuous function such that $\displaystyle\lim_{|z|\to+\infty} \frac{W(z)}{|z|} = +\infty$. Then there exist $u_a, u_b \in \mathbf{R}$ such that the problem

$$\min\Big\{\int_{(a,b)} W(u')\,dt : u(a) = u_a,\ u(b) = u_b\Big\}$$

admits infinitely many solutions in $W^{1,\infty}(a,b)$.

Hint: there exist $z_1 < z_2$ such that $W(z) \geq W(z_1) + (W(z_2) - W(z_1))(z - z_1)/(z_2 - z_1)$ for all z. Choose u_a and u_b such that $(b-a)z_1 < u_b - u_a < (b-a)z_2$ and look for solutions whose gradients take only the values z_1 and z_2.

2.2 Let $F_j(u) = \int_a^b a_j(t)|u'|^p\,dt$ with $0 < c_1 \leq a_j \leq c_2$. Characterize the Γ-convergence of F_j in $L^p(a,b)$ (use Theorem 2.35).

2.3 Prove that the lower-semicontinuous envelope with respect to the $L^2(-1,1)$-convergence of the functional defined by $F(u) = \int_{-1}^1 t^2|u'|^2\,dt$ on $W^{1,2}(-1,1)$ is finite on $W^{1,2}((-1,1)\setminus\{0\})$.

Hint: if $u(t) = t/|t|$ then take $u_j(t) = (-1 \vee (jt)) \wedge 1$ and note that $u_j \to u$ and $\lim_j F(u_j) = 0$. Generalize the construction to arbitrary u.

2.4 Prove that the lower-semicontinuous envelope with respect to the $L^1(-1,1)$-convergence of the functional defined by $F(u) = \int_{-1}^1 \sqrt{|t|}\,|u'|^2\,dt$ on $W^{1,2}(-1,1)$ is finite only on $W^{1,2}(-1,1)$.

Hint: from the inequality

$$|v(t) - v(s)| = \Big|\int_s^t v'\,d\tau\Big| \leq \Big(\int_s^t \frac{1}{\sqrt{|\tau|}}d\tau\Big)^{1/2}\Big(\int_s^t \sqrt{|\tau|}\,|v'|^2 d\tau\Big)^{1/2}$$

deduce that if $\sup_j F(u_j) < +\infty$ then (u_j) is equicontinuous. From this deduce that if also $u_j \to u$ in $L^2(-1,1)$ then the convergence is uniform and in $W^{1,2}_{\rm loc}((-1,1)\setminus\{0\})$.

2.5 Prove that if f is convex then $F(u) = \int_a^b f(u)\,dt$ is lower semicontinuous with respect to the convergence of L^1 functions in the sense of distributions; i.e. $F(u) \leq \liminf_j F(u_j)$ if $u_j, u \in L^1(a,b)$ and $\lim_j \int_a^b \varphi(u - u_j)\,dt = 0$ for all $\varphi \in C_0^\infty(a,b)$.

Hint: fix $\varphi \in C_0^\infty(a,b)$ and repeat the argument of Proposition 2.13 for the functional $u \mapsto \int_a^b \varphi f(u)\,dt$. Let $\varphi \to 1$.

2.6 Compute the Γ-limit of $F_j(u) = \int_0^1 a_j(t)|u'|^2\,dt$ with a_j as in Section 2.8.1 with and without boundary conditions. Deduce that the boundary conditions in 0 are not 'compatible' with this Γ-limit.

Comments on Chapter 2

A suggested reading for weak convergence methods are the lecture notes by Müller (1999) and the book by Evans (1990).

A proof of the de la Vallée Poussin criterion can be found in Dellacherie and Meyer (1975). A different criterion for the weak compactness in $L^1(\Omega)$ of a sequence (u_j) is its *equi-integrability*: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all j and measurable E with $|E| \leq \delta$ we have $\int_\Omega |u_j|\,dx \leq \varepsilon$ (*Dunford Pettis criterion*).

The space $L^1(\Omega)$ is often viewed as a subspace of the space $\mathcal{M}(\Omega)$ of Radon measures on Ω; this is naturally driven by coerciveness arguments since bounded sequences in $L^1(\Omega)$ are precompact in the weak* topology of $\mathcal{M}(\Omega)$. A variational theory for integral functionals on $\mathcal{M}(\Omega)$ has been developed by Bouchitté and Buttazzo (1993); their Γ-convergence is characterized in Amar and Braides (1998). There are many proofs of the lower semicontinuity of convex integrals. We have presented one that does not rely on Hahn Banach's Theorem (and in particular it does not involve the axiom of choice!).

The argument leading to the sufficiency of convexity for lower semicontinuity in Sobolev spaces can be repeated in any dimension and for vector-valued u. The necessity argument fails if u is vector valued (see Chapter 12). We refer to Buttazzo *et al.* (1998) for a treatment of one-dimensional variational problems and of Euler Lagrange equations in particular. The characterization Theorem 2.35 is due to Marcellini and Sbordone (1977). Sequences of non-equiuniformly elliptic problems may give as a Γ-limit a functional defined on measures; this connection is explored in Buttazzo and Freddi (1991). The result in Section 2.8.2 is taken from Garroni *et al.* (2001), where also applications to *dielectric breakdown* are given. This result illustrates from a variational standpoint the derivation of viscosity solutions for the *infinity laplacian*. For more information see, for example, Crandall *et al.* (2001) and the survey by Barron (1999) and the references cited there. Failure of the upper growth condition for sequences of quadratic functionals may result in non-local Γ-limits that can be expressed as *Dirichlet forms* (see Mosco (1994)).

There are countless interesting applications of Γ-convergence in the framework of integral functionals that range from singular perturbation problems to Control Theory, from reinforcement problems in Continuum Mechanics to Optimal Design, etc.; we refer to the guide to the bibliography of Dal Maso (1993) for a partial list.

3

SOME HOMOGENIZATION PROBLEMS

The terminology 'homogenization' stands for a great variety of asymptotic problems where the solutions exhibit a highly-oscillating behaviour, most of which lying outside the variational framework. In the case of minimum problems of integral type, computing the Γ-limit allows to obtain the 'effective' behaviour of these problems by means of 'averaged' quantities. The possibility of this description often relies in finding suitable *homogenization formulas* to describe the limit integrand in terms of an 'optimization process' among a class of perturbations of a linear function.

3.1 A direct approach

Some types of homogenization problems can be translated in the asymptotic study of one-dimensional energies of the type

$$F_\varepsilon(u) = \int_a^b f\Big(\frac{t}{\varepsilon}, u'(t)\Big)\,dt. \tag{3.1}$$

This is, for example, the case when modelling the behaviour of a medium with rapidly-varying conductivity, in which case u is interpreted as the electric potential. In this case homogenization results can be directly obtained from the general characterization result Theorem 2.35. At other times we are interested in oscillations in the target space; that is, in functionals of the form

$$F_\varepsilon(u) = \int_a^b f\Big(\frac{u(t)}{\varepsilon}, u'(t)\Big)\,dt. \tag{3.2}$$

If only averaged (i.e. weakly converging) quantities are of interest, we may substitute the functional F_ε by its Γ-limit, which (if suitable growth conditions are satisfied) we expect to be a functional of the form $\int_a^b \psi(u')\,dt$, with the form of ψ independent of the interval (a,b).

We prefer first to show in this section a 'direct' approach that can be generalized to higher dimension both in the target and in the reference configuration. In order to highlight the differences with the 'indirect' approach by convex analysis, we treat the case of general functionals of the form with $u : (a,b) \to \mathbf{R}^N$. Note that the framework of the problem changes only in that we consider N copies of a Sobolev space $W^{1,p}(I)$.

In order to characterize the function ψ, the simple idea is to use its convexity (which holds by the lower semicontinuity of the Γ-limit), to express it as a minimum value (by Jensen's inequality):

$$\psi(z) = \min\Big\{\int_0^1 \psi(u'(y)+z)\,dy : \ u \in (W_0^{1,p}(0,1))^N\Big\}. \tag{3.4}$$

On the other hand, by the properties of Γ-convergence we argue that this minimum is the limit of the minima of the approximating functionals, thus obtaining

$$\begin{aligned}\psi(z) &= \lim_{\varepsilon\to 0+} \min\Big\{\int_0^1 f\Big(\frac{s}{\varepsilon}, \frac{u(s)+zs}{\varepsilon}, u'(s)+z\Big)\,ds : \ u \in (W_0^{1,p}(0,1))^N\Big\}\\ &= \lim_{T\to+\infty} \min\Big\{\frac{1}{T}\int_0^T f(t, v+zt, v'(t)+z)\,dt : \ v \in (W_0^{1,p}(0,T))^N\Big\}\end{aligned} \tag{3.5}$$

(the last equality just follows from the change of variables $s = \varepsilon t$, with $T = 1/\varepsilon$, and correspondingly choosing $v(t) = u(\varepsilon t)/\varepsilon$). The right-hand side of this last equality is the *asymptotic homogenization formula* for ψ. Of course to complete this reasoning it still remains to check that the Γ-limit does exist and that this formula makes sense. This argument can be refined and made into a proof, and more handy formulas obtained.

Theorem 3.1 (homogenization). *Let $1 < p < \infty$ and let $f : \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N \to [0,+\infty)$ be a Borel function satisfying the growth condition*

$$c_1|z|^p - c_2 \le f(t,s,z) \le c_3(1+|z|^p)$$

for all t, s, z and some $c_i > 0$, and such that

(i) *$f(\cdot, s, z)$ is 1-periodic for all $s, z \in \mathbf{R}^N$;*

(ii) *$f(t,\cdot,z)$ is 1-periodic for all $t \in \mathbf{R}$, $z \in \mathbf{R}^N$;*

For all $\varepsilon > 0$, let $F_\varepsilon : (W^{1,p}(a,b))^N \to [0,+\infty)$ be defined by (3.3). Then, there exists a convex $f_{\rm hom} : \mathbf{R}^N \to \mathbf{R}$ such that, if $F_{\rm hom} : (W^{1,p}(a,b))^N \to [0,+\infty)$ is defined by setting

$$F_{\rm hom}(u) = \int_a^b f_{\rm hom}(u')\,dt, \tag{3.6}$$

then we have $F_{\rm hom} = \Gamma(L^p(a,b))\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon$, and $f_{\rm hom}$ satisfies

$$f_{\rm hom}(z) = \lim_{T\to+\infty} \inf\Big\{\frac{1}{T}\int_0^T f(y, u(y)+zy, u'(y)+z)\,dy : \ u \in (W_0^{1,p}(0,T))^N\Big\} \tag{3.7}$$

for all $z \in \mathbf{R}^N$.

Proof By the argument in (3.4) and (3.5) formula (3.7) follows from the convergence of minima once we prove the Γ-convergence for all (a,b) (and for $(0,1)$ in particular). In order to prove this fact, we provide an alternative formula for $f_{\rm hom}$ from which the Γ-convergence can be easily obtained. We set

$$g(z)=\liminf_{T\to+\infty}\inf\Big\{\frac{1}{T}\int_x^{x+T}f(y,u(y)+zy,u'(y)+z)\,dy:\\ u\in(W^{1,p}(x,x+T))^N,\ u(x)=u(x+T),x\in\mathbf{R}\Big\}\qquad(3.8)$$

for all $z\in\mathbf{R}^N$ and define $f_{\rm hom}=g^{**}$. Formula (3.8) is suggested by localizing the lower bound for the Γ-liminf, by the argument in (3.4) and (3.5), now assuming that the limit in T may not exist and the outcome may be dependent of the interval in which the argument is applied.

By the growth conditions from below we can consider (u_ε) converging to u weakly in $(W^{1,p}(a,b))^N$ (and in particular, uniformly). We fix $n\geq 1$ and consider the points

$$t_j^n=a+\frac{b-a}{n}j\qquad j=0,\dots,n.$$

Proceeding as in Proposition 2.37 we may suppose that $u_\varepsilon(t_j^n)=u(t_j^n)$ for all j. Having set $v_\varepsilon(s)=u_\varepsilon(\varepsilon s)/\varepsilon$, we then compute

$$\begin{aligned}F_\varepsilon(u_\varepsilon)&=\sum_{j=1}^n\int_{t_{j-1}^n}^{t_j^n}f\Big(\frac{t}{\varepsilon},\frac{u_\varepsilon(t)}{\varepsilon},u_\varepsilon'(t)\Big)\,dt=\sum_{j=1}^n\varepsilon\int_{t_{j-1}^n/\varepsilon}^{t_j^n/\varepsilon}f\Big(s,\frac{u_\varepsilon(\varepsilon s)}{\varepsilon},u_\varepsilon'(\varepsilon s)\Big)\,ds\\&=\sum_{j=1}^n\varepsilon\int_{t_{j-1}^n/\varepsilon}^{t_j^n/\varepsilon}f(s,v_\varepsilon(s),v_\varepsilon'(s))\,ds\\&\geq\sum_{j=1}^n(t_j^n-t_{j-1}^n)\inf\Big\{\frac{\varepsilon}{t_j^n-t_{j-1}^n}\int_{t_{j-1}^n/\varepsilon}^{t_j^n/\varepsilon}f(y,v(y)+z_j^ny,v'(y)+z_j^n)\,dy:\\&\qquad v\in\Big(W_0^{1,p}\Big(\frac{t_{j-1}^n}{\varepsilon},\frac{t_j^n}{\varepsilon}\Big)\Big)^N,v\Big(\frac{t_{j-1}^n}{\varepsilon}\Big)=v\Big(\frac{t_j^n}{\varepsilon}\Big)\Big\},\end{aligned}$$

where

$$z_j^n:=\frac{u_\varepsilon(t_j^n)-u_\varepsilon(t_{j-1}^n)}{t_j^n-t_{j-1}^n}=\frac{u(t_j^n)-u(t_{j-1}^n)}{t_j^n-t_{j-1}^n}.$$

Let $v_n\in(W^{1,p}(a,b))^N$ be the piecewise-affine interpolation of u related to the points (t_j^n), defined by $v_n(a)=u(a)$ and $v_n'=z_j^n$ on (t_{j-1}^n,t_j^n). The inequalities above show that

$$\liminf_{\varepsilon\to0+}F_\varepsilon(u_\varepsilon)\geq\sum_{j=1}^n(t_j^n-t_{j-1}^n)f_{\rm hom}(z_j^n)=\int_a^b f_{\rm hom}(v_n')\,dt=F_{\rm hom}(v_n).$$

Since $v_n\rightharpoonup u$ and $f_{\rm hom}$ is convex we have $\liminf_n F_{\rm hom}(v_n)\geq F_{\rm hom}(u)$, and the liminf inequality is achieved.

By Remark 2.32(ii), it suffices to show that $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u) \le \int_a^b g(u')\,dt$ (upon taking the lower semicontinuous envelope of both sides). As usual, it is enough to prove the limsup inequality when $u(t) = zt$ since we easily construct recovery sequences for piecewise-affine target functions by following that proof on each interval where u is affine, and eventually for all $W^{1,p}$ functions proceeding by density. By density it suffices to deal with $z \in \mathbf{Q}$ and take $u(t) = zt$. We fix $\eta > 0$, $T > \frac{1}{\eta}$, $x \in \mathbf{R}$ and $v \in (W^{1,p}(x, x+T))^N$ such that $v(x) = v(x+T)$ and

$$\int_x^{x+T} f(t, v(t) + zt, v'(t) + z)\,dy \le T\,g(z) + \eta.$$

Then, we choose $m_T \in \mathbf{N}$ with $m_T \le c(z)$ such that $k = [T + m_T]$ (the integer part of $T + m_T$) satisfies $kz \in \mathbf{N}$, and extend v to the k-periodic function such that $v(t) = v(x+T)$ on $(x+T, x+k)$. Set $u_\varepsilon(t) = zt + \varepsilon v(t/\varepsilon)$, which converges to u. We may compute, using Example 2.4,

$$\begin{aligned}\limsup_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon) &= (b-a)\frac{1}{k}\int_x^{x+k} f(s, v(s) + zs, v'(s) + z)\,ds \\ &\le (b-a)g(z) + 2(b-a)\eta c(1 + |z|^p),\end{aligned}$$

as desired. □

3.2 Different homogenization formulas

The proof in the previous section gives a 'homogenization formula' for the limit that can be improved and simplified if the integrands fall within the class dealt with in Theorem 2.35. We here give some equivalent formulas; note that some of them, namely (3.9) and (3.12) have no counterpart in the higher-dimensional case. The proof of the following theorem also includes an alternative existence argument for the Γ-limit based on the compactness Theorem 2.35.

Theorem 3.2 (convex homogenization). *Let $f : \mathbf{R} \times \mathbf{R} \to [0, +\infty)$ be a Borel function satisfying the growth condition* (2.25) *such that*

(i) *$f(\cdot, z)$ is 1-periodic for all $z \in \mathbf{R}$;*

(ii) *$f(t, \cdot)$ is convex for all $t \in \mathbf{R}$,*

and let F_ε be defined by (3.1). *Then there exists a convex $f_{\rm hom} : \mathbf{R} \to \mathbf{R}$ such that, if $F_{\rm hom} : W^{1,p}(a,b) \to [0,+\infty)$ is defined as in* (3.6) *then we have $F_{\rm hom} = \Gamma(L^p(a,b))\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon$, and $f_{\rm hom}$ satisfies each of the following formulas:*

$$f_{\rm hom}(z) = \min\Big\{\int_0^1 f(t, u'(t) + z)\,dt : \ u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ 1-periodic}\Big\} \tag{3.9}$$

$$= \inf_{k\in\mathbf{N}} \inf\Big\{\frac{1}{k}\int_0^k f(t, u'(t) + z)\,dt : u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ } k\text{-periodic}\Big\} \tag{3.10}$$

$$= \lim_{T\to+\infty} \inf\Big\{\frac{1}{T}\int_0^T f(t, u'(t) + z)\,dt : \ u \in W^{1,p}_0(0,T)\Big\} \tag{3.11}$$

$$= \sup\Big\{z^* z - \int_0^1 f^*(t, z^*)\,dt : z^* \in \mathbf{R}\Big\} \tag{3.12}$$

for all $z \in \mathbf{R}$.

Compared with Theorem 3.1, formula (3.10) considers as test function all periodic perturbations with integer period, while formula (3.9) further simplifies by taking into account only 1-periodic perturbations.

Proof With fixed (ε_j) converging to 0, we can apply Theorem 2.35 to (f_j) defined by $f_j(t,z) = f(t/\varepsilon_j, z)$, noting that the weak* limit of $f_j^*(\cdot, z)$ is simply $\int_0^1 f^*(s,z)\,ds$. By the arbitrariness of (ε_j) the Γ-convergence of the whole family (F_ε) is proven, as well as formula (3.12).

In order to prove the other formulas, note that, by the convexity of $f_{\rm hom}$ and Jensen's inequality, we have

$$\begin{aligned} f_{\rm hom}(z) &= \min\Big\{\int_0^1 f_{\rm hom}(z+u')\,ds :\ u \in W_0^{1,p}(0,1)\Big\} \\ &= \min\Big\{\int_0^1 f_{\rm hom}(z+u')\,ds :\ u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ 1-periodic}\Big\}. \end{aligned} \tag{3.13}$$

By the Γ-convergence of F_ε to $F_{\rm hom}$ and by the equi-coerciveness of (F_ε) we have the convergence of minimum problems

$$\begin{aligned} f_{\rm hom}(z) &= \min\Big\{\int_0^1 f_{\rm hom}(z+u')\,ds :\ u \in W_0^{1,p}(0,1)\Big\} \\ &= \lim_{\varepsilon\to 0+} \min\Big\{\int_0^1 f\Big(\frac{t}{\varepsilon}, z+u'\Big)\,dt :\ u \in W_0^{1,p}(0,1)\Big\}, \end{aligned} \tag{3.14}$$

which proves (3.11) upon setting $T = 1/\varepsilon$ and changing variables, using the scaling $v(s) = u(\varepsilon s)/\varepsilon$, and the convergence of minimum problems

$$f_{\rm hom}(z) = \min\Big\{\int_0^1 f_{\rm hom}(z+u')\,ds :\ u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ 1-periodic}\Big\} \tag{3.15}$$

$$= \lim_{k\to+\infty} \min\Big\{\int_0^1 f(kt, z+u')\,dt :\ u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ 1-periodic}\Big\},$$

which proves (3.10) upon changing variables and noticing that the limit is actually an infimum.

Finally, it remains to prove that for all $k \in \mathbf{N}$

$$\begin{aligned} &\min\Big\{\int_0^1 f(s, z+u')\,ds :\ u \in W^{1,p}_{\rm loc}(\mathbf{R}) \text{ 1-periodic}\Big\} \\ &\le \frac{1}{k} \min\Big\{\int_0^k f(t, z+u')\,dt :\ u \in W^{1,p}_{\rm loc}(\mathbf{R})\ k\text{-periodic}\Big\}, \end{aligned} \tag{3.16}$$

as the converse inequality is trivial. With fixed $k \in \mathbf{N}$ and $u \in W^{1,p}_{\rm loc}(\mathbf{R})$ k-periodic, let

$$u_k(t) = \frac{1}{k}\sum_{i=1}^{k} u(t-i).$$

The function u_k is 1-periodic, and, by the convexity of $f(t,\cdot)$ we have

$$\begin{aligned}\int_0^1 f(s, z+u_k')\,ds &= \frac{1}{k}\int_0^k f(s,z+u_k')\,ds = \frac{1}{k}\int_0^k f\Big(s,\frac{1}{k}\sum_{i=1}^{k}(z+u'(s-i))\Big)\,ds\\ &\le \frac{1}{k^2}\sum_{i=1}^{k}\int_0^k f(s,z+u'(s-i))\,ds\\ &= \frac{1}{k^2}\sum_{i=1}^{k}\int_0^k f(s,z+u'(s))\,ds = \frac{1}{k}\int_0^k f(s,z+u')\,ds, \quad (3.17)\end{aligned}$$

so that (3.16) is proved by the arbitrariness of u. □

3.3 Limits of oscillating Riemannian metrics

Riemannian metrics on (a subset of) $\mathbf{R}^N$ are characterized by their *energy functional*: the energy of a curve $u : (a,b) \to \mathbf{R}^N$ given by

$$E(u) = \int_a^b \sum_{i,j=1}^{N} a_{ij}(u(t))u_i' u_j'\,dt,$$

where (a_{ij}) is a $N \times N$ matrix of bounded measurable functions such that

$$\alpha|z|^2 \le \sum_{i,j=1}^{N} a_{ij}(s) z_i z_j \le \beta|z|^2 \tag{3.18}$$

for all $z \in \mathbf{R}^N$ and $s \in \mathbf{R}^N$, with $\alpha, \beta > 0$.

The description of the behaviour of Riemannian metrics in a finely-periodic environment can be stated as the computation of a Γ-limit by considering 1-periodic a_{ij} and scaling the energy functional, obtaining

$$E_\varepsilon(u) = \int_a^b \sum_{i,j=1}^{N} a_{ij}\Big(\frac{u}{\varepsilon}\Big)u_i' u_j'\,dt \tag{3.19}$$

defined on curves $u : [a,b] \to \mathbf{R}^N$. In this way we are led to a homogenization problem, where the oscillation is not in the reference configuration, but on the target space. Nevertheless, the form of the limit energy, which is to be considered as a *Finsler metric* (i.e., not necessarily corresponding to a quadratic form), is still an integral functional by Theorem 3.1; that we can restate in this particular situation as: if $N \in \mathbf{N}$ and $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty)$ is a Borel function, 1-periodic in the first component and satisfying a 2-growth condition, then there exists a convex function $\varphi : \mathbf{R}^N \to [0, +\infty)$ such that for every bounded open subset I of $\mathbf{R}$ and $u = (u_1, \ldots, u_N) \in (W^{1,2}(I))^N$ the limit

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \to 0} \int_I \sum_{i,j=1}^N f\Big(\frac{u(t)}{\varepsilon}, u'(t)\Big)\, dt = \int_I \varphi(u'(t))\, dt \tag{3.20}$$

exists, and the function φ satisfies

$$\varphi(z) = \lim_{T \to +\infty} \inf\Big\{\frac{1}{T} \int_0^t f(u(t) + zt, u'(t) + z)\, dt : u \in (W_0^{1,2}(0,T))^N\Big\} \tag{3.21}$$

for all $z \in \mathbf{R}^N$. The main part of this section is to show with an example that the function φ may indeed not be a quadratic function.

Example 3.3 We take $N = 2$ and $f(s, z) = a(s)|z|^2$, where $a : \mathbf{R}^2 \to \{\alpha, \beta\}$ is the 1-periodic function defined on $[0,1]^2$ by

$$a(s) = \begin{cases} \beta & \text{if } s \in (0,1)^2 \\ \alpha & \text{if } s \in \partial(0,1)^2, \end{cases} \tag{3.22}$$

where $\alpha, \beta > 0$ with

$$2\alpha \leq \beta; \tag{3.23}$$

that is, $a(s_1, s_2) = \alpha$ if $s_1 \in \mathbf{Z}$ or $s_2 \in \mathbf{Z}$, $a(s) = \beta$ otherwise, so that we may think of it as representing an energetically-convenient network structure in an isotropic medium. The condition $\beta \geq 2\alpha$ assures that the 'minimal paths' according to the energy E_ε (with $a_{ij}(s) = a(s)\delta_{ij}$) will tend to lie in the region where the coefficient a equals α, and the limit Finsler metric will be given by $\varphi(z_1, z_2) = \alpha(|z_1| + |z_2|)^2$. Note that in this case this (anisotropic) Finsler metric is obtained as the limit of *isotropic* Riemannian metrics.

In order to check the form for φ, we make use of formula (3.21), which can be rewritten as

$$\varphi(z) = \lim_{T \to +\infty} \inf\Big\{\frac{1}{T} \int_0^t a(u(t) + zt)|u'(t) + z|^2\, dt : u \in (W_0^{1,2}(0,T))^2\Big\}. \tag{3.24}$$

With $z \in \mathbf{R}^N$, $t > 0$ and $u \in (W_0^{1,2}(0,T))^N$ fixed, we set

$$I_1 = \{t \in (0,T) : \ u_1(t) + z_1 t \in \mathbf{Z}\},$$
$$I_2 = \{t \in (0,T) : \ u_2(t) + z_2 t \in \mathbf{Z}\} \setminus I_1, \qquad I_3 = (0,T) \setminus (I_1 \cup I_2),$$
$$s_1 = \frac{|I_1|}{T}, \quad s_2 = \frac{|I_2|}{T}, \quad s_3 = \frac{|I_3|}{T}.$$

We define $z', z'' \in \mathbf{R}^2$ as follows (note that $u' + z$ is parallel to $(1,0)$ on I_1 and to $(0,1)$ on I_2):

$$T(0, z_1') = \int_{I_1} (u' + z)\, dt, \qquad T(z_2', 0) = \int_{I_2} (u' + z)\, dt, \qquad Tz'' = \int_{I_3} (u' + z)\, dt.$$

Since $u \in (W_0^{1,2}(0,T))^N$ we have

$$z' + z'' = \frac{1}{T} \int_0^T (u' + z)\, dt = z.$$

Using Jensen's inequality, (3.23) and the convexity inequality

$$\frac{A^2}{x} + \frac{B^2}{y} \geq \frac{(A+B)^2}{x+y}, \tag{3.25}$$

valid if $x, y > 0$, we obtain

$$\begin{aligned}
&\frac{1}{T} \int_0^T a(u(t) + zt) |u'(t) + z|^2\, dt \\
&= \frac{\alpha}{T} \int_{I_1} |u'(t) + z|^2\, dt + \frac{\alpha}{T} \int_{I_2} |u'(t) + z|^2\, dt + \frac{\beta}{T} \int_{I_3} |u'(t) + z|^2\, dt \\
&\geq \alpha \Big(\frac{|z_1'|^2}{s_1} + \frac{|z_2'|^2}{s_2} \Big) + \beta \frac{|z''|^2}{s_3} \geq \alpha \frac{(|z_1'| + |z_2'|)^2}{s_1 + s_2} + \beta \frac{|z''|^2}{s_3} \\
&\geq \alpha \frac{(|z_1'| + |z_2'|)^2}{s_1 + s_2} + \frac{\beta}{2} \frac{(|z_1''| + |z_2''|)^2}{s_3} \geq \alpha \frac{(|z_1'| + |z_2'|)^2}{s_1 + s_2} + \alpha \frac{(|z_1''| + |z_2''|)^2}{s_3} \\
&\geq \alpha (|z_1'| + |z_2'| + |z_1''| + |z_2''|)^2 \geq \alpha (|z_1| + |z_2|)^2.
\end{aligned}$$

In the last line we have used the fact that $s_1 + s_2 + s_3 = 1$. In this way we have proved a lower bound inequality for φ.

To give an upper bound for φ, since it is continuous, being convex, it is enough to check it for $z \in \mathbf{Q}^2$. Moreover, since φ is homogeneous of degree two

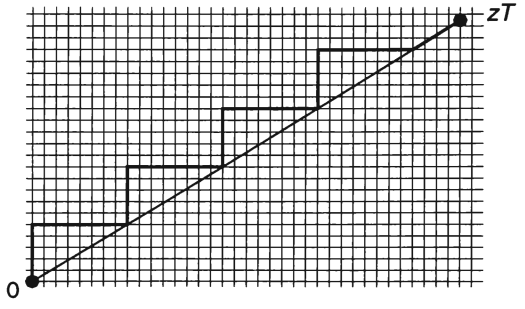

FIG. 3.1. Constructing optimal curves

and symmetric with respect to the axes we can consider just $z = (n_1, n_2) \in \mathbf{N}^2$. Let $T > 0$ and let $v \in (W^{1,2}_{\rm loc}(\mathbf{R}))^2$ be defined by

$$v(t) = \begin{cases} zk + ((n_1+n_2)(t-k), 0) & \text{if } t \in \Big[k, k+\frac{n_1}{n_1+n_2}\Big),\ k \in \mathbf{Z} \\ z(k+1) + (0, (n_1+n_2)(t-k-1)) & \text{if } t \in \Big[k+\frac{n_1}{n_1+n_2}, k+1\Big),\ k \in \mathbf{Z}. \end{cases}$$

Note that for all t we have $v_1(t) \in \mathbf{Z}$ or $v_2(t) \in \mathbf{Z}$ and that $v(k) = zk$ if $k \in \mathbf{Z}$, so that we can use as test function in (3.24) the function (pictured in Fig. 3.1)

$$u(t) = \begin{cases} v(t) - zt & \text{if } t \in (0, [T]) \\ (0,0) & \text{if } t \in ([T], T), \end{cases}$$

and obtain

$$\frac{1}{T}\int_0^T a(u(t)+zt)|u'(t)+z|^2\,dt \le \frac{1}{T}\Big(\int_0^{[T]} \alpha|n_1+n_2|^2\,dt + \int_{[T]}^T \beta|z|^2\,dt\Big)$$

Letting $T \to +\infty$ we get $\varphi(z) \le \alpha|n_1+n_2|^2$, as desired.

Example 3.4 Another example can be constructed by taking $f(s,z) = a(s)|z|^2$ and $a : \mathbf{R}^2 \to \{\alpha, \beta\}$ is the 1-periodic function defined on $[0,1)^2$ by

$$a(s) = \begin{cases} \beta & \text{if } s \in (0,\frac12)\times(\frac12,1) \text{ or } s \in (\frac12,1)\times(0,\frac12) \\ \alpha & \text{otherwise,} \end{cases} \tag{3.26}$$

where $4\alpha < \beta$. The matrix (a_{ij}) is related to a *chessboard-type* structure on $\mathbf{R}^2$ (see Fig. 0.7).

Elementary geometric reasonings show that (for $z_1, z_2 \in \mathbf{N}$, $z_1 > z_2$) a minimal (piecewise-affine) curve is given by the segment joining $(0,0)$ and (z_2, z_2) and the segment joining (z_2, z_2) and (z_1, z_2). In the general case symmetry arguments show that

$$\varphi(z) = \alpha\big((\sqrt{2}-1)\min\{|z_1|,|z_2|\} + \max\{|z_1|,|z_2|\}\big)^2.$$

The same formula for φ holds on the whole $\mathbf{R}^2$ since φ is continuous and positively homogeneous of degree two.

3.4 Homogenization of Hamilton Jacobi equations

We can derive as an example from Theorem 3.1 a homogenization result for Hamilton Jacobi equations. This convergence result will be expressed through the behaviour of the viscosity solutions of these equations, for whose general theory we refer to the related bibliography. These solutions can be expressed by means of some minimum problems involving functionals related to the Legendre transforms of the Hamiltonians. The simple idea to describe their convergence is to describe the behaviour of those minimum problems by Γ-convergence.

Let $H : \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N \to \mathbf{R}$ be a continuous function satisfying the hypotheses of Theorem 3.1 with $p = 2$, and such that $H(t, x, \cdot)$ is convex for every t and x, and let φ be a bounded and uniformly continuous function in $\mathbf{R}^N$. We study the limiting behaviour of the viscosity solutions (defined below) of the Cauchy problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + H\Big(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}, Du_\varepsilon\Big) = 0 & \text{in } \mathbf{R}^N \times [0, +\infty) \\ u_\varepsilon(x, 0) = \varphi(x) & \text{in } \mathbf{R}^N. \end{cases} \tag{3.27}$$

Even though this is stated as a problem of Partial Differential Equations, in this case the notion of solution will involve only a system of ordinary differential equations or more precisely the solution of one-dimensional variational problems.

Definition 3.5 *Let H be a continuous function satisfying a growth condition of order 2, and such that $H(t, x, \cdot)$ is convex, and let φ be a given bounded and uniformly continuous function in $\mathbf{R}^N$. Then, the (unique)* viscosity solution *of the* Hamilton Jacobi equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + H(t, x, Dv) = 0 & \textit{in } \mathbf{R}^N \times [0, +\infty) \\ v(x, 0) = \varphi(x) & \textit{in } \mathbf{R}^N, \end{cases} \tag{3.28}$$

is the function constructed as follows. Let L be the Legendre transform *(conjugate) of H, defined as*

$$L(t, x, z) = \sup\{\langle z, z'\rangle - H(t, x, z') : z' \in \mathbf{R}^N\}.$$

We define for $x, y \in \mathbf{R}^N$ and $0 \le s < t$

$$S(x, t; y, s) = \inf\Big\{\int_s^t L(\tau, u(\tau), u'(\tau))\, d\tau : \\ u(s) = y,\ u(t) = x,\ u \in (\mathrm{W}^{1,\infty}(s, t))^N\Big\}.$$

Then v is given by

$$v(x, t) = \inf\{\varphi(y) + S(x, t; y, s)\ :\ y \in \mathbf{R}^N, 0 \le s < t\}.$$

This last equality is usually referred to as the Lax formula.

We can consider the Legendre transform of our periodic H and apply Theorem 3.1 to the functionals $\int_s^t L(\tau/\varepsilon, u/\varepsilon, u')d\tau$. We then obtain that the related homogenized functional $\int_s^t L_{\text{hom}}(u')d\tau$ is described by the integrand

$$L_{\text{hom}}(z) = \lim_{T\to+\infty} \frac{1}{T} \inf\Big\{\int_0^T L(\tau, u(\tau) + z\tau, u'(\tau) + z)\, d\tau :\ u \in (\mathrm{W}_0^{1,2}(0, T))^N\Big\}. \tag{3.29}$$

We can now define the *homogenized Hamiltonian* H_{hom} as the Legendre transform of L_{hom}, i.e.

$$H_{\mathrm{hom}}(z) = \sup\{\langle z, z'\rangle - L_{\mathrm{hom}}(z') : z' \in \mathbf{R}^N\},$$

and state the convergence result as follows.

Theorem 3.6 *Let φ be a given bounded and uniformly continuous function in $\mathbf{R}^N$, and let u_ε be the unique viscosity solution of (3.27). Then as $\varepsilon \to 0$, the family (u_ε) converges uniformly on compact sets to the unique viscosity solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + H_{\mathrm{hom}}(Du) = 0 & \textit{in } \mathbf{R}^N \times [0, +\infty) \\ u(x,0) = \varphi(x) & \textit{in } \mathbf{R}^N. \end{cases} \tag{3.30}$$

Proof Following Definition 3.5, we set for $x, y \in \mathbf{R}^N$ and $0 \leq s < t$

$$\begin{aligned} &S_\varepsilon(x,t;y,s) \\ &= \inf\Big\{\int_s^t L\Big(\frac{\tau}{\varepsilon}, \frac{u}{\varepsilon}, u'\Big)\, d\tau : u(s) = y,\ u(t) = x,\ u \in (\mathrm{W}^{1,\infty}(s,t))^N\Big\} \\ &= \inf\Big\{\int_s^t L\Big(\frac{\tau}{\varepsilon}, \frac{u}{\varepsilon}, u'\Big)\, d\tau : u(\tau) - \Big(\frac{y-x}{s-t}(\tau - s) + y\Big) \in (\mathrm{W}_0^{1,2}(s,t))^N\Big\}. \end{aligned}$$

Then the unique viscosity solution to problem (3.27) is

$$u_\varepsilon(x,t) = \inf\{\varphi(y) + S_\varepsilon(x,t;y,s) \ :\ y \in \mathbf{R}, 0 \leq s < t\}.$$

By Theorem 3.1 and Proposition 2.37, we have that for every $x, y \in \mathbf{R}^N$ and $0 \leq s < t$

$$\begin{aligned} &\lim_{\varepsilon \to 0} S_\varepsilon(x,t;y,s) \\ &= \min\Big\{\int_s^t L_{\mathrm{hom}}(u')\, d\tau \ :\ u(\tau) - \Big(\frac{y-x}{s-t}(\tau - s) + y\Big) \in (\mathrm{W}_0^{1,2}(s,t))^N\Big\} \\ &= (t-s) L_{\mathrm{hom}}\Big(\frac{y-x}{s-t}\Big), \end{aligned}$$

the last equality following from the convexity of L_{hom} and Jensen's inequality. By the growth hypothesis on L we obtain that the functions $S_\varepsilon(x,t;\cdot,\cdot)$ are equicontinuous in $\{y \in \mathbf{R}^N, 0 \leq s \leq t - \eta\}$, and then $u_\varepsilon(x,t) \to u(x,t)$ pointwise, where

$$u(x,t) = \inf\Big\{\varphi(y) + (t-s)\, L_{\mathrm{hom}}\Big(\frac{y-x}{s-t}\Big) \ :\ y \in \mathbf{R}^N, 0 \leq s < t\Big\}.$$

Since the functions u_ε are equicontinuous on compact sets, the convergence is uniform on bounded sets by Ascoli Arzelà's Theorem. Again by the Lax formula, and by the definition of H_{hom}, u is the unique viscosity solution of (3.30). □

We now examine an example which gives a flavour of how homogenization may change some features of the Hamilton Jacobi equation.

Example 3.7 Let $N = 1$, let $g : \mathbf{R} \to \mathbf{R}$ be a continuous 1-periodic function with $g(0) = 0 = \min g$ and let

$$H(x, z) = \tfrac{1}{2}|z|^2 - g(x).$$

The corresponding Lagrangian is then simply $L(x, z) = \frac{1}{2}|z|^2 + g(x)$. We examine the behaviour of the homogenized L_{hom} in 0. We, clearly, have $L_{\text{hom}}(0) = 0$. Moreover, we can estimate $L_{\text{hom}}(z)$ for $z \neq 0$ by remarking that for all $u \in W_0^{1,\infty}(0, T)$, if we set $v(\tau) = u(\tau) + z\tau$, we have

$$\frac{1}{T}\int_0^T \left(\frac{1}{2}|v'|^2 + g(v)\right) d\tau \geq \left|\frac{1}{T}\int_0^T \sqrt{2g(v)}v'\, d\tau\right| = \left|\frac{1}{T}\int_0^{Tz} \sqrt{2g(s)}\, ds\right|.$$

This last quantity tends to $|z| \int_0^1 \sqrt{2g(s)}\, ds$, so that by (3.29) we obtain

$$L_{\text{hom}}(z) \geq |z| \int_0^1 \sqrt{2g(s)}\, ds. \tag{3.31}$$

This estimate together with the convexity of L_{hom} shows that this function exhibits a corner point at 0, and hence, an easy computation shows that the corresponding H_{hom} is 0 on a neighbourhood of 0 (see Fig. 3.2).

3.5 Exercises

3.1 Compute f_{hom} when $f(t, z) = \frac{1}{p}a(t)|z|^p$. In particular, consider the case $a(t) = \begin{cases} \alpha & \text{if } 0 \leq t < s, \\ \beta & \text{if } s \leq t < 1. \end{cases}$

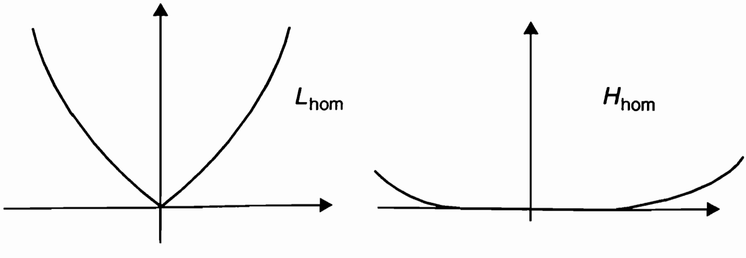

FIG. 3.2. Homogenized Lagrangian and homogenized Hamiltonian

3.2 (homogenization with constraint on the gradient). Compute the homogenized formula (3.9) with ($\alpha \leq \beta$)

$$f(t,z) = \begin{cases} \alpha z^2 & \text{if } 0 \leq t < s \text{ and } |z| \leq M, \\ \beta z^2 & \text{if } s \leq t < 1 \text{ and } |z| \leq M \\ +\infty & \text{otherwise.} \end{cases}$$

Hint: by Euler's equation the minimal u' is constant on $(0,1/2)$ and $(1/2,1)$. We obtain then

$$f_{\rm hom}(z) = \min\Big\{\frac{\alpha}{2}x^2 + \frac{\beta}{2}(2z-x)^2 : |x| \leq M, \ |2z-x| \leq M\Big\}.$$

A calculus exercise yields

$$f_{\rm hom}(z) = \begin{cases} \frac{2\alpha\beta}{\alpha+\beta} z^2 & \text{if } |z| \leq M\frac{\alpha+\beta}{2\beta} \\ \frac{\alpha}{2}M^2 + \frac{\beta}{2}(2|z|-M)^2 & \text{if } M\frac{\alpha+\beta}{2\beta} < |z| \leq M \\ +\infty & \text{if } |z| > M. \end{cases}$$

Hence, for $|z|$ small enough the constraint on the gradient is not felt and the minimizers for the unconstrained and the constrained problem coincide. As $|z|$ increases the effect of the constraint is more evident: for $|z| = M$ the only test functions are the constants so that the outcome is just the average value $(\alpha+\beta)M^2/2$.

Comments on Chapter 3

We refer to the book by Braides and Defranceschi (1998), which is completely devoted to the homogenization of integral functionals, for more information on technical issues in homogenization by Γ-convergence and references. Homogenization of differential equations, raising many more interesting issues can be successfully studied with other techniques; see, for example, Bensoussan *et al.* (1978), Tartar (1979, 1990), Allaire (1992), Zhikov *et al.* (1994), Milton (2002). The result in Section 3.1 can be seen as a particular case of a work of E (1991). The homogenization of functionals $\int_a^b f(x/\varepsilon, Du)\,dx$ gives somewhat trivial results in the one-dimensional case, while it raises interesting questions in higher dimensions (see Chapter 12). The limit of oscillating Riemannian metrics was first studied by Acerbi and Buttazzo (1983) and is connected to some geometric notion of convergence of distances (see Burago (1992)). A work by Braides *et al.* (2002a) shows that *all Finsler metrics* can be approximated by isotropic Riemannian metrics. A simplified treatment of Example 3.4 can be found in Braides and Defranceschi (1998). The homogenization of Hamilton Jacobi equations through Lax's formula stems from a note by Lions *et al.* (1987). Recent advances can be found in Evans and Gomes (2001). We refer the reader interested in viscosity solutions to the user's guide by Crandall *et al.* (1992).

4

FROM DISCRETE SYSTEMS TO INTEGRAL FUNCTIONALS

In the previous chapters we have dealt with classes of functionals that are 'stable by Γ-convergence', with a few exceptions. In this chapter we study a first class of problems where the Γ-limit has a form different from that of the approximating functionals: precisely, we show how some types of energies defined on discrete functions have as their Γ-limit an integral functional of the form described in the previous chapters. This limit process can also be seen in the reverse way: given an integral energy, we show different ways to approximate it with discrete systems.

The discrete energies we consider depend on a parameter $n \in \mathbf{N}$ and have the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \varphi_n^j(u_{i+j} - u_i), \tag{4.1}$$

where $\{u_i\} = \{u_0, u_1, \dots, u_n\}$, with $u_i \in \mathbf{R}$. We introduce as a set of parameters the points $x_i^n = i\lambda_n$ of a lattice of lattice spacing $\lambda_n = L/n$, so that we may picture the set $\{x_i^n\}$ as the reference configuration of an array of material points, and u_i as representing the displacement of the ith point. An interpretation with a physical flavour of the energy E_n is as the internal interaction energy of this chain of $n+1$ (ordered) material points, under the assumption that the points may move only along one axis and that the interaction energy densities depend only on the distance between the two points $u_{i+j} - u_i$ in the deformed configuration and on the order j of the interaction (i.e. on the distance in the reference configuration). The function φ_n^j can be thought of as the energy density of the interaction of points with distance j lattice spacings in the reference lattice. A special case is when $\varphi_n^j = 0$ if $j > 1$, in which each point interacts with its 'nearest neighbour' only.

We will show that, under some growth conditions, upon suitably identifying discrete functions $\{u_i\}$ with some interpolations, the free energies E_n Γ-converge to a limit energy F, which is defined on a Sobolev space and takes the form

$$F(u) = \int_0^L \psi(u')\, dt. \tag{4.2}$$

Even though the description of this limit passage can be performed in a much more general setting, for the time being, we will treat the case when the limit is defined in a Sobolev space only. As an application we may describe the behaviour of problems of the form

$$\min\Big\{E_n(\{u_i\}) - \sum_{i=0}^{n} \lambda_n u_i f_i : \ u_0 = U_0, \ u_n = U_L\Big\} \tag{4.3}$$

(and similar), and show that for a quite general class of energies these problems have a limit continuous counterpart. Here $\{f_i\}$ represents a (discretization of an) external force and U_0, U_L are the boundary conditions at the endpoints of the interval $(0, L)$. From the Γ-convergence result we obtain that minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min\Big\{\int_0^L (\psi(u') - fu)\,dt : \ u(0) = U_0, \ u(L) = U_L\Big\}. \tag{4.4}$$

4.1 Discrete functionals

As anticipated above, in order to define a limit energy on a continuum we parameterize our discrete functions on a single interval $(0, L)$. Set

$$\lambda_n = \frac{L}{n}, \qquad x_i^n = \frac{i}{n}L = i\lambda_n, \qquad i = 0, 1, \ldots, n. \tag{4.5}$$

We denote $I_n = \{x_0^n, \ldots, x_n^n\}$ and by $\mathcal{A}_n(0, L)$ the set of functions $u : I_n \to \mathbf{R}$. If n is fixed and $u \in \mathcal{A}_n(0, L)$, we equivalently use the notation

$$u_i = u(x_i^n).$$

We will study the limit as $n \to +\infty$ of sequences (E_n) with $E_n : \mathcal{A}_n(0, L) \to [0, +\infty]$ of the form (4.1)

Remark 4.1 From elementary calculus we have that E_n is lower semicontinuous if each φ_n^j is lower semicontinuous, and that bounded sets of $\mathcal{A}_n(0, L)$ are precompact.

Since each functional E_n is defined on a different function space, the first step is to identify each $\mathcal{A}_n(0, L)$ with a subspace of a common space of functions defined on $(0, L)$. In order to identify each discrete function with a continuous counterpart, we extend u by $\tilde{u} : (0, L) \to \mathbf{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{4.6}$$

In this case, $\mathcal{A}_n(0, L)$ is identified with those continuous $u \in W^{1,1}(0, L)$ (actually, in $W^{1,\infty}(0, L)$) such that u is affine on each interval (x_{i-1}, x_i). Note, moreover, that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n} \tag{4.7}$$

on (x_{i-1}, x_i). If no confusion is possible, we will simply write u in place of $\tilde{u}$. With this identification in mind each functional $E_n : \mathcal{A}_n(0,L) \to [0,+\infty)$ may be identified with the functional $F_n : L^1(0,L) \to [0,+\infty]$ given by

$$F_n(u) = \begin{cases} E_n(u) & \text{if } u \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

Definition 4.2 (convergence of discrete functions and energies). With the identifications above we will say that u_n *converge to* u (respectively, in L^1, in measure, in $W^{1,1}$, etc.) if $\tilde{u}_n$ converge to u (respectively, in L^1, in measure, weakly in $W^{1,1}$, etc.), and we will say that E_n Γ*-converge to* F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.) if F_n defined in (4.8) Γ-converge to F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.).

4.2 Continuous limits

Since we will treat limit functionals defined on Sobolev spaces, it is convenient to rewrite the dependence of the energy densities in (4.1) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n} \Big), \tag{4.9}$$

where

$$\psi_n^j(z) = \frac{1}{\lambda_n} \varphi_n^j(j\lambda_n z), \tag{4.10}$$

and $K_n \in \{1, \dots, n\}$ (which means we suppose $\varphi_n^j = 0$ if $j > K_n$; that is, we neglect interactions of sufficiently-high order).

We now investigate the effects of the passage to the limit by describing more in detail the limit energies in the two cases of nearest-neighbour ($K_n = K = 1$) and next-to-nearest-neighbour ($K_n = K = 2$) interactions.

4.2.1 *Nearest-neighbour interactions: a convexification principle*

In the case $K = 1$ the functionals take the form

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n \Big(\frac{u_{i+1} - u_i}{\lambda_n} \Big). \tag{4.11}$$

The 'integral counterpart' of E_n is given, using (4.7), simply by

$$F_n(u) = \begin{cases} \displaystyle\int_0^L \psi_n(u')\,dx & \text{if } u \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.12}$$

We now show that in this simpler case the only effect of the passage from the discrete setting to the continuum is the convexification of the integrand. Note that the convexification is not due to a relaxation process at fixed n as in Sobolev spaces (since the lower semicontinuity of E_n is linked to the lower semicontinuity of ψ_n and not to its convexity — see Remark 4.1) but really to the limit as $n \to +\infty$.

Theorem 4.3 (limits of discrete systems of nearest-neighbour interactions). *Let $1 < p < +\infty$, and let $\psi_n : \mathbf{R} \to [0, +\infty)$ be locally equi-bounded Borel functions satisfying*

$$c_1|z|^p - c_2 \le \psi_n(z) \tag{4.13}$$

for all z, and suppose that there exists the limit $\psi = \lim_n \psi_n^{**}$. Then the Γ-limit of E_n with respect to the convergence in $L^p(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,p}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{4.14}$$

on $L^p(0,L)$. In particular if $\psi_n = \varphi$ independently of n then $\psi = \varphi^{**}$.

Proof By Theorem 2.20, we have Γ-$\liminf_j F_j(u) \ge F(u)$. Conversely, fixed $u \in W^{1,\infty}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. If each ψ_n is convex then we have by Jensen's inequality

$$\int_{x_i^n}^{x_{i+1}^n} \psi_n(u')\,dt \ge \lambda_n \psi_n\Big(\frac{1}{\lambda_n}\int_{x_i^n}^{x_{i+1}^n} u'\,dt\Big) = \lambda_n \psi_n\Big(\frac{u(x_{i+1}^n) - u(x_i^n)}{\lambda_n}\Big);$$

hence, summing up, letting $n \to +\infty$, and using the pointwise convergence of ψ_n^{**} to ψ, we get

$$\int_0^L \psi(u')\,dt = \lim_n \int_0^L \psi_n(u')\,dt \ge \limsup_n E_n(u_n).$$

By a density argument we recover the same inequality on the whole $W^{1,p}(0,L)$.

If ψ_n is not convex a direct construction is needed. It is sufficient to deal with the case of a linear target function $u(t) = zt$, since by repeating that construction we can easily deal with the case of u piecewise affine and then the general case follows by density. By Exercise 4.1 and Proposition 1.32 we may suppose that each ψ_n is lower semicontinuous. By Remark 2.17(c) we find z_n^1, z_n^2 and $t_n \in [0,1]$ such that $t_n z_n^1 + (1-t_n)z_n^2 = z$ and $\psi_n^{**}(z) = t_n\psi_n(z_n^1) + (1-t_n)\psi_n(z_n^2)$. Upon extracting subsequences, we may suppose that $t_n \to t$, $z_n^1 \to z_1$ and $z_n^2 \to z_2$, with $tz_1 + (1-t)z_2 = z$. Let $T_n \in \mathbf{N}$ with $\lim_n T_n = +\infty$ and $\lim_n \lambda_n T_n = 0$. Let $K_n \in \{0, \dots, T_n\}$ be such that $\lim_n K_n/T_n = t$. We define $u_n \in \mathcal{A}_n(0,L)$ as $u_n(t) = zt + v_n(t)$, where v_n is the piecewise-affine $\lambda_n T_n$-periodic function with

$$v_n'(t) = \begin{cases} (z_n^1 - z) & \text{if } 0 \le t < \lambda_n K_n \\ (z_n^2 - z) & \text{if } \lambda_n K_n \le t < \lambda_n T_n. \end{cases}$$

Note that $\|v_n\|_\infty = |z_n^1 - z||\lambda_n K_n| = O(T_n K_n) = o(1)$, so that $u_n \to u$. Moreover

$$\lim_n E_n(u_n) = \lim_n L\Big(\frac{K_n}{T_n}\psi_n(z_n^1) + \Big(1 - \frac{K_n}{T_n}\Big)\psi_n(z_n^2)\Big) = L\psi(z)$$

as desired.

By (4.13) the sequence (E_n) is equi-coercive on bounded sets of $L^p(0,L)$ with respect to the weak convergence in $W^{1,p}(0,L)$, from which the complete thesis is easily deduced. □

Remark 4.4 (form of the recovery sequence). The construction of the recovery sequence in the proof of Theorem 4.3 shows that oscillations often must be introduced to obtain an optimal behaviour, as in the continuous case. In the discrete setting, oscillations are not possible at scale λ_n or lower, due to the constraint $u_n \in \mathcal{A}_n(0,L)$, so that an additional scale $\delta_n = \lambda_n T_n >> \lambda_n$ must be introduced.

4.2.2 *Next-to-nearest neighbour interactions: non-convex relaxation*

In the non-convex setting, the case $K = 2$ offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case $K \geq 3$. Essentially, the way the limit continuum theory is obtained is by first 'integrating-out' the contribution due to nearest neighbours obtaining an energy with underlining lattice of double spacing, and then by applying the nearest-neighbour theorem to the resulting functional.

Theorem 4.5 (limits of discrete systems of next-to-nearest-neighbour interactions). *Let $1 < p < +\infty$ and let $\psi_n^1, \psi_n^2 : \mathbf{R} \to [0,+\infty)$ be locally equi-bounded Borel functions such that*

$$c_1|z|^p - c_2 \leq \psi_n^j(z) \tag{4.15}$$

for all z, and let $E_n(u) : \mathcal{A}_n(0,L) \to [0,+\infty)$ be given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2\Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big) \tag{4.16}$$

Let $\tilde\psi_n : \mathbf{R} \to [0,+\infty)$ be defined by

$$\begin{aligned}\tilde\psi_n(z) &= \psi_n^2(z) + \tfrac{1}{2}\inf\{\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi_n^2(z) + \tfrac{1}{2}(\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\Big\},\end{aligned} \tag{4.17}$$

(minimization of the nearest-neighbour interaction) *and suppose that*

$$\psi = \lim_n \tilde\psi_n^{**} \tag{4.18}$$

(which is not restrictive up to subsequences). Then the Γ-limit of E_n with respect to the convergence in $L^p(0,L)$ is given by F defined by (4.14) *on $L^p(0,L)$.*

Remark 4.6 (i) The growth conditions on ψ_n^2 can be weakened, by requiring that $\psi_n^2 : \mathbf{R} \to \mathbf{R}$ and $-c_1'|z|^p - c_3' \le \psi_n^2 \le c_2'(1+|z|^p)$, provided that we still have $\liminf_{|z|\to\infty} \frac{\psi(z)}{|z|^p} > 0$.

(ii) If ψ_n^1 is convex then $\tilde{\psi}_n = \psi_n^1 + \psi_n^2$; if also ψ_n^2 is convex then $\psi = \lim_n(\psi_n^1 + \psi_n^2)$.

Proof For the sake of notational simplicity, we deal with the case of $\psi_n^k = \psi^k$ independent of n, the proof capturing the main features and being essentially the same as that of the general case. We write $\tilde{\psi} = \tilde{\psi}_n$, so that $\psi = \tilde{\psi}^{**}$.

Let $v \in \mathcal{A}_n(0,L)$. By regrouping the terms in (4.16) we have

$$
\begin{aligned}
E_n(v) = & \sum_{\substack{i=0\\ i \text{ even}}}^{n-2} \lambda_n\Big(\psi^2\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{v_{i+2}-v_{i+1}}{\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{v_{i+1}-v_i}{\lambda_n}\Big)\Big) \\
& + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} \lambda_n\Big(\psi^2\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{v_{i+2}-v_{i+1}}{\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{v_{i+1}-v_i}{\lambda_n}\Big)\Big) \\
& + \frac{1}{2}\psi^1\Big(\frac{v_n-v_{n-1}}{\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{v_1-v_0}{\lambda_n}\Big) \\
\ge & \frac{1}{2}\Big(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\tilde{\psi}\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big) + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde{\psi}\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big)\Big) \\
\ge & \frac{1}{2}\Big(\sum_{\substack{i=0\\ i \text{ even}}}^{n-2} 2\lambda_n\tilde{\psi}^{**}\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big) + \sum_{\substack{i=0\\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde{\psi}^{**}\Big(\frac{v_{i+2}-v_i}{2\lambda_n}\Big)\Big) \\
= & \frac{1}{2}\Big(\int_0^{2\lambda_n[n/2]} \psi(\tilde{v}_1')\,dt + \int_{\lambda_n}^{(1+2[n-1/2])\lambda_n} \psi(\tilde{v}_2')\,dt\Big),
\end{aligned}
\tag{4.19}
$$

where $\tilde{v}_k$, respectively, with $k=1,2$, are the continuous piecewise-affine functions such that

$$
\tilde{v}_k' = \frac{v_{i+2}-v_i}{2\lambda_n} \qquad \text{on } (x_i^n, x_{i+2}^n) \tag{4.20}
$$

for i, respectively, even or odd.

Let now $u_n \to u$ in $L^p(0,L)$ and $\sup_n E_n(u_n) < +\infty$; then $u_n \rightharpoonup u$ in $W^{1,p}(0,L)$. Let $u_{k,n}$ be defined as in (4.20) with u_n in place of v; we then deduce that $u_{k,n} \rightharpoonup u$ as $n \to +\infty$, for $k=1,2$. For every fixed $\eta > 0$ by (4.19) we obtain

$$
\begin{aligned}
\liminf_n E_n(u_n) \ge & \frac{1}{2}\Big(\liminf_n \int_\eta^{L-\eta} \psi(u_{1,n}')\,dt + \liminf_n \int_\eta^{L-\eta} \psi(u_{2,n}')\,dt\Big) \\
\ge & \int_\eta^{L-\eta} \psi(u')\,dt,
\end{aligned}
$$

and the liminf inequality follows by the arbitrariness of $\eta > 0$.

Now we prove the limsup inequality. By an easy relaxation argument as in Example 4.1, it suffices to treat the case when $\tilde{\psi}$ is lower semicontinuous, $u(x) = zx$ and $\psi(z) = \tilde{\psi}(z)$. With fixed $\eta > 0$ let z_1, z_2 be such that

$$z_1 + z_2 = 2z \qquad \text{and} \qquad \psi^2(z) + \frac{1}{2}(\psi^1(z_1) + \psi^2(z_2)) \leq \tilde{\psi}(z) + \eta.$$

We define the recovery sequence u_n as

$$u_n(x_i^n) = \begin{cases} zx_i^n & \text{if } i \text{ is even} \\ z(i-1)\lambda_n + z_1^n\lambda_n & \text{if } i \text{ is odd.} \end{cases}$$

We then have

$$\begin{aligned} E_n(u_n) &= \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n}\Big) \\ &\leq \frac{L}{2}(\psi^1(z_1) + \psi^1(z_2)) + L\psi^2(z) \leq L\psi(z) + L\eta = F(u) + L\eta, \end{aligned}$$

and the limsup inequality follows by the arbitrariness of η. □

Remark 4.7 (multiple-scale effects). The formula defining ψ highlights a double-scale effect pictured in the recovery sequence of Fig. 4.1. The operation of 'inf-convolution' (4.17) highlights oscillations on the scale λ_n, while the convexification of $\tilde{\psi}$ acts at a much larger scale (see Remark 4.4).

4.2.3 *Long-range interactions: homogenization*

In the case of more than two interactions simple formulas as those obtained in the previous sections do not hold. We do not treat this case in detail, but only outline a description of the limit and its analogies with the homogenization of integral energies.

For the sake of simplicity, we suppose that our functionals are of the form (4.9) with $K_n = K$, and that $\psi_n^j = \psi^j$ are independent of n and $c_1|z|^p \leq$

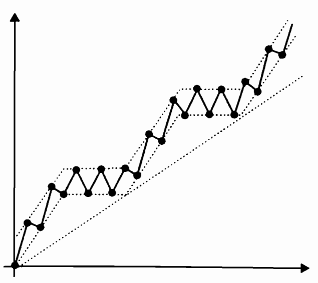

FIG. 4.1. Recovery sequence with a double scale

$\psi^j(z) \le c_2(1+|z|^p)$ for all z and j. We then have the following result, in which it is worth noting that formula (4.21) is completely analogous to the asymptotic homogenization formula (3.11), the condition $u \in W_0^{1,p}(0,N)$ being replaced by a condition on the first K and the last K points of the interval.

Theorem 4.8 (limits of discrete systems with long-range interactions). *The functionals E_n Γ-converge to the functional F of the form* (4.14), *where the integrand ψ satisfies the* homogenization formula

$$\psi(z) = \lim_N \inf\Big\{\frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N-j}\psi^j(v(i+j)-v(i)) : v:\{0,\dots,N\}\to\mathbf{R},$$
$$v(i) = iz \textit{ if } i \in \{0,\dots,K\}\cup\{N-K,\dots,N\}\Big\} \tag{4.21}$$

for all $z \in \mathbf{R}$.

Proof We just give an hint of the proof: it can be performed similarly to that of the Homogenization Theorem 3.1. The form of ψ can be deduced from the convergence of minimum problems.

The condition $v(i) = iz$ if $i \in \{0,\dots,K\}\cup\{N-K,\dots,N\}$ allows to easily construct a recovery sequence for $u(t) = zt$. In fact, it suffices to fix $\eta > 0$ and choose $N \in \mathbf{N}$, $N > 1/\eta$, and $v : \{0,\dots,N\} \to \mathbf{R}$ such that the boundary conditions are satisfied and

$$\frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N-j}\psi^j(v(i+j)-v(i)) \le \psi(z)+\eta.$$

We extend v to the whole $\mathbf{Z}$ in such a way that $i \mapsto v(i) - iz$ is a N-periodic function and set $u_n(x_n^i) = \lambda_n v(i)$. We have $u_n \to u$ and

$$\begin{aligned}
\limsup_n E_n(u_n) &= L\frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N}\psi^j(v(i+j)-v(i))\\
&\le L\psi(z) + L\eta + L\frac{1}{N}\sum_{j=1}^{K}\sum_{i=N-j+1}^{N}\psi^j(v(i+j)-v(i))\\
&= L\psi(z) + L\eta + L\frac{1}{N}\sum_{j=1}^{K}\sum_{i=N-j+1}^{N}\psi^j(zj)\\
&= L\psi(z) + L\eta + L\frac{1}{N}\sum_{j=1}^{K}(j-1)\psi^j(zj)\\
&\le L\psi(z) + L\eta + L\frac{1}{N}\sum_{j=1}^{K}(j-1)c_2(1+|zj|^p)
\end{aligned}$$

$$\leq L\psi(z) + L\eta c(1 + |z|^p),$$

which proves the limsup inequality by the arbitrariness of η. □

4.2.4 *Convergence of minimum problems*

From Theorem 4.5 we immediately deduce the following theorem.

Theorem 4.9 *Let E_n and F be given by Theorem* 4.5, $f \in L^1(0,L)$ *and* $d > 0$. *Then the minimum values* $m_n = \min\{E_n(u)+\int_0^L fu\,dt : u(0)=0,\ u(L)=d\}$ *converge to* $m = \min\{F(u) + \int_0^L fu\,dt : u(0)=0, u(L)=d\}$, *and from each sequence of minimizers of m_n we can extract a subsequence converging to a minimizer of m.*

Proof Since the sequence of functionals (E_n) is equi-coercive, it suffices to show that the boundary conditions do not change the form of the Γ-limit; i.e., that for all $u \in W^{1,p}(0,L)$ such that $u(0) = 0$ and $u(L) = d$ and for all $\varepsilon > 0$ there exists a sequence u_n such that $u_n(0) = 0$, $u_n(L) = d$ and $\limsup_n E_n(u_n) \leq F(u) + \varepsilon$. This can be done similarly to Proposition 2.37 (we leave to the reader the transposition of that proposition to the discrete setting). □

4.3 Exercises

4.1 Let $n \in \mathbf{N}$ be fixed and let E_n be of the form (4.1). Prove that the relaxation of E_n is obtained by taking $\mathrm{sc}\psi_n^j$ in place of ψ_n^j.

4.2 Compute the limit in Theorem 4.5 when $\psi_n^1(z) = \psi^1(z) = 2(|z| - 1)^2$ and $\psi_n^2(z) = \psi^2(z) = z^2$.

4.3 Compute the limit in Theorem 4.5 when $\psi_n^1(z) = \psi^1(z) = 2(|z| - 1)^2$ and $\psi_n^2(z) = \psi^2(z) = -z^2$.

Comments on Chapter 4

The treatment of non-convex discrete systems by Γ-convergence is a very recent subject. In the framework of Sobolev spaces we refer to Braides *et al.* (2002b) and Pagano and Paroni (2002), where an application to the theory of phase transitions is given. A Γ-convergence result for long-range quadratic discrete interactions is given by Piatnitski and Remy (2001). In Braides (2000) it is shown that if the maximum order of interaction K is not bounded then the limit may be a non-local Dirichlet form type energy. A mechanical analyisis of nonlinear discrete interactions is given by Puglisi and Truskinovsky (2000). The treatment of systems in dimension higher than one is a subject of active research. In this respect, note that the 'homogenization formula' in Section 4.2.3 can be generalized to the many-dimensional setting, not relying on a one-dimensional formulation. This issue is linked to the work by Blanc *et al.* (2001) and Friesecke and Theil (2002).

5

SEGMENTATION PROBLEMS

In this section we develop a general theory for another class of variational problems, providing lower semicontinuity and Γ-convergence results in a parallel way to that followed for integral functionals. The type of energies we consider are of the form

$$F(u) = \sum_{t \in S(u)} \theta(u(t-), u(t+)), \tag{5.1}$$

defined when u is a piecewise-constant function; here, $S(u)$ denotes the set of points where u is discontinuous and $u(t\pm)$ are the limit values of u at t. In the form (5.1) such functionals are the one-dimensional simpler analogue of interfacial energies in higher dimensions. These functionals are used, for example, to model some problems in signal reconstruction, fracture mechanics, phase transitions, etc. The combination of energies of this type with integral functionals gives rise to free-discontinuity problems (see Chapter 7).

We can draw some comparisons between these energies and the theory for integral functionals:

— If we have a converging sequence of piecewise-constant functions with constant number of discontinuities then their limit may have the same number of discontinuities or fewer. The latter case must be considered as a type of *weak* convergence,

— The qualitative condition entailing the lower semicontinuity of F is the *subadditivity* of the 'integrand' θ. This notion is in a sense less easy to grasp than convexity and its analysis is the starting point for the understanding of much more complex notions in higher dimension,

— Relaxation and Γ-convergence are expressed in terms of the *subadditive envelopes* of the integrands,

— The problems are not stable under the addition of boundary data. The notion of *boundary value must be relaxed.*

We note that even though a little more exotic than the problems of integral type, those defined on piecewise-constant functions provide a simpler setting which is slightly more complex than that of a finite-dimensional Euclidean space but already allows us to make some non-trivial observations about lower semicontinuity and Γ-convergence.

Remark 5.1 We additionally note that in the same way as illustrated in this chapter, we also can consider vector-valued $u : (a,b) \to \mathbf{R}^k$ and the corresponding segmentation problems. The notion of subadditivity remains unchanged, and it can be easily seen that all the results of this chapter still hold in this more general context. The reader is encouraged to check this fact as an exercise.

5.1 Model problems

We will keep in mind two types of model situations. The first one comes from a problem of *signal reconstruction*: Given a datum $g \in L^p(a,b)$ (which we can think of as a corrupted signal) find the 'best' piecewise-constant approximation of g; that is, it must have the 'least' possible underlying segmentation and at the same time be the closest possible to g. This approximation can be thought of as a 'reconstruction' of the original signal. This problem can be translated in the following terms: Find a finite subset $S = \{t_1, \dots, t_N\}$ of (a,b) with $t_1 < \cdots < t_N$ (completed by $t_0 = a$ and $t_{N+1} = b$) and constants $c_1, \dots, c_{N+1}$ such that the quantity

$$\alpha\,\#(S) + \beta \sum_{i=1}^{N+1} \int_{(t_{i-1},t_i)} |c_i - g(x)|^p \, dx \tag{5.2}$$

is minimal ($\alpha, \beta > 0$ are called *contrast parameters*) (see Fig. 5.1).

The second type of problems comes from mechanics: Let (a,b) parameterize a (one-dimensional) bar composed of a homogeneous material, which is *rigid* (i.e. cannot be elongated or compressed) and *brittle* (i.e. it can break). If we force some boundary displacements, the bar will then break in one point, or many points if the material is prone to 'micro-cracking'. Let S denote the finite set parameterizing the fracture points, and let u denote the displacement field of the bar, which will be piecewise constant. The energy released by the fracture will be of the form

$$\sum_{t \in S} \vartheta(u(t+) - u(t-)), \tag{5.3}$$

where ϑ is a function which describes the behaviour of the material subject to fracture, which we think to depend only on the size of the fracture opening. The number of fracture points and the openings will be determined by minimizing this free energy. Note that a particular case is when ϑ is a constant α, which in

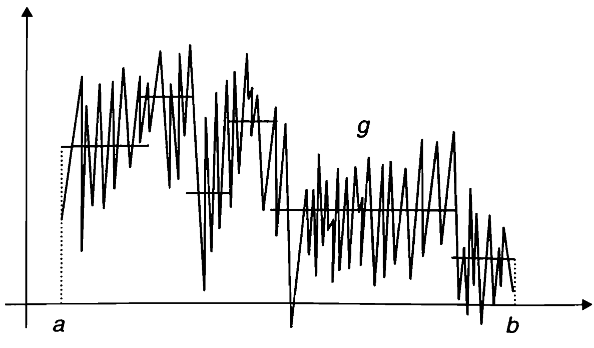

FIG. 5.1. An optimal segmentation

physical terms can be interpreted as supposing that a fixed amount is required to break the atomic bonds, after having done which, no interaction occurs between the two sides of the fracture (*Griffith's theory of fracture*). In this case the energy in (5.3) has the form of the first term in (5.2). Other theories of fracture take into account the presence of a 'cohesive zone' (i.e. the energy depends on the size of the crack, for small values of the opening; this happens in *Barenblatt's theory*).

The existence and description of solutions to both these problem can be obtained 'by hand'. However, in order to provide a framework that can be generalized to more general dimensions, we give a complete characterization of lower semicontinuous functionals modelled on (5.2) and (5.3) to obtain in particular solutions via the direct methods of the Calculus of Variations.

5.2 The space of piecewise-constant functions

We now introduce the precise definitions and describe the topology of the space of piecewise-constant functions.

Definition 5.2 *We say that a function* $u : (a, b) \to \mathbf{R}$ *is* piecewise constant on (a, b) *if there exist points* $a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b$ *such that*

$$u(t) \text{ is constant a.e. on } (t_{i-1}, t_i) \text{ for all } i = 1, \ldots, N+1. \tag{5.4}$$

The subspace of $L^\infty(a, b)$ *of all such* u *is denoted by* $PC(a, b)$. *If* $u \in PC(a, b)$ *we define* $S(u)$ *as the minimal set* $\{t_1, \ldots, t_N\} \subset (a, b)$ *such that* (5.4) *holds.*

At all points $t \in (a, b)$ *we define the values* $u(t+)$ *and* $u(t-)$ *as the values taken a.e. by* u *on* $(t, t+\varepsilon)$ *and* $(t-\varepsilon, t)$, *respectively, for* ε *small enough, or, equivalently,*

$$u(t+) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} u(s)\, ds, \qquad u(t-) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} u(s)\, ds.$$

In the same way we define $u(a+)$ *and* $u(b-)$. *We finally define the functions* $u^+ : [a, b) \to \mathbf{R}$ *and* $u^- : (a, b] \to \mathbf{R}$ *as* $u^\pm(t) = u(t\pm)$.

5.2.1 *Coerciveness conditions*

The easiest way to obtain coerciveness is to impose a bound on the number of discontinuity points.

Proposition 5.3 *Let* (u_j) *be a sequence in* $PC(a, b)$ *such that*

$$\sup_j \#(S(u_j)) < +\infty. \tag{5.5}$$

(i) (closure) *if* (u_j) *converges to* u *a.e. then* $u \in PC(a, b)$; *moreover,* $u_j \to u$ *in measure and* $\#(S(u)) \le \liminf_j \#(S(u_j))$;

(ii) (compactness) *if for all* I *open subsets of* (a, b) $\liminf_j \inf_I |u_j| < +\infty$ *then there exists a subsequence of* (u_j) *converging a.e. Note that in particular, this condition is satisfied if* (u_j) *is bounded in* $L^1(a, b)$.

Proof Upon extracting a subsequence we suppose $S(u_j) = \{t_j^1, \dots, t_j^N\}$ with N independent of j, and such that $t_j^k \to t^k$ for all $k = 1, \dots, N$. Let $S = \{t^1, \dots, t^N\} \subset [a, b]$. There exist $a = a_0 < a_1 \cdots < a_M = b$ such that we can write

$$(a,b) \setminus S = \bigcup_{i=1}^{M} (a_{i-1}, a_i).$$

With fixed $0 < \eta < \min_i (a_i - a_{i-1})/2$, we have that u_j equals a constant, which we denote by c_i^j, a.e. on $(a_{i-1} + \eta, a_i - \eta)$ for all $i = 1, \dots, M$ and for all j large enough.

(i) If (u_j) converges to u a.e. then there exists the limit $\lim_j c_i^j = c_i$, and we have $u = c_i$ a.e. on $(a_{i-1} + \eta, a_i - \eta)$. By the arbitrariness of η we obtain $u = c_i$ on (a_{i-1}, a_i), $u \in PC(a, b)$ and $S(u) \subset S$. Note that this (sub)sequence converges also in measure. Since we can apply this reasoning to all subsequences of the original sequence (u_j) we conclude that $u_j \to u$ in measure. Note that we can choose $N = \liminf_j \#(S(u_j))$ so that we have $\#(S(u)) \le \liminf_j \#(S(u_j))$.

(ii) If for all I open subsets of (a, b) $\liminf_j \inf_I |u_j|$ is bounded, in particular, upon choosing a subsequence, for all $i = 1, \dots, M$ the sequence (c_i^j) is bounded. Upon extracting a further subsequence we can then suppose that $c_i^j \to c^i$ for all $i = 1, \dots, M$. If we set $u = c^i$ on (a_{i-1}, a_i) then $u_j \to u$ a.e. □

Remark 5.4 If $u_j \in PC(a, b)$, and $u_j \to u$ a.e., in general this convergence is not in $L^1(a, b)$, even if $\sup_j \int_{a,b} |u_j| \, dt < +\infty$. Take for example $u_j = j\chi_{(0,1/j)}$. Note in particular that, even though we have local uniform convergence on $(a, b) \setminus S$, the functions u_j may not be equibounded.

5.2.2 *Functionals on piecewise-constant functions*

We will treat functionals $F : PC(a, b) \to [0, +\infty]$ of the form (5.1), with $\theta : \mathbf{R}^2 \setminus \Delta \to [0, +\infty]$, where $\Delta = \{(a, a) : \ a \in \mathbf{R}\}$.

Remark 5.5 If $\theta \ge c > 0$ then by Proposition 5.3 the set

$$\Big\{ u \in PC(a,b) : F(u) \le c_1, \int_{(a,b)} |u| \, dt \le c_2 \Big\}$$

is a precompact set with respect to the convergence in measure for all c_1 and c_2.

If we take θ equal to the constant α we have $F(u) = \alpha \#(S(u))$. Note that the functional to minimize in (5.2) turns out to be $F(u) + \beta \int_{(a,b)} |u - g|^p \, dx$ for $u \in PC(a, b)$.

5.3 Lower semicontinuity conditions: subadditivity

We study the lower semicontinuity of F with respect to a.e. convergence on $PC(a, b)$. Let $u \in PC(a, b)$; we can figure out two different ways of approximating u a.e. by a sequence (u_j) (see Fig. 5.2), which give different types of information on θ once we suppose that F is lower semicontinuous.

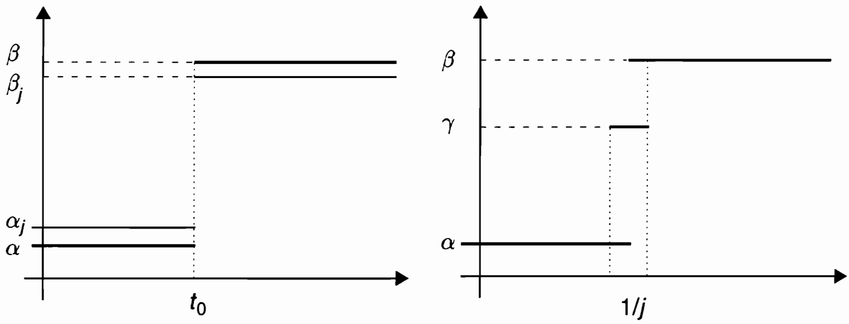

FIG. 5.2. Approximation of a jump function

(i) $u_j \to u$ a.e. and $\#(S(u)) = \#(S(u_j))$. In particular, we may apply the lower semicontinuity inequality for F to

$$u(t) = \begin{cases} \alpha & t < t_0 \\ \beta & t > t_0, \end{cases} \qquad u_j(t) = \begin{cases} \alpha_j & t < t_0 \\ \beta_j & t > t_0, \end{cases}$$

which gives

$$\theta(\alpha,\beta) \le \liminf_j \theta(\alpha_j,\beta_j) \qquad \text{if } \alpha_j \to \alpha,\ \beta_j \to \beta \tag{5.6}$$

for all distinct $\alpha, \beta \in \mathbf{R}$; that is, θ is lower semicontinuous in $\mathbf{R}^2 \setminus \Delta$;

(ii) $u_j \to u$ a.e and $\#(S(u)) < \#(S(u_j))$. In particular, we may apply the lower semicontinuity inequality for F to

$$u(t) = \begin{cases} \alpha & t < t_0 \\ \beta & t > t_0, \end{cases} \qquad u_j(t) = \begin{cases} \alpha & t < t_0 - 1/j \\ \gamma & t_0 - 1/j < t < t_0 + 1/j \\ \beta & t > t_0 + 1/j, \end{cases}$$

which gives

$$\theta(\alpha,\beta) \le \theta(\alpha,\gamma) + \theta(\gamma,\beta) \tag{5.7}$$

for all distinct $\alpha, \beta, \gamma \in \mathbf{R}$; that is, θ is *subadditive* in $\mathbf{R}^2 \setminus \Delta$. Note that in both cases $u_j \to u$ in $L^1(a,b)$.

Remark 5.6 (i) We may sometime prefer to consider θ as defined on the whole $\mathbf{R}^2$, even though its value on Δ is never taken into account. The most convenient way is by setting $\theta(\alpha,\beta) = 0$ if $\alpha = \beta$. Note that θ is subadditive and l.s.c. on $\mathbf{R}^2 \setminus \Delta$ if and only if this extension is subadditive and l.s.c. The non-restrictive condition $\theta(\alpha,\alpha) = 0$ may be sometimes handy in definitions and proofs. On the other hand at times it is useful to consider continuous θ also on Δ, and that it the reason why the zero extension on Δ is not done systematically.

(ii) Note that combined conditions (5.6) and (5.7) are equivalent to requiring that for all $\alpha, \beta \in \mathbf{R}$ and for all sequences of natural numbers (N_j), and real numbers $\alpha_j^0, \dots, \alpha_j^{N_j}$ with $\alpha_j^0 \to \alpha$ and $\alpha_j^{N_j} \to \beta$ we have (θ extended on Δ as in (i))

$$\theta(\alpha,\beta) \le \liminf_j \sum_{i=1}^{N_j} \theta(\alpha_j^{i-1}, \alpha_j^i). \tag{5.8}$$

Indeed, from (5.8) we obtain (5.6) by choosing $N_j = 1$ for all j, and we obtain (5.7) by choosing $N_j = 2$, $\alpha_j^0 = \alpha$, $\alpha_j^1 = \gamma$ and $\alpha_j^2 = \beta$ for all j. Conversely, we obtain

$$\theta(\alpha,\beta) \le \liminf_j \theta(\alpha_j^0, \alpha_j^{N_j})$$

by applying (5.6), while from (5.7) we get

$$\theta(\alpha_j^0, \alpha_j^{N_j}) \le \sum_{i=1}^{N_j} \theta(\alpha_j^{i-1}, \alpha_j^i),$$

and we deduce (5.8) by combining these two inequalities.

(iii) If (θ_i) is a family of subadditive functions then $\theta = \sup_i \theta_i$ is subadditive. In fact, by subadditivity

$$\theta_i(\alpha,\beta) \le \theta_i(\alpha,\gamma) + \theta_i(\gamma,\beta) \le \theta(\alpha,\gamma) + \theta(\gamma,\beta),$$

and the subadditivity inequality for θ follows by taking the supremum on i. Note moreover that, if each θ_i is also l.s.c. then θ is subadditive and l.s.c.

Example 5.7 Easy examples as the following (whose proof is left as an exercise) show that the structure of subadditive functions is very different from that of convex function, for example. From (i) and (ii) in particular we note that no regularity or growth conditions can be directly deduced from subadditivity.

(i) If $c \le \theta \le 2c$ then θ is subadditive.

(ii) If $\phi : \mathbf{R} \to [0, +\infty]$ and $\theta(\alpha,\beta) = \phi(\alpha) + \phi(\beta)$ then θ is subadditive. Moreover if ϕ is l.s.c then θ is also l.s.c.

(iii) Another general example of a subadditive function is given by

$$\theta(\alpha,\beta) = \left| \int_\alpha^\beta \varphi(t)\,dt \right|,$$

where φ is any integrable function.

Theorem 5.8 *Let $\theta : \mathbf{R}^2 \to [0, +\infty]$, and let $F : PC(a,b) \to [0, +\infty]$ be given by* (5.1). *Then the following conditions are equivalent:*

(i) *F is l.s.c. on $PC(a,b)$ with respect to the a.e. convergence;*

(ii) *θ is lower semicontinuous and subadditive.*

Proof The implication (i) $\Longrightarrow$ (ii) is proven by (5.6) and (5.7) above.

To check that (ii) implies (i), let $u_j \to u$ a.e. and let $t \in S(u)$. Choose $0 < \varepsilon < \inf\{|t-s| : t, s \in S(u), t \ne s\}/2$ such that $u_j(t \pm \varepsilon) \to u(t \pm \varepsilon) = u(t\pm)$,

$t \pm \varepsilon \notin S(u_j)$ for all j, and $[t-\varepsilon, t+\varepsilon] \cap S(u) = \{t\}$. Then we have, using Remark 5.6(ii),

$$\begin{aligned}\theta(u(t-), u(t+)) &= \theta(u(t-\varepsilon), u(t+\varepsilon)) \\ &\leq \liminf_j \sum_{s \in S(u_j) \cap (t-\varepsilon, t+\varepsilon)} \theta(u_j(s-), u_j(s+)).\end{aligned}$$

We have the thesis by summing up for $t \in S(u)$. □

Remark 5.9 Note that since, up to subsequences, L^1 convergence implies convergence in measure, which in turn implies a.e. convergence, in Theorem 5.8 we can equivalently replace a.e. convergence with convergence in measure or the $L^1(a,b)$-convergence.

Example 5.10 (existence of optimal segmentations). Let $\theta \geq c$ be lower semicontinuous and subadditive and let $g \in L^p(a,b)$ for some $p \geq 1$. Then there exists a solution to the minimization problem

$$\min\Big\{F(u) + \beta \int_{(a,b)} |u-g|^p \, dt : \ u \in PC(a,b)\Big\}$$

for all $\beta > 0$. By taking $u = 0$ we obtain that a bound for the minimum value above is $\int_{(a,b)} |g|^p \, dt$. It suffices then to use the direct method of the calculus of variations, applying Theorem 5.8, Remark 5.5 and using Fatou's Lemma for the term $\int_{(a,b)} |u-g|^p \, dt$ along a minimizing sequence.

We can apply this example to $F(u) = \alpha\#(S(u))$ and obtain existence for the first model problem in Section 5.1.

5.4 Relaxation and Γ-convergence

In the space of functionals on piecewise-constant functions it will be possible to characterize the lower-semicontinuous envelope and Γ-convergence in terms of subadditivity properties. We will simplify our analysis by restricting to translation-invariant functionals.

5.4.1 *Translation-invariant functionals*

We deal with the case of those functionals as in (5.1) which are invariant by addition of a constant; that is, $F(u+c) = F(u)$. In this case $\theta(\alpha,\beta) = \theta(0, \beta - \alpha) =: \vartheta(\beta - \alpha)$; hence, we can rewrite F as

$$F(u) = \sum_{t \in S(u)} \vartheta(u(t+) - u(t-)), \tag{5.9}$$

where $\vartheta : \mathbf{R} \setminus \{0\} \to [0, +\infty]$. As before, we tacitly set $\vartheta(0) = 0$ when needed.

Definition 5.11 *A function $\vartheta : \mathbf{R} \to [0,+\infty]$ is* subadditive *if*

$$\vartheta(\alpha+\beta) \le \vartheta(\alpha)+\vartheta(\beta) \tag{5.10}$$

for all $\alpha, \beta \in \mathbf{R}$.

Example 5.12 The functions 1, $|\sin z|$, $|z|$, $\min\{1,|z|\}$, $\max\{1,|z|\}$, $\arctan|z|$ are subadditive. We leave the verification of this as an exercise.

Remark 5.13 The function $\vartheta : \mathbf{R} \to [0,+\infty]$ is subadditive if and only if the function $\theta : \mathbf{R}^2 \to [0,+\infty]$ defined by $\theta(\alpha,\beta) = \vartheta(\beta-\alpha)$ is subadditive in the sense of (5.7). Moreover, θ is l.s.c. if and only if ϑ is l.s.c. Hence, F is l.s.c. with respect to a.e. convergence if and only if ϑ is subadditive and l.s.c.

By Remark 5.6(ii) ϑ is subadditive and l.s.c. if and only if for all $z \in \mathbf{R}$, for all sequences of natural numbers (N_j), and real numbers $z_j^0, \dots, z_j^{N_j}$ with

$$\lim_j \sum_{i=1}^{N_j} z_j^i = z,$$

we have (ϑ extended as $\vartheta(0) = 0$ if necessary)

$$\vartheta(z) \le \liminf_j \sum_{i=1}^{N_j} \vartheta(z_j^i). \tag{5.11}$$

Remark 5.14 (i) If ϑ satisfies (5.10) for $\alpha\beta > 0$ and ϑ is non-increasing on $(-\infty,0)$ and non-decreasing on $(0,+\infty)$ then ϑ is subadditive.

(ii) If ϑ is concave on $(-\infty,0)$ and on $(0,+\infty)$ then ϑ is subadditive. In particular $\vartheta(z) = \phi(|z|)$, with $\phi : (0,+\infty) \to [0,+\infty)$ concave, is subadditive. In fact, by concavity, if $a > 0$ and $b > 0$ then

$$\vartheta(a) \ge \frac{a}{a+b}\vartheta(a+b) + \frac{b}{a+b}\vartheta(0) \ge \frac{a}{a+b}\vartheta(a+b),$$

$$\vartheta(b) \ge \frac{b}{a+b}\vartheta(a+b) + \frac{a}{a+b}\vartheta(0) \ge \frac{b}{a+b}\vartheta(a+b),$$

from which (5.10) follows. Similarly we check (5.10) for $a < 0$, $b < 0$. As ϑ is non-increasing on $(-\infty,0)$ and non-decreasing on $(0,+\infty)$, by (i) we obtain the subadditivity of ϑ

5.4.2 *Properties of subadditive functions on* $\mathbf{R}$

Subadditive functions on $\mathbf{R}$ enjoy some additional properties.

Remark 5.15 Let ϑ be subadditive.

(i) If $k \in \mathbf{N}$, $k \ge 1$ then $\vartheta(kz) \le k\vartheta(z)$ for all $z \in \mathbf{R}$ (by induction).

(ii) If ϑ is locally bounded then there exists $C \ge 0$ such that $\vartheta(z) \le C(1+|z|)$ for all $z \in \mathbf{R}$. Indeed (for $z > 0$), let $k = 1 + [z]$; then by (i)

$$\vartheta(z) = \vartheta\Big(k\frac{z}{k}\Big) \le k\vartheta\Big(\frac{z}{k}\Big) \le Ck \le C(1+|z|),$$

where $C = \sup\{\vartheta(s) : 0 \le s \le 1\}$.

(iii) If the limit $C = \lim_{t\to 0+} \vartheta(t)/t$ exists then $\vartheta(z) \le Cz$ for $z \ge 0$. Indeed, with fixed $\varepsilon > 0$, write

$$\begin{aligned}\vartheta(z) &= \vartheta\Big(\varepsilon\Big[\frac{z}{\varepsilon}\Big] + \Big(z - \varepsilon\Big[\frac{z}{\varepsilon}\Big]\Big)\Big) \\ &\le \Big[\frac{z}{\varepsilon}\Big]\vartheta(\varepsilon) + \vartheta\Big(z - \varepsilon\Big[\frac{z}{\varepsilon}\Big]\Big) \le z\sup\Big\{\frac{\vartheta(t)}{t} :\ 0 < t \le \varepsilon\Big\},\end{aligned}$$

and then let $\varepsilon \to 0$. Similarly, if $\lim_{\varepsilon\to 0} \vartheta(z)/|z| = C$ then $\vartheta(z) \le C|z|$ on $\mathbf{R}$. Note that in this case ϑ is *Lipschitz continuous* on $\mathbf{R}$ since

$$\vartheta(z_1) - \vartheta(z_2) \le \vartheta(z_1 - z_2) \le C|z_1 - z_2|.$$

Note that actually C is the *best Lipschitz constant* for ϑ.

(iv) If, after setting $\vartheta(0) = 0$, ϑ is convex then it is positively homogeneous of degree one. Indeed, by the convexity of ϑ the limit $C = \lim_{t\to 0+} \vartheta(t)/t$ exists and $\vartheta(z) \ge Cz$ for $z \ge 0$. The converse inequality holds by (iii). Similarly for $z \le 0$.

(v) A continuous subadditive function need not be uniformly continuous: take, for example, $\varphi(t) = 3 + \sin(t^2)$.

5.4.3 *Relaxation: subadditive envelopes*

We face now the problem of characterizing the lower-semicontinuous envelope of a segmentation energy. Clearly, this envelope can be estimated 'from below' by segmentation functionals with subadditive and semicontinuous energy densities. It is then reasonable to introduce the following definition.

Definition 5.16 *Let* $\vartheta : \mathbf{R}\setminus\{0\} \to [0,+\infty]$. *The* lower semicontinuous and subadditive envelope *of* ϑ *is*

$$\operatorname{sub}\vartheta(z) = \sup\{\phi(z) :\ \phi \le \vartheta,\ \phi \textit{ is l.s.c. and subadditive}\} \tag{5.12}$$

for $z \ne 0$. *We may set* $\operatorname{sub}\vartheta(0) = 0$ *if needed.*

Proposition 5.17 (characterization of the subadditive envelope). Let $\vartheta : \mathbf{R}\setminus\{0\} \to [0,+\infty]$. Then we have

$$\operatorname{sub}\vartheta(z) = \inf\Big\{\liminf_j \sum_{i=1}^{N_j} \vartheta(z_j^i) :\ \lim_j \sum_{i=1}^{N_j} z_j^i = z\Big\} \tag{5.13}$$

for all $z \in \mathbf{R}$, $z \ne 0$. Moreover, if ϑ is uniformly continuous then

$$\operatorname{sub}\vartheta(z) = \inf\Big\{\sum_{i=1}^{N} \vartheta(z_j) :\ \sum_{j=1}^{N} z_j = z\Big\}. \tag{5.14}$$

Proof Let $\varphi(z)$ denote the right hand-side of (5.13). By Remark 5.6(iii) $\operatorname{sub}\vartheta$ is subadditive and l.s.c. Clearly, by (5.11) and the inequality $\operatorname{sub}\vartheta \le \vartheta$ we have $\operatorname{sub}\vartheta \le \varphi \le \vartheta$. It suffices to remark that φ is subadditive and l.s.c.; both properties are easily deduced from its definition. □

Remark 5.18 (i) If ϑ is L-Lipschitz, then so is also $\operatorname{sub}\vartheta$. In fact, let $s, t \in \mathbf{R}$; for every $\eta > 0$ there exist $t_1, \ldots, t_m$ such that $\sum t_j = t$, and $\sum \vartheta(t_j) \le \operatorname{sub}\vartheta(t) + \eta$. We then define $s_j = t_j + \frac{s-t}{m}$, so that we have $\sum s_j = s$, and

$$\operatorname{sub}\vartheta(s) \le \sum \vartheta(s_j) \le \sum \vartheta(t_j) + L|t-s| \le \operatorname{sub}\vartheta(t) + L|t-s| + \eta.$$

This shows that $\operatorname{sub}\vartheta(s) \le \operatorname{sub}\vartheta(t) + L|t-s|$. In the same way we obtain $\operatorname{sub}\vartheta(t) \le \operatorname{sub}\vartheta(s) + L|t-s|$.

(ii) If ϑ is convex on $\mathbf{R} \setminus \{0\}$ then $\operatorname{sub}\vartheta$ can be computed more easily:

$$\operatorname{sub}\vartheta(x) = \inf\Big\{k\vartheta\Big(\frac{x}{k}\Big) : k = 1, 2, \ldots\Big\}.$$

This follows immediately by the convexity inequality $k\vartheta(\frac{y}{k}) \le \sum_{j=1}^{k} \vartheta(y_j)$, whenever $y = \sum_j y_j$. If ϑ is convex on $\mathbf{R}$ and $\vartheta(0) = 0$ then

$$\operatorname{sub}\vartheta(z) = \begin{cases} \vartheta'(0+)z & \text{if } z > 0 \\ \vartheta'(0-)z & \text{if } z < 0 \end{cases}$$

In particular, if $\vartheta'(0) = 0$ then $\operatorname{sub}\vartheta = 0$.

Example 5.19 (i) If we take $\vartheta(t) = 1 + t^2$, it can be immediately seen that ϑ is not subadditive (e.g. by Remark 5.15(ii)). Since ϑ is convex we have, by Remark 5.18(ii),

$$\operatorname{sub}\vartheta(t) = \min\Big\{k + \frac{t^2}{k} : k = 1, 2, \ldots\Big\}$$

(see Fig. 5.3). Note that ϑ is C^1 but not Lipschitz continuous, while $\operatorname{sub}\vartheta$ is Lipschitz continuous but not C^1. Note also that $\operatorname{sub}\vartheta$ is asymptotic to $2|t|$ as $t \to \pm\infty$.

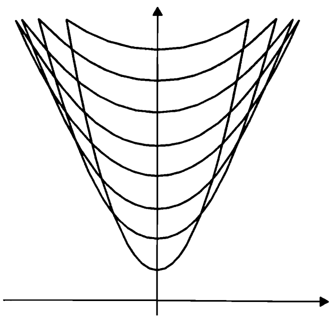

FIG. 5.3. Subadditive envelope of $1 + t^2$

(ii) If $\vartheta(t) = (2|t| - 1) \vee 1$, then $\operatorname{sub}\vartheta$ is even and continuous, and in $[0, +\infty)$ we have (see Fig. 5.4)

$$\operatorname{sub}\vartheta(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ k + 2(t-k) & \text{if } k \leq t \leq k + \frac{1}{2},\ k = 1, 2, \dots. \\ k & \text{if } k - \frac{1}{2} \leq t \leq k,\ k = 2, 3, \dots \end{cases}$$

In this case we have

$$|t| \leq \operatorname{sub}\vartheta(t) \leq |t| + \tfrac{1}{2}$$

for $|t| \geq \frac{1}{2}$. We have $\operatorname{sub}\vartheta(t) = |t|$ for $t = \pm 1, \pm 2, \dots$, $\operatorname{sub}\vartheta(t) = |t| + \frac{1}{2}$ for $t = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$, and hence $\operatorname{sub}\vartheta$ is not asymptotic to a linear function as $t \to \pm\infty$.

(iii) If $\vartheta(t) = |t - 1|$, then by Remark 5.18(ii) we have $\operatorname{sub}\vartheta(t) = \min\{|t-k| : k = 1, 2, \dots\}$; that is,

$$\operatorname{sub}\vartheta(t) = \begin{cases} 1 - t & \text{if } t \leq 1 \\ \operatorname{dist}(t, \mathbf{N}) & \text{if } t \geq 1 \end{cases}$$

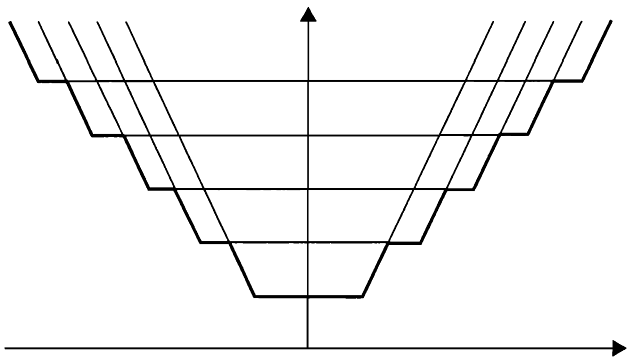

FIG. 5.4. Subadditive envelope of $\max\{2|z| - 1, 1\}$

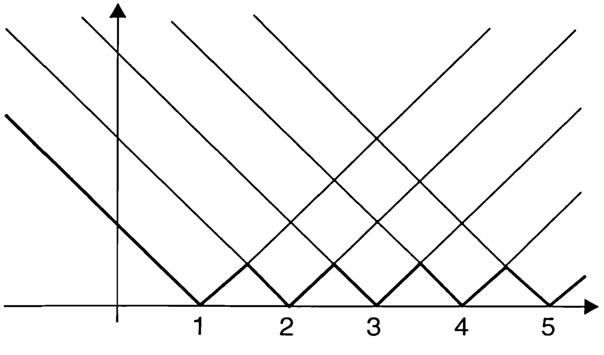

FIG. 5.5. Subadditive envelope of $|z - 1|$

(see Fig. 5.5). Notice that in this case the limit $\lim_{t\to+\infty} \operatorname{sub}\vartheta(t)$ does not exist.

(iv) If $\vartheta(t) = ||t|-1|$, then we have $\operatorname{sub}\vartheta(t) = \operatorname{dist}(t, \mathbf{Z})$. In fact, it can be easily checked that $\overline{\vartheta}(t) = \operatorname{dist}(t, \mathbf{Z})$ is subadditive, and hence $\overline{\vartheta}(t) \leq \operatorname{sub}\vartheta(t)$. Moreover $\operatorname{sub}\vartheta(t) = 0$ on $\mathbf{Z}$ (since $\operatorname{sub}\vartheta(0) \leq \vartheta(1)+\vartheta(-1) = 0$, and $\operatorname{sub}\vartheta(\pm k) \leq k\vartheta(\pm 1) = 0$, for $k = 1, 2, \ldots$), and hence by Remark 5.18 we have also $\operatorname{sub}\vartheta(t) \leq \overline{\vartheta}(t)$.

We now show that the lower-semicontinuous envelope for functionals is equivalent to the subadditive and lower-semicontinuous envelope for their 'integrands'.

Theorem 5.20 (relaxation of segmentation energies). *Let $\vartheta: \mathbf{R} \to [0, +\infty]$, and let $F : PC(a, b) \to [0, +\infty]$ be given by (5.9). Then the lower semicontinuous envelope of F with respect to convergence in measure is given by*

$$\operatorname{sc}F(u) = \sum_{t\in S(u)} \operatorname{sub}\vartheta(u(t+) - u(t-)), \tag{5.15}$$

on $PC(a, b)$, where $\operatorname{sub}\vartheta$ is the subadditive and lower semicontinuous envelope of ϑ. Moreover, $\operatorname{sc}F$ coincides with the lower semicontinuous envelope of F with respect to $L^1(a, b)$-convergence.

Proof Let Φ be given by the right hand-side of (5.15). As $\operatorname{sub}\vartheta$ is subadditive and l.s.c., and $\operatorname{sub}\vartheta \leq \vartheta$ we have $\Phi \leq \operatorname{sc}F$ by Theorem 5.8. Conversely, we have to check that for all $u \in PC(a, b)$ and for all $\varepsilon > 0$ there exist u_j converging to u in $L^1(a, b)$ such that $\Phi(u) \geq \liminf_j F(u_j) - \varepsilon$. We deal with the case

$$u(t) = \begin{cases} \alpha & t < t_0 \\ \beta & t \geq t_0 \end{cases}$$

only, the general case being easily dealt with by repeating the argument for this function. Let $z = \beta - \alpha$; by (5.13) there exist z_j^i such that

$$\operatorname{sub}\vartheta(z) \geq \sum_{i=1}^{N_j} \vartheta(z_j^i) - \varepsilon, \qquad \Big|\sum_{i=1}^{N_j} z_j^i - z\Big| < \frac{1}{j}.$$

Let $M_j \in \mathbf{N}$ be such that $\sup_i(N_j|z_j^i|/M_j) \leq 1/j$. Define

$$u_j(t) = \begin{cases} \alpha & t < t_0 \\ \alpha + \sum_{i=1}^{k} z_j^i & t_0 + \frac{k-1}{M_j} < t < t_0 + \frac{k}{M_j},\ k = 1, \ldots, N_j - 1 \\ \alpha + \sum_{i=1}^{N_j} z_j^i & t > t_0 + \frac{N_j - 1}{M_j}. \end{cases}$$

We easily get $\lim_j \int_{(a,b)} |u_j - u|\, dt = 0$ and

$$\lim_j F(u_j) = \limsup_j \sum_{i=1}^{N_j} \vartheta(z_j^i) \leq \operatorname{sub}\vartheta(z) + \varepsilon = \Phi(u) + \varepsilon,$$

as required. □

5.4.4 Γ-*convergence*

We now gather all the information we have obtained in the previous sections to state a general Γ-convergence result for translation-invariant functionals.

Theorem 5.21 (Γ-convergence of segmentation problems). *For all $j \in \mathbf{N}$ let $\vartheta_j : \mathbf{R} \to [0, +\infty]$, and let $F_j : PC(a,b) \to [0, +\infty]$ be given by*

$$F_j(u) = \sum_{t \in S(u)} \vartheta_j(u(t+) - u(t-)). \tag{5.16}$$

Then the sequence (F_j) Γ-converges to some functional F with respect to convergence in measure if and only if the sequence $(\operatorname{sub} \vartheta_j)$ *(extended by* $\operatorname{sub} \vartheta_j(0) = 0$*) Γ-converges in $\mathbf{R}$ to some function ϑ. In this case*

$$F(u) = \sum_{t \in S(u)} \vartheta(u(t+) - u(t-)). \tag{5.17}$$

Moreover, F coincides with the Γ-limit of (F_j) with respect to the $L^1(a,b)$-convergence.

Proof Note that by the relaxation theorem above, it suffices to check the case when the functions ϑ_j are lower semicontinuous and subadditive. Moreover, since Γ-convergence is compact, it suffices to check that if $\vartheta = \Gamma\text{-}\lim_j \vartheta_j$ in $\mathbf{R}$ then $F = \Gamma\text{-}\lim_j F_j$, with F given as above. The proof of the lower semicontinuity inequality can be easily obtained as in Theorem 5.8: let $u_j \to u$ in measure. Upon passing to a subsequence, we can suppose that $u_j \to u$ a.e. Let $t \in S(u)$. Choose $0 < \varepsilon < \inf\{|t-s| : t, s \in S(u), t \neq s\}/2$ such that $u_j(t \pm \varepsilon) \to u(t \pm \varepsilon) = u(t\pm)$, $t \pm \varepsilon \notin S(u_j)$ for all j, and $[t-\varepsilon, t+\varepsilon] \cap S(u) = \{t\}$. Then we have, by the subadditivity of ϑ_j and the lower semicontinuity inequality for the Γ-convergence of ϑ_j to ϑ,

$$\begin{aligned} \vartheta(u(t+) - u(t-)) &= \vartheta(u(t+\varepsilon) - u(t-\varepsilon)) \\ &\leq \liminf_j \vartheta_j(u_j(t+\varepsilon) - u(t-\varepsilon)) \\ &\leq \liminf_j \sum_{s \in S(u_j) \cap (t-\varepsilon, t+\varepsilon)} \vartheta_j(u_j(s+) - u_j(s-)). \end{aligned}$$

We have the thesis by summing up for $t \in S(u)$.

To prove the opposite inequality, let $u \in PC(a,b)$ with $S(u) = \{t_1, \dots, t_N\}$ $(t_i < t_{i+1})$, and let $z_i = u(t_i+) - u(t_i-)$; by the Γ-convergence of ϑ_j to ϑ there exists sequences (z_j^i) such that $z_j^i \to z_i$ and

$$\vartheta(z_i) = \lim_j \vartheta_j(z_j^i).$$

Define

$$u_j(t) = u(t) + \sum_{\{i:\ t_i<t\}} \Big(z_j^i - z_i\Big).$$

We have $u_j \to u$ in $L^1(a,b)$, and

$$\lim_j F_j(u_j) = \lim_j \sum_i \vartheta_j(z_j^i) = \sum_i \vartheta(z_i) = F(u),$$

as required. □

Remark 5.22 Clearly, we may have $\operatorname{sub}\vartheta_j \to \vartheta$ even if (ϑ_j) does not converge pointwise, or converges to ϑ' with $\operatorname{sub}\vartheta' \neq \vartheta$. A trivial example can be constructed as follows (see also Exercise 5.4). Let (z_j) be a dense sequence in $\mathbf{R}$ and let ϑ_j be defined on $\mathbf{R}\setminus\{0\}$ by

$$\vartheta_j(z) = \begin{cases} 1 & \text{if } z = z_k \text{ for some } k \geq j \\ 2 & \text{otherwise.} \end{cases}$$

Then $\operatorname{sub}\vartheta_j = 1$ but $\vartheta_j \to 2$.

5.4.5 *Boundary values*

We begin with an example to show how in general boundary values cannot be simply added to functionals.

Example 5.23 As a simple illustration consider the problem of the relaxation of the functional

$$F(u) = \sum_{S(u)} \vartheta(u^+ - u^-), \qquad \vartheta(z) = \begin{cases} 2 & \text{if } z \in \mathbf{Q}, \\ 1 & \text{otherwise} \end{cases}$$

on the space

$$V = \{u \in PC(a,b):\ u(a+) = u_a,\ u(b-) = u_b\}.$$

Clearly $\operatorname{sub}\vartheta = 1$, but the lower semicontinuous envelope of F is not $\#(S(u))$. In fact, if $u_b - u_a$ is rational and not zero then each function in V must have at least either one rational jump or two irrational ones, so that $F(u) \geq 2$ for all $u \in V$.

In the previous example the function ϑ was highly discontinuous. In the case of continuous ϑ we can easily describe the Γ-convergence of boundary value problems as follows.

Proposition 5.24 *Let $\vartheta_j : \mathbf{R} \to [0,+\infty)$ be a family of equiuniformly continuous functions such that* $\operatorname{sub}\vartheta_j$ *converges to ϑ. Then the Γ-limit of the functionals*

$$F_j(u) = \sum_{S(u)} \vartheta_j(u^+ - u^-)$$

defined on $V = \{u \in PC(a,b) : \ u(a+) = u_a, \ u(b-) = u_b\}$ *is described by the functional*

$$F(u) = \sum_{S(u)} \vartheta(u^+ - u^-) + \vartheta(u(a+) - u_a) + \vartheta(u_b - u(b-))$$

defined on $PC(a,b)$. *Note that this result is non-trivial also if* $\vartheta_j = \vartheta$ *and* ϑ *is subadditive.*

Proof For the sake of notational simplicity we treat the case of subadditive $\vartheta_j = \vartheta$ independent of j, which already bears all the interesting features, and whose proof does not differ significantly from the general case.

Let $u_j \to u$ with $u_j \in V$. Consider the extended functions

$$v_j = \begin{cases} u_a & \text{on } (-\infty, a] \\ u_j & \text{on } (a,b) \\ u_b & \text{on } [b, +\infty), \end{cases} \qquad v = \begin{cases} u_a & \text{on } (-\infty, a] \\ u & \text{on } (a,b) \\ u_b & \text{on } [b, +\infty). \end{cases}$$

Fix $(a', b') \supset [a,b]$. We then have $v_j \to v$ on (a', b'). By applying the lower semicontinuity theorem on the interval (a', b') we then get $F(u) \le \liminf_j F_j(u_j)$.

If $u \in PC(a,b) \setminus V$ then it suffices to choose as recovery sequence

$$u_j(t) = v\Big(\frac{b+a}{2} + \Big(t - \frac{b+a}{2}\Big)\Big(1 + \frac{1}{j}\Big)\Big).$$

If $u \in V$ then we trivially take $u_j = u$. □

5.5 Exercises

5.1 Find subadditive $\vartheta_1, \vartheta_2 : \mathbf{R} \to [0, +\infty]$ such that $\min\{\vartheta_1, \vartheta_2\}$ is not subadditive.

A solution: take $\vartheta_1(z) = \mathrm{dist}\,(z, \mathbf{Z})$ and $\vartheta_2(z) = \mathrm{dist}\,(z, \frac{3}{2}\mathbf{Z})$.

5.2 Prove that the functional defined on $PC(-1,1)$ by

$$F(u) = \sum_{t \in S(u)} \vartheta(t, u(t+) - u(t-)), \text{ where } \vartheta(t,z) = \begin{cases} 1 & \text{if } t \neq 0, \\ g(z) & \text{if } t = 0 \end{cases}$$

(g any lower semicontinuous function with $0 \le g \le 1$) is lower semicontinuous with respect to the $L^1(-1,1)$-convergence (Note that in this case it is not necessary to require that g is subadditive).

5.3 Let $\varphi : [0, +\infty) \to [0, +\infty)$ be convex with $\varphi(0) = 0$, let $\psi : [0, +\infty) \to [0, +\infty)$ be concave, and let $\vartheta(z) = \min\{\psi(|z|), \varphi(|z|)\}$. Compute $\mathrm{sub}\,\vartheta$.

Hint: (see Fig. 5.6)

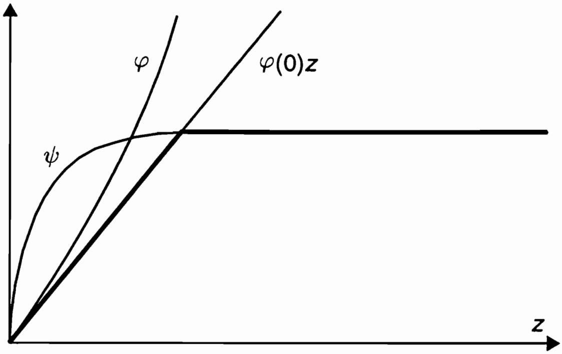

FIG. 5.6. The subadditive envelope in Exercise 5.3

5.4 Let ϑ_j be defined on $\mathbf{R}\setminus\{0\}$ by $\vartheta_j(z) = \begin{cases} 3 & \text{if } |z| \leq j \\ 1 & \text{otherwise.} \end{cases}$ Check that sub $\vartheta_j \to 2$ but $\vartheta_j \to 3$.

5.5 Describe the lower semicontinuous envelope of the functional F in Example 5.23.

5.6 (homogenization of segmentation problems). Compute the Γ-limit of (F_ε) as $\varepsilon \to 0+$, where $F_\varepsilon(u) = \sum_{t\in S(u)} a\left(\frac{t}{\varepsilon}\right)$ and $1 \leq a \leq 2$ is a periodic function.

Hint: the idea is that it is convenient to have discontinuities where $a(t/\varepsilon)$ is least, and that this set of t gets dense as $\varepsilon \to 0$. The limit F is then simply $F(u) = (\inf a)\,\#(S(u))$.

Comments on Chapter 5

Functions in $PC(a,b)$ (or, more precisely, their distributional derivatives) can be identified with atomic measures: sum of Dirac masses concentrated on the jump points with coefficient the size of the corresponding jump. In this sense, more jumps concurring in one point can be viewed as a weak* convergence of measures. Also, the subadditive condition can be viewed as a particular case of the conditions ensuring the lower semicontinuity of functionals defined on measures (see Bouchitté and Buttazzo (1993)). In the context of Fracture Mechanics, criteria for fracture initiation have been first given by Griffith (1920). The introduction of energy densities depending on the opening of the fracture is commonly traced back to Barenblatt (1962) and Dugdale. The interpretation of non-subadditive energy densities as a reason for micro cracking is proposed by Del Piero and Truskinovsky (2001).

Caccioppoli partitions Segmentation problems have a very simple formulation in dimension one. Their many-dimensional counterparts on the contrary involve a great deal of technical machinery: a (meaningful) piecewise-constant function on a set $\Omega \subset \mathbf{R}^N$ can be written as

$$u(x) = \sum_i c_i \chi_{E_i}(x),$$

where (E_i) is a Caccioppoli partition (i.e., a partition of Ω into sets of finite perimeter — see Appendix A) and c_i are constants. Integral functionals on these functions take the form

$$F(u) = \sum_{i \neq j} \int_{\partial^* E_i \cap \partial^* E_j} \varphi_{ij}(x, \nu_{ij})\, d\mathcal{H}^{N-1}$$

for suitable φ_{ij}, where ν_{ij} is the (suitably-defined) normal to $\partial^* E_i \cap \partial^* E_j$ (see Appendix A for notation). Minimum problems on Caccioppoli partitions in a variational framework have been studied, for example, by Congedo and Tamanini (1991). A complete study of functionals on these partitions when the c_i above are fixed and finite is performed by Ambrosio and Braides (1990) and Braides and Chiadò Piat (1996). In particular, there it is shown that the arguments leading to subadditivity can be adapted to the general case giving a new condition called *BV-ellipticity*, whose statement is very similar to quasiconvexity (see Chapter 12). Note that even if $\{c_i\} = \{0, 1\}$ (i.e. F are functionals on sets of finite perimeter) these energies are not trivial and the study of their structure has been useful for example in problems in statistical mechanics (see Bodineau *et al.* (2000)). In this case we have only one $\varphi = \varphi_{ij}$ and if $\varphi = \varphi(\nu)$ then the condition of BV-ellipticity simplifies to the convexity of (the positively homogeneous of degree one extension from S^{N-1} to $\mathbf{R}^N$ of) φ. The study of partition problems is a crucial part of the treatment of general free-discontinuity problems (see Chapter 7). Finally, it must be noted that we may consider also curvature-dependent functionals on sets of finite-perimeter, which exhibit interesting non-local phenomena (see Bellettini *et al.* (1993)) and are applied to image reconstruction problems (see Mumford (1993)).

6

PHASE-TRANSITION PROBLEMS

In this chapter we characterize the successive Γ-limits of a singular perturbation by a gradient term of a non-convex integral functional, of the type

$$\int_a^b (W(u) + \varepsilon^2 |u'|^2)\, dt,$$

showing how to obtain a segmentation problem by a development by Γ-convergence. In this way we will introduce the Γ-convergence approach to a classic problem in phase-transition theory; that is, to justify sharp discontinuities between two phases as limits of smooth interfaces (see Example 0.1). Conversely, the same process can be seen as an approximation procedure for some types of segmentation problems by 'smooth' problems, that is the basis of many other approximation results.

6.1 Phase transitions as segmentation problems

A particular case of the segmentation functionals treated in the previous chapter are those in which we add the constraint $u(t) \in Z$ a.e., where Z is some fixed set in $\mathbf{R}$. We will baptize this set as the *set of phases* of u. In some cases, especially when more than two phases are present, it is more natural to consider Z as a subset of $\mathbf{R}^k$. For the sake of simplicity we treat the case $k = 1$ only, but it must be kept in mind that most of the reasonings of this chapter still hold for general k (see also Remark 5.1). We may consider energies defined on functions satisfying the constraint $u \in Z$ a.e., of the form

$$F(u) = \sum_{t \in S(u)} \theta(u^-(t), u^+(t)) \qquad u \in PC(a, b),\ u(t) \in Z \text{ a.e.} \tag{6.1}$$

with $\theta : Z \times Z \to [0, +\infty]$ (with the condition $\theta(\alpha, \alpha) = 0$ for all $\alpha \in Z$ if needed). We may view this functional as defined on the whole $PC(a, b)$ simply by setting

$$\theta(\alpha, \beta) = +\infty \qquad \text{if } \alpha \notin Z \text{ or } \beta \notin Z, \tag{6.2}$$

so that with F defined by the formula in (6.1) on the whole $PC(a, b)$ we have $F(u) = +\infty$ if $u \in PC(a, b)$ does not satisfy $u \in Z$ a.e. To the functional F extended in this way we can apply the lower-semicontinuity Theorem 5.8. If Z is closed then the constraint $u \in Z$ a.e. is closed with respect to a.e. convergence and we see that the conditions on θ provided by Theorem 5.8 can be expressed as conditions on Z:

(i) θ is l.s.c. on $Z \times Z$

(ii) θ is subadditive on $Z \times Z$.

A particular case is when Z is *discrete* (in particular, if Z is finite), in which case the lower-semicontinuity condition is trivially satisfied.

6.2 Gradient theory for phase-transition problems

In this section we will approximate phase transition energies as above, when Z is finite, by 'standard' integral functionals. The heuristic idea is to construct a family of energies depending on a parameter ε such that the uniform boundedness of these energies implies in the limit as $\varepsilon \to 0$ both the constraint $u \in Z$ and that u is piecewise constant. In order to penalize the distance from Z we consider a (continuous) function $W : \mathbf{R} \to [0, +\infty)$ such that $Z = \{W = 0\}$, so that the 'closeness of u to Z' is quantitatively translated in the 'smallness' of $\int W(u)\, dt$. By introducing the small parameter ε, we can consider an energy of the form

$$\int_a^b \frac{W(u(t))}{\varepsilon}\, dt.$$

Even though the boundedness of this energy for ε small does imply that u is close to Z, it does not forbid u to 'oscillate' wildly between different values in Z. In order that u 'resembles' also a function in $PC(a,b)$ we must bound the number of these oscillations. A possible way to do this is to add a higher-order term containing the derivative of u. We are then led to considering an energy of the form

$$\int_a^b \frac{W(u(t))}{\varepsilon}\, dt + \varepsilon \int_a^b |u'(t)|^2\, dt.$$

The coefficient ε in front of the derivative term is explained as follows: consider an interval $(t, t+\delta)$ and suppose that at the endpoints of this interval u is close to two different elements of Z. On this interval the contribution of the first integral is then of order δ/ε, while the contribution of the second one is of order ε/δ (since the derivative will be of order $1/\delta$), so that

$$\int_t^{t+\delta} \frac{W(u(s))}{\varepsilon}\, ds + \varepsilon \int_t^{t+\delta} |u'(s)|^2\, ds \approx \frac{\delta}{\varepsilon} + \frac{\varepsilon}{\delta},$$

and this quantity is minimal (and positive!) when $\varepsilon = \delta$. This implies that if the energy is bounded then the number of such intervals is correspondingly bounded, so that u resembles a piecewise-constant function.

We will make this heuristic argument rigorous by using the language of Γ-convergence when, in addition, the function θ is *additive* on Z; that is,

$$\theta(\alpha, \beta) = \theta(\alpha, \gamma) + \theta(\gamma, \beta) \tag{6.3}$$

if $\alpha, \beta, \gamma \in Z$ and $\alpha < \gamma < \beta$. This result will follow clearly from Theorem 6.4.

Let $W : \mathbf{R} \to [0, +\infty)$ be a C^1 function such that the set $Z = \{W = 0\}$ is a finite set of points. Suppose moreover that

$$\limsup_{|s|\to+\infty} W(s) > 0 \tag{6.4}$$

(a condition of this type is to be imposed since otherwise our functionals are not coercive even on constant functions). We will consider the functionals F_ε defined on $W^{1,2}(a,b)$ by

$$F_\varepsilon(u) = \int_a^b \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt \tag{6.5}$$

and $F_\varepsilon(u) = +\infty$ if $u \notin C^1(a,b)$.

The first problem is the choice of the topology in which to frame our limit problem. We then have to examine the compactness properties of minimizing sequences for the functionals F_ε. Before giving a detailed proof of this result, we spend a few words describing the main idea: let (u_j) be a sequence with equibounded $F_{\varepsilon_j}(u_j)$. Since in particular $\int W(u_j)\,dt = O(\varepsilon_j)$ we deduce that the function u_j is 'close' to Z except for a set of measure $O(\varepsilon_j)$. To deduce that u_j is close to some piecewise-affine function it suffices to show that the number of 'transitions' of u_j between two different points of Z is bounded. To check this it suffices then to prove that each time we have such a transition we 'pay' in the energy at least a fixed price. The first thing is then to estimate this transition energy.

Remark 6.1 (formulas for the phase-transition energy density). With fixed j, let $s < t$. We want to estimate the contribution of the integration on (s,t) in $F_j(u_j)$ in terms of $w = u_j(s)$ and $z = u_j(t)$. We will first give a lower bound for this contribution by minimizing over all possible 'profiles' (that is, functions with the same boundary data) and eventually proving that this estimate is sharp by exhibiting an 'optimal profile'.

A first piece of information is obtained by a scaling argument by which we eliminate the dependence on ε_j of the energy density: we define

$$v(\tau) = u_j(\varepsilon_j\tau)$$

so that

$$\int_s^t \Big(\frac{W(u_j)}{\varepsilon_j} + \varepsilon_j|u_j'|^2\Big)d\tau = \int_{s/\varepsilon_j}^{t/\varepsilon_j} (W(v) + |v'|^2)d\tau.$$

If we set $T = T(\varepsilon_j) = (t-s)/2\varepsilon_j$ we have the estimate

$$\begin{aligned}&\int_s^t \Big(\frac{W(u_j)}{\varepsilon_j} + \varepsilon_j|u_j'|^2\Big)d\tau \qquad (6.6)\\ &\geq \inf\Big\{\int_{-T}^T (W(v) + |v'|^2)d\tau : v \in W^{1,2}(-T,T),\ v(-T) = w,\ v(T) = z\Big\}.\end{aligned}$$

We finally integrate out the dependence on T: if we set

$$\vartheta_1(w,z) \tag{6.7}$$
$$= \inf_{T>0} \inf\Big\{\int_{-T}^{T}(W(v)+|v'|^2)d\tau : v\in W^{1,2}(-T,T),\ v(-T)=w,\ v(T)=z\Big\},$$

then we have

$$\int_s^t\Big(\frac{W(u_j)}{\varepsilon_j}+\varepsilon_j|u_j'|^2\Big)d\tau \geq \vartheta_1(w,z).$$

This first estimate is based only on the observation of the scaling properties of F_{ε_j}, and can also be used for other types of problems. In our case it can be further simplified by using the simple algebraic inequality $x^2+y^2\geq 2xy$, which gives

$$\begin{aligned}\int_{-T}^{T}(W(v)+|v'|^2)d\tau &\geq 2\int_{-T}^{T}\sqrt{W(v(\tau))}|v'(\tau)|d\tau\\ &\geq \Big|2\int_{-T}^{T}\sqrt{W(v(\tau))}v(\tau)d\tau\Big|\\ &= 2\Big|\int_{v(-T)}^{v(T)}\sqrt{W(r)}\,dr\Big| = 2\Big|\int_w^z\sqrt{W(r)}\,dr\Big| \end{aligned}\tag{6.8}$$

for all T and all test functions v in the definition of ϑ_1 (we have also used the change of variables $r=v(\tau)$). Hence, if we set

$$\vartheta(w,z) = 2\Big|\int_w^z\sqrt{W(r)}\,dr\Big|, \tag{6.9}$$

we have

$$\vartheta_1(w,z)\geq\vartheta(w,z). \tag{6.10}$$

Note that by the continuity of W we have $\vartheta(w,z)\geq c$ for all $w,z\in Z$ with $w\neq z$. This inequality provides the main ingredient both for the argument of the equi-coerciveness of the sequence F_{ε_j} and for the lower bound of the Γ-limit.

Now, let $w,z\in Z$. In this case, another way to express an estimate on ϑ is by comparison with the energy density

$$\vartheta_2(w,z)$$

$$= \inf\Big\{\int_{-\infty}^{+\infty}(W(v)+|v'|^2)d\tau : v\in W^{1,2}_{\rm loc}(\mathbf{R}), v(-\infty)=w, v(+\infty)=z\Big\}, \tag{6.11}$$

where the values at $\pm\infty$ are understood as the existence of the corresponding limits. If no other point of Z falls between w and z, a particular test function for (6.11) is the solution v of the ordinary differential equation

$$\begin{cases} v' = \sqrt{W(v)} \\ v(0) = \dfrac{w+z}{2} \end{cases}$$

satisfying $v \in [w, z]$. (It is a good elementary exercise in ordinary differential equations to prove the existence of this solution on $\mathbf{R}$, for example, by using the local Lipschitz continuity of $\sqrt{W}$.) The choice for this particular test function is suggested by the remark that it is optimal for the first inequality in (6.8) since $v' > 0$ and we have

$$W(v(\tau)) + |v'(\tau)|^2 = 2\sqrt{W(v(\tau))}v'(\tau)$$

for all τ. Plugging v in (6.11) we get

$$\vartheta_2(w, z) \leq 2\Big|\int_{-\infty}^{+\infty} \sqrt{W(v(\tau))}v'(\tau)\,d\tau\Big| = \vartheta(w, z).$$

In the general case, this inequality is easily obtained by remarking that ϑ_2 is subadditive and ϑ is additive.

On the other hand, the same argument of (6.8) shows the converse inequality, so that in particular v realizes the minimum for $\vartheta_1(w, z)$. In this case, we can use v to test the minimum problem (6.6) for T large (upon slightly modifying v to match the boundary conditions), so that $\vartheta_2 \geq \vartheta_1$.

We conclude that we have the equalities

$$\vartheta(w, z) = \vartheta_1(w, z) = \vartheta_2(w, z) \tag{6.12}$$

for $w, z \in Z$.

Lemma 6.2 (equi-coerciveness) *If (ε_j) is a sequence of positive numbers converging to 0 and $\sup_j F_{\varepsilon_j}(u_j) < +\infty$ then there exists a subsequence of (u_j) converging in $L^1(a, b)$ to some function $u \in PC(a, b)$ which satisfies $u \in Z$ a.e.*

Proof First, note that u_j tends to Z in measure; that is, that for all $\eta > 0$ the measure of the set $I_j^\eta = \{t \in (a, b) : \text{dist}\,(u_j(t), Z) > \eta\}$ tends to 0 as $j \to +\infty$. In fact,

$$|I_j^\eta| \min\{W(s) : \text{dist}\,(u_j(t), Z) > \eta\} \leq \int_{I_j^\eta} W(u_j)\,ds \leq \varepsilon_j \sup_j F_{\varepsilon_j}(u_j).$$

With fixed $N \in \mathbf{N}$, let $x_N^i := a + i(b-a)/N$ for $i = 0, \ldots, N$. We show that, except for a number of indices independent of N, upon extracting a subsequence, the functions u_j converge in measure to a constant on $[x_N^i, x_N^{i+1}]$. To show this it suffices to show that the oscillation of u_j on $[x_N^i, x_N^{i+1}]$ is smaller than d, the minimal distance between points in Z given by

$$d = \min\{|x - y| : x, y \in Z, x \neq y\},$$

since already $u_j \to Z$ in measure.

Now, with fixed i and j, let $s,t \in [x_N^i, x_N^{i+1}]$ be such that $u_j(s) = \min\{u_j(\tau) : \tau \in [x_N^i, x_N^{i+1}]\}$ and $u_j(t) = \max\{u_j(\tau : \tau \in [x_N^i, x_N^{i+1}]\}$. By the previous remark we have

$$\int_s^t \Big(\frac{W(u_j)}{\varepsilon_j} + \varepsilon_j|u_j'|^2\Big)d\tau \geq \vartheta(u_j(s), u_j(t)).$$

Note that since $u \to Z$ in measure then for j sufficiently large we must have $u_j(s) < \max Z + 1$ and $u_j(t) > \min Z - 1$; that is, the pair $(u_j(s), u_j(t))$ belongs to the closed triangle

$$T = \{(x,y) \in \mathbf{R}^2 : x \leq y,\ y \leq \max Z + 1,\ x \geq \min Z - 1\},$$

so that the function

$$\omega(\rho) = \min\Big\{2\int_x^y \sqrt{W(s)}\,ds : (x,y) \in T,\ y - x = \rho\Big\},$$

is strictly positive for $\rho > 0$. Let

$$J_j = \Big\{i \in \{0,\dots,N\} : \rho\Big(\max_{[x_N^{i-1}, x_N^i]} u_j - \min_{[x_N^{i-1}, x_N^i]} u_j\Big) \geq \rho(d)\Big\},$$

which coincides with the set of indices i for which the oscillation of u_j on $[x_N^i, x_N^{i+1}]$ is larger than d. From (6.8) we then deduce that

$$\#(J_j) \leq F_{\varepsilon_j}(u_j),$$

is equibounded, independently of N, as desired. With fixed N, upon passing to a subsequence we can suppose that the set $J_j = J(N)$ itself is independent of j. We deduce then that on the complement of $\bigcup_{i\in J(N)}[x_N^i, x_N^{i+1}]$ the functions u_j converge in measure to a piecewise-constant function u. Since this reasoning is independent of N we conclude that the convergence is on the whole (a,b) and that $u \in PC(a,b)$. Finally, it can be easily seen that (u_j) is bounded in $L^\infty(a,b)$ and hence from convergence in measure, we deduce, up to a further subsequence, the L^1 convergence. □

Remark 6.3 By refining the proof of the lemma above we get that for every $\eta > 0$ there exists a finite set $S = S_\eta$ such that the oscillation of u_j is definitively less than η on each fixed compact subset of $(a,b) \setminus S$.

Now that we have checked the equi-coerciveness with respect to the $L^1(a,b)$ convergence, we can compute the Γ-limit with respect to this topology.

Theorem 6.4 *Let W and F_ε be defined as above. Then there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon$ with respect to the $L^1(a,b)$ convergence, and it equals the functional F defined on $L^1(a,b)$ by*

$$F(u) = \begin{cases} \displaystyle\sum_{S(u)} \vartheta(u^+, u^-) & \text{if } u \in PC(a,b) \text{ and } u \in Z \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases} \tag{6.13}$$

where

$$\vartheta(z,w) = \left|2\int_z^w \sqrt{W(s)}\,ds\right|. \tag{6.14}$$

Proof We first prove the liminf inequality; i.e, that if $u_\varepsilon \to u$ in $L^1(a,b)$ and $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ then $F(u) \le \liminf_\varepsilon F_\varepsilon(u_\varepsilon)$. By the coercivity property just proven we already have that $u \in PC(a,b)$ and $u \in Z$ a.e.

Now, let $a < s < t < b$ and $u_\varepsilon(s) \to w$, $u_\varepsilon(t) \to z$. We then have, as in inequality (6.8)

$$\int_s^t \Big(\frac{W(u_\varepsilon)}{\varepsilon} + \varepsilon|u'|^2\Big)\,d\tau \ge \left|2\int_{u_\varepsilon(s)}^{u_\varepsilon(t)} \sqrt{W(s)}\,ds\right|, \tag{6.15}$$

and the last term tends to $\vartheta(w,z)$. If we partition (a,b) into subintervals (s_i, s_{i+1}) containing at most one point of $S(u)$ and such that u_ε converges to u at all endpoints (except possibly the first and the last) and use the inequality above we easily deduce that $\lim_\varepsilon F_\varepsilon(u_\varepsilon) \ge F(u)$ as desired.

To check the limsup inequality for the Γ-limit, it will suffice to deal with the case

$$u = \begin{cases} w & \text{if } t < t_0 \\ z & \text{if } t \ge t_0. \end{cases}$$

Fix $\eta > 0$; by (6.12) there exist $T > 0$ and $v_T \in W^{1,2}(-T,T)$ such that $v(-T) = w$, $v(T) = z$, and

$$\int_{-T}^{T} \Big(W(v) + |v'|^2\Big)\,dt \le \theta(w,z) + \eta. \tag{6.16}$$

Then, a recovery sequence can be constructed by taking

$$u_\varepsilon(t) = \begin{cases} w & \text{if } t < t_0 - \varepsilon T \\ v_T(\varepsilon t) & \text{if } t_0 - \varepsilon T \le t \le t_0 + \varepsilon T \\ z & \text{if } t > t_0 + \varepsilon T. \end{cases}$$

Then, we have

$$\begin{aligned} \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \int_{-T}^{T} \Big(W(v) + |v'|^2\Big)\,dt \\ &\le \theta(w,z) + \eta \le F(u) + \eta. \end{aligned}$$

Since this construction modifies the target function u only on a small neighbourhood of the discontinuity set $S(u)$ it can be repeated for an arbitrary $u \in PC(a,b)$ with $u \in Z$ a.e. □

Remark 6.5 We can consider $W : \mathbf{R} \to [0,+\infty)$ vanishing only at 0 and 1. In this case the functional F is finite only on characteristic functions of a finite number of segments contained in (a,b), and on these functions we have

$$F(u) = c_W\,\#(S(u)),$$

where $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$.

6.3 Gradient theory as a development by Γ-convergence

We may state the Γ-convergence result of the gradient theory of phase transitions as a development by Γ-convergence of a perturbation of a non-convex energy. Let

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_a^b \Big(W(u) + \varepsilon^2 |u'|^2\Big)\,dt & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise,} \end{cases} \tag{6.17}$$

where $W : \mathbf{R} \to [0,+\infty)$ is such that $\{W=0\} = \{0,1\}$. For the sake of simplicity we also suppose that W satisfy a 2-growth condition

$$c_1 s^2 - c_2 \le W(s) \le c_3(1+s^2) \tag{6.18}$$

for all $s \in \mathbf{R}$. We leave to the reader the generalization to more general growth conditions.

From the growth conditions on W a priori we only know that F_ε are equi-coercive with respect to the weak convergence of $L^2(a,b)$, and hence we compute their Γ-limit with respect to that convergence. By the Relaxation Theorem 2.18 if we set

$$F(u) = \int_a^b W^{**}(u)\,dt \tag{6.19}$$

on $L^2(a,b)$, we have $F = \Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon$ with respect to the weak L^2 convergence. If fact, the lim inf inequality is trivial, while if $u \in W^{1,2}(a,b)$ and we take as recovery sequence $u_\varepsilon = u$ we get that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u) \le \int_a^b W(u)\,dt$$

on $W^{1,2}(a,b)$. By density the same inequality holds on $L^2(a,b)$. The desired inequality follows by taking the lower semicontinuous envelope of both sides.

This Γ-convergence result is stable by adding a 'volume constraint' as in the following proposition. In the case of phase transitions this is a natural constraint since it prescribes the volume of the phases.

Proposition 6.6 *Let $d \in \mathbf{R}$ and let F^d_ε be defined on $L^2(a,b)$ by*

$$F^d_\varepsilon(u) = \begin{cases} F_\varepsilon(u) & \textit{if } u \in W^{1,2}(a,b) \textit{ and } \int_a^b u\,dt = d(b-a) \\ +\infty & \textit{otherwise.} \end{cases} \tag{6.20}$$

Then the Γ-limit of F^d_ε with respect to the weak L^2 convergence is

$$F^d(u) = \begin{cases} F(u) & \textit{if } u \in L^2(a,b) \textit{ and } \int_a^b u\,dt = d(b-a) \\ +\infty & \textit{otherwise.} \end{cases} \tag{6.21}$$

on $L^2(a,b)$.

Proof Since the constraint $\int_a^b u\,dt = d(b-a)$ is closed for the weak L^2 convergence the liminf inequality is trivial. To check the limsup inequality, let $u \in L^2(a,b)$ satisfy the integral constraint, and let $v_\varepsilon \in W^{1,2}(a,b)$ be such that $F(u) = \lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon)$, and $v_\varepsilon \to u$ in $L^2(a,b)$. In particular,

$$d_\varepsilon(b-a) =: \int_a^b v_\varepsilon\,dt \to \int_a^b u\,dt = d(b-a). \tag{6.22}$$

To obtain a recovery sequence for our problem we have to modify v_ε in order to satisfy the constraint, and still have the same limit. We can modify v_ε only on $(b-\varepsilon, b)$ by setting

$$u_\varepsilon(t) = \begin{cases} v_\varepsilon(t) & \text{if } a < t \le b-\varepsilon \\ v_\varepsilon(t) + \frac{2}{\varepsilon}(d-d_\varepsilon)(t-b+\varepsilon) & \text{if } b-\varepsilon < t \le b. \end{cases}$$

Note that $u_\varepsilon \to u$ in $L^2(a,b)$, $\int_a^b u_\varepsilon\,dt = d(b-a)$, and that

$$\lim_{\varepsilon\to 0+} \int_a^b W(u_\varepsilon)\,dt = \lim_{\varepsilon\to 0+} \int_a^b W(v_\varepsilon)\,dt.$$

Moreover,

$$\begin{aligned} &\lim_{\varepsilon\to 0+} \varepsilon^2 \Big| \int_a^b (|v_\varepsilon'|^2 - |u_\varepsilon'|^2)\,dt \Big| \\ &= \lim_{\varepsilon\to 0+} \varepsilon^2 \Big| \int_{b-\varepsilon}^b (|v_\varepsilon'|^2 - |u_\varepsilon'|^2)\,dt \Big| \\ &\le \lim_{\varepsilon\to 0+} \varepsilon^2 \int_{b-\varepsilon}^b \Big(\frac{2}{\varepsilon}|v_\varepsilon'||d-d_\varepsilon| + \frac{4}{\varepsilon^2}|d-d_\varepsilon|^2\Big)\,dt = 0, \end{aligned}$$

which shows that (u_ε) is a recovery sequence. □

We are interested now in the description of problems of the form

$$m_\varepsilon = \min\Big\{ \int_a^b (W(v) + \varepsilon^2|v'|^2)\,dt : \ v \in W^{1,2}(a,b), \ \int_a^b v\,dt = d(b-a) \Big\} \tag{6.23}$$

with $0 < d < 1$. Note that for such values of d the Γ-limit provides little information, since the corresponding limit problem

$$m = \min\Big\{ \int_a^b W^{**}(v)\,dt : \ v \in L^2(a,b) \ \int_a^b v\,dt = d(b-a) \Big\} \tag{6.24}$$

gives $m = 0$ and all functions u satisfying the integral constraint and such that $0 \le u \le 1$ a.e. are minimizers.

We can study the *first-order* Γ-limit of F^d_ε; that is, the Γ-limit of

$$G^d_\varepsilon(u) = \frac{F^d_\varepsilon(u) - m}{\varepsilon}; \tag{6.25}$$

that is, of

$$G^d_\varepsilon(u) = \begin{cases} \displaystyle\int_a^b \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt & \text{if } u \in W^{1,2}(a,b) \text{ and } \int_a^b u\,dt = d(b-a) \\ +\infty & \text{otherwise.} \end{cases} \tag{6.26}$$

The Γ-limit of these functionals is again compatible with the integral constraint.

Theorem 6.7 *The functionals G^d_ε as above Γ-converge with respect to the weak L^2 convergence to the functional*

$$F^{(1)}(u) = \begin{cases} c_W\,\#(S(u)) & \textit{if } u \in PC(a,b),\ u \in \{0,1\} \textit{ a.e. and } \int_a^b u\,dt = d(b-a) \\ +\infty & \textit{otherwise,} \end{cases} \tag{6.27}$$

where $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$. The convergence takes place also with respect to the strong L^2 convergence.

Proof Since sequences with bounded energy are precompact in $L^1(a,b)$ by Theorem 6.4 the lim inf inequality it trivial. We have to construct a recovery sequence for $u \in PC(a,b)$ with $u \in \{0,1\}$ a.e. and $\int_a^b u\,dt = d(b-a)$. It suffices to deal with the case

$$u(t) = \begin{cases} 0 & \text{if } t \le t_0 \\ 1 & \text{if } t > t_0. \end{cases}$$

By the proof of Theorem 6.4 for all fixed $\eta > 0$ we find $T > 0$ and a sequence $v_\varepsilon \to u$ of the form $v_\varepsilon(t) = v((t-t_0)/\varepsilon)$ such that $v_\varepsilon(t) = u(t)$ if $|t - t_0| \ge \varepsilon$ and

$$\lim_{\varepsilon\to 0+} \int_a^b \Big(\frac{W(v_\varepsilon)}{\varepsilon} + \varepsilon|v'_\varepsilon|^2\Big)\,dt \le c_W + \eta.$$

Let $d_\varepsilon(b-a) = \int_a^b u_\varepsilon\,dt$. The desired recovery sequence (u_ε) is obtained by setting $u_\varepsilon(t) = v(t - t_\varepsilon)$, where $t_\varepsilon = t_0 + d_\varepsilon - d$. □

We finally deduce the convergence of minimum problems.

Theorem 6.8 *Let u_ε be a minimizer of* (6.23). *Then, upon extraction of a subsequence u_ε converges in $L^2(a,b)$ to a function u which minimizes both* (6.23) *and*

$$\min\Big\{\#(S(v)) :\ v \in PC(a,b),\ v \in \{0,1\},\ \int_a^b v\,dt = d(b-a)\Big\}. \tag{6.28}$$

Proof The convergence result immediately follows from the general Theorem 1.47 and the compactness Lemma 6.2. □

Remark 6.9 We can adapt this result to the study of problems in a general situation when the non-convex energy density does not necessarily have many minima. More precisely, the convergence result above can be used to study the behaviour of minimum problems

$$\tilde{m}_\varepsilon = \min\Big\{\int_a^b (f(v) + \varepsilon^2|v'|^2)\,dt:\ v \in W^{1,2}(a,b),\ \int_a^b v\,dt = d(b-a)\Big\}, \tag{6.29}$$

where f is a C^1 non-convex energy density satisfying a 2-growth condition and d is such that $f^{**}(d) < f(d)$ (see Example 0.1). Note that there exist c_1 and c_2 such that $W(s) = f(s) - c_1 s - c_2$ is a non-negative function with minimum value 0. Suppose for the sake of simplicity that 0 is attained precisely at two points. Upon a change of variables we can suppose these two points to be 0 and 1. Since

$$\tilde{m}_\varepsilon = \min_{v\in W^{1,2}(a,b)}\Big\{\int_a^b (W(v) + \varepsilon^2|v'|^2)\,dt:\ \int_a^b v\,dt = d(b-a)\Big\} \\ +(b-a)(c_1 d + c_2),$$

we can apply then the corollary above to obtain that these minimum values converge to

$$\tilde{m} = \min\Big\{\int_a^b f^{**}(v)\,dt:\ v \in L^2(a,b)\ \int_a^b v\,dt = d(b-a)\Big\} = f^{**}(d). \tag{6.30}$$

Moreover, upon extraction of subsequences, minimizers (u_ε) converge to minimizers both of (6.30) and of (6.28).

Comments on Chapter 6

The result presented in this chapter can be generalized to n-dimensional energies of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \varepsilon\int_\Omega |Du|^2\,dx\,, \tag{6.31}$$

with $\Omega \subset \mathbf{R}^n$, by a slicing procedure (see Chapter 15). It is essentially due to Modica and Mortola (1977); its interpretation as a gradient theory for phase transitions can be found in Modica (1987). The approach as a development by Γ-convergence is presented by Anzellotti and Baldo (1993).

This is a very interesting example of a result that can be generalized in many ways, in some cases providing different results and rising difficult technical issues. For example, the gradient term can be substituted by a non-local difference term obtaining functionals of the form (here $\Omega = \mathbf{R}^n$)

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbf{R}^n} W(u)\,dx + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{1}{\varepsilon^{n+1}} J\Big(\frac{x-y}{\varepsilon}\Big)(u(x)-u(y))^2\,dy\,dx.$$

with J a positive interaction potential (see Alberti and Bellettini (1998)), and this result can be applied to the study of *Ising systems* with Kac potentials (see Alberti *et al.* (1996)). The nonlocal term above can itself be seen as a particular case of a two-parameter energy of the form

$$F_\varepsilon(u,v) = \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \frac{\alpha}{\varepsilon}\int_\Omega (u-v)^2\,dx + \varepsilon\int_\Omega |Dv|^2\,dx$$

where the effect of v has been integrated out (see Solci and Vitali (2001), Ren and Truskinovsky (2001)). Note that the case $\alpha = +\infty$ gives the original Modica and Mortola functional. Furthermore, we can consider vector-valued u (see e.g. Baldo (1990)) or non-isotropic perturbations (see e.g. Sternberg (1988)).

In a different direction, the first gradient can be substituted by higher-order gradients (see Fonseca and Mantegazza (2000)), considering, for example, (in the one-dimensional formulation)

$$F_\varepsilon(u) = \int_a^b \Big(\frac{1}{\varepsilon}W(u) + \varepsilon^3|u''|^2\Big)\,dt.$$

Note that in this case the optimal-profile problem cannot be simplified by using the Modica and Mortola 'trick' as in (6.8).

Also higher-order problems involving functionals of the form

$$F_\varepsilon(u) = \int_\Omega \Big(\frac{1}{\varepsilon}W(Du) + \varepsilon|D^2u|^2\Big)\,dx$$

can be addressed (here $D^2u = D(Du)$ denotes the tensor of the second derivatives). Note that the gradient constraint forbids to directly reduce to a vector problem by setting $v = Du$. In this case, the determination of the Γ-limit is technically much more complex and in part still open (see Conti *et al.* (2002b) for two-well W and u vector-valued, Aviles and Giga (1999), Ortiz and Gioia (1994), Ambrosio *et al.* (1999), De Simone *et al.* (2001) for the case $W(z) = (|z|-1)^2)$ and u scalar).

This theory can be combined with other geometric or physical models, obtaining interesting applications, for example, in ferromagnetism (see Anzellotti *et al.* (1991)), to capillarity phenomena (see Alberti *et al.* (1998)) or to functionals on sets of finite perimeter (see Braides and Malchiodi (2002)).

An important variation is the case of the *Ginzburg Landau energies*, in which we take $u : \Omega \to \mathbf{R}^2$ and W vanishing on $\{|u| = 1\}$ in (6.31). In this case, by making a different scaling we obtain a non-trivial Γ-limit, whose domain is the set of functions $u : \Omega \to S^1$ with singularities of codimension two (see Alberti (2001)). For a study of minimizers of such energies if $\Omega \subset \mathbf{R}^2$ we refer to the book of Bethuel *et al.* (1994).

7

FREE-DISCONTINUITY PROBLEMS

In this chapter we consider minimization problems for functionals whose natural domains are sets of functions which admit a finite number of discontinuities. The set of these discontinuities will be an unknown of the problems, and for this reason they will be called *free-discontinuity problems.* In their treatment we combine the theories for integral functionals and for segmentation energies. The main issue of this chapter is to show that for a class of free-discontinuity energies the integral part and the segmentation part can be in a sense 'decoupled'.

7.1 Piecewise-Sobolev functions

To have a precise statement of free-discontinuity problems, it will be useful to define some spaces of piecewise weakly-differentiable functions.

Definition 7.1 *Let* $1 \leq p \leq +\infty$. *We define the space* $P\text{-}W^{1,p}(a,b)$ *of* piecewise-$W^{1,p}$ functions *on the bounded interval* (a,b) *as the sum*

$$P\text{-}W^{1,p}(a,b) = W^{1,p}(a,b) + PC(a,b), \tag{7.1}$$

that is, $u \in P\text{-}W^{1,p}(a,b)$ *if and only if* $v \in W^{1,p}(a,b)$ *and* $w \in PC(a,b)$ *exist such that* $u = v + w$. *Note that* $W^{1,p}(a,b) \cap PC(a,b)$ *equals the set of constant functions, so that* u *and* v *are uniquely determined up to an additive constant. The function* u *inherits the notation valid for* v *and* w*; namely, we define the* jump set *of* u *and the* weak derivative *of* u *as*

$$S(u) = S(w) \qquad \text{and} \qquad u' = v', \tag{7.2}$$

respectively. Moreover, the left- and right-hand side values *of* u *are defined by*

$$u^{\pm}(x) = v(x) + w^{\pm}(x). \tag{7.3}$$

Remark 7.2 Clearly, $u \in P\text{-}W^{1,p}(a,b)$ if and only if there exist $a = t_0 < t_1 < \ldots < t_N = b$ such that $u \in W^{1,p}(t_{i-1}, t_i)$ for all $i = 1, \ldots, N$. With this definition $S(u)$ is interpreted as the minimal of such sets of points, and $u \in L^2(a,b)$ is defined piecewise on $(a,b) \setminus S(u)$.

7.2 Some model problems

Even though the treatment of minimization problems for functionals defined on $P\text{-}W^{1,p}(a,b)$ with $p > 1$ will be easily dealt with by combining the results that we have already proved for functionals defined on Sobolev functions and on piecewise-constant functions we illustrate their importance with two examples.

7.2.1 *Signal reconstruction: the Mumford–Shah functional*

As for functionals defined on piecewise-constant functions a model for signal reconstruction can be introduced using piecewise-Sobolev functions. Mumford and Shah proposed a model which can be translated in dimension one in the following (see Fig. 7.1): Given a datum g (the *distorted signal*) recover the original piecewise-smooth signal u by solving the problem

$$\min\Big\{c_1\int_{(a,b)}|u'|^2\,dt+c_2\#(S(u))+c_3\int_{(a,b)}|u-g|^2\,dt:\ u\in P\text{-}W^{1,2}(a,b)\Big\}. \quad (7.4)$$

The parameters c_1, c_2, c_3 are *tuning parameters.* A large c_1 penalizes high gradients (in a sense, we can regard the corresponding segmentation problem in Chapter 2, where functions with non-zero gradients are not allowed, as that in (7.4) with $c_1 = +\infty$), a large c_2 forbids the introduction of too many discontinuity points (over-segmentation), and c_3 controls the fidelity of u to g.

7.2.2 *Fracture mechanics: the Griffith functional*

A simple approach to some problems in the mechanics of brittle solids is that proposed by Griffith, which can be stated more or less like this: Each time a crack is created, an energy is spent proportional to the area of the fracture site. We consider as an example that of a brittle elastic bar subject to a forced displacement at its ends, so that volume integrals become line integrals and surface discontinuities turn into jumps. In this case, if g denotes the external body forces acting on the bar, the deformation u of the bar at equilibrium will solve the following problem:

$$\min\Big\{\int_{(a,b)}f(u')\,dt+\lambda\#(S(u))-\int_{(a,b)}gu\,dt:$$
$$u(a)=u_a,\ u(b)=u_b,\ u^+>u^-\text{ on }S(u)\Big\}. \quad (7.5)$$

on the space of functions $u\in P\text{-}W^{1,p}(a,b)$, for some $p>1$. The function f represents the elastic response of the bar in the unfractured region, while the

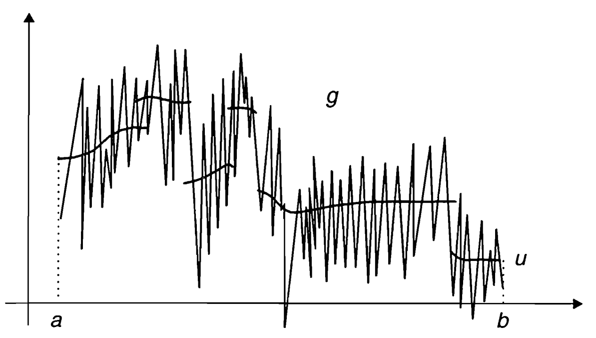

FIG. 7.1. Piecewise-smooth segmentation

condition $u^+ > u^-$ derives from the impenetrability of matter.

7.3 Functionals on piecewise-Sobolev functions

We consider energies on $P\text{-}W^{1,p}(a,b)$ of the form

$$F(u) = \int_{(a,b)} f(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-). \tag{7.6}$$

Lower-semicontinuity and coerciveness properties for such functionals will easily follow from the corresponding properties on $W^{1,p}(a,b)$ and $PC(a,b)$.

Theorem 7.3 *Let $p > 1$.*

(i) (coerciveness) *If (u_j) is a sequence in $P\text{-}W^{1,p}(a,b)$ such that*

$$\sup_j \Big(\int_{(a,b)} |u_j'|^p\,dt + \#(S(u_j)) \Big) < +\infty \tag{7.7}$$

and for all open sets $I \subset (a,b)$ we have $\liminf_j \inf_I |u_j| < +\infty$, then there exists a subsequence of (u_j) (not relabelled) converging in measure to some $u \in P\text{-}W^{1,p}(a,b)$. Moreover, we can write $u_j = v_j + w_j$ with $v_j \in W^{1,p}(a,b)$ and $w_j \in PC(a,b)$, with v_j weakly converging in $W^{1,p}(a,b)$ and w_j converging in measure.

(ii) (lower semicontinuity) *If $f : \mathbf{R} \to [0 + \infty]$ is convex and lower semicontinuous, and if $\vartheta : \mathbf{R} \to [0 + \infty]$ is subadditive and lower semicontinuous then the functional F defined in* (7.6) *is lower semicontinuous on $P\text{-}W^{1,p}(a,b)$ with respect to convergence in measure along sequences (u_j) satisfying* (7.7).

Proof (i) Let $v_j \in W^{1,p}(a,b)$ be defined by

$$v_j(t) = \int_a^t u_j'(s)\,ds.$$

Since $v_j(a) = 0$ for all j, the sequence (v_j) is bounded in $W^{1,p}(a,b)$ by Poincaré's inequality, and hence we can extract a weakly converging subsequence (that we still denote by (v_j)) that weakly converges to some v in $W^{1,p}(a,b)$. Now, set $w_j = u_j - v_j \in PC(a,b)$. Since $v_j \to v$ in $L^\infty(a,b)$ the sequence (w_j) satisfies the hypotheses of Proposition 5.3, so that, upon extracting a subsequence, it converges in measure to some $w \in PC(a,b)$. The sequence (u_j) satisfies the required properties with $u = v + w$.

(ii) Let (u_j) satisfy (7.7) and $u_j \to u$ in measure. Then by (i) we can write $u_j = v_j + w_j$ with $v_j \in W^{1,p}(a,b)$ and $w_j \in PC(a,b)$, $v_j \to v$ weakly in $W^{1,p}(a,b)$ and $w_j \to w \in PC(a,b)$ in measure. By Proposition 2.13 and Theorem 5.8 we then get

$$F(u) = F(v) + F(w) \le \liminf_j F(v_j) + \liminf_j F(w_j) \le \liminf_j F(u_j)$$

as desired. □

Corollary 7.4 *Let $f, \vartheta : \mathbf{R} \to [0, +\infty]$ be functions satisfying*

$$c|z|^p \le f(z) \qquad \text{and} \qquad c \le \vartheta(z) \tag{7.8}$$

for all $z \in \mathbf{R}$, then the functional F defined in (7.6) *is lower semicontinuous on P-$W^{1,p}(a,b)$ with respect to the convergence in measure if and only if f is convex and lower semicontinuous and ϑ is subadditive and lower semicontinuous.*

Proof Let F be lower semicontinuous. Then also its restrictions to $W^{1,p}(a,b)$ and to $PC(a,b)$ are lower semicontinuous; hence, we deduce that f is convex and lower semicontinuous and ϑ is subadditive and lower semicontinuous by Proposition 2.11 and Theorem 5.8. The converse is a immediate consequence of Theorem 7.3. □

Remark 7.5 (non-convexity of free-discontinuity energies). It is apparent that the functionals introduced above are never convex, except for trivial cases. It is instructive to check the lack of convexity of the Mumford–Shah functional

$$F(u) = c_1 \int_{(a,b)} |u'|^2\, dt + c_2 \#(S(u))$$

by testing it, for example, on the functions $u_1 = \chi_{[t_1,+\infty)}$ and $u_2 = \chi_{[t_2,+\infty)}$, with $t_1, t_2 \in (a,b)$:

$$2c_2 = F\Big(\frac{u_1+u_2}{2}\Big) > \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2) = c_2.$$

7.4 Examples of existence results

As examples of an application of the lower semicontinuity theorems on the space P-$W^{1,p}(a,b)$ we prove the existence of solutions for the problems outlined in Section 7.2.

Example 7.6 (existence for Image Reconstruction problems). We use the notation of Section 7.2.1. Let $g \in L^2(a,b)$ and let (u_j) be a minimizing sequence for the problem

$$m = \inf\Big\{F(u) + c_3 \int_{(a,b)} |u-g|^2\, dt : \; u \in P\text{-}W^{1,2}(a,b)\Big\}, \tag{7.9}$$

where

$$F(u) = c_1 \int_{(a,b)} |u'|^2\, dt + c_2 \#(S(u)).$$

By taking $u = 0$ as a test function, we obtain that $m \le \int_{(a,b)} |g|^2\, dt$. Moreover, we immediately get that (u_j) is bounded in $L^2(a,b)$; hence, it satisfies the hypotheses of Theorem 7.3(i). Thus we can suppose that $u_j \to u \in P$-$W^{1,p}(a,b)$ in measure and a.e., so that by Theorem 7.3(ii) (with $p = 2$, $f(z) = |z|^2$ and $\vartheta(z) = 1$) $F(u) \le \liminf_j F(u_j)$, and by Fatou's Lemma

$$\int_{(a,b)} |u - g|^2\, dt \le \liminf_j \int_{(a,b)} |u_j - g|^2\, dt,$$

so that u is a minimum point for (7.9).

Example 7.7 (existence for problems in Fracture Mechanics). We use the notation of Section 7.2.2. In this case we may have to specify the boundary conditions better, as $S(u)$ may tend to a or b; that is, the elastic bar may break at its ends. In view of Theorem 5.24, the minimization problem with relaxed boundary conditions takes the form

$$m = \inf\Big\{F(u) - \int_{(a,b)} gu\, dt + \vartheta(u(a+) - u_a) + \vartheta(u_b - u(b-)) : \ u \in P\text{-}W^{1,p}(a,b)\Big\}, \tag{7.10}$$

where

$$F(u) = \int_{(a,b)} f(u')\, dt + \lambda\, \#(S(u)),$$

f is some convex function, which we suppose satisfies $f(z) \ge |z|^p - c$, and ϑ is defined by

$$\vartheta(z) = \begin{cases} +\infty & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ 1 & \text{if } z > 0. \end{cases}$$

Note that this definition of ϑ takes care of the impenetrability condition, which needs not be repeated in the statement of the minimum problem in the form (7.10).

We deal with the case $u_b > u_a$, and suppose $f(0) = 0$ and $\lambda = 1$. We may use $u = (u_a + u_b)/2$ as a test function, obtaining

$$m \le F(u) + \vartheta(u(a+) - u_a) + \vartheta(u_b - u(b-)) = 2\,\vartheta\Big(\frac{u_b - u_a}{2}\Big) = 2.$$

Let (u_j) be a minimizing sequence for (7.10). We set $u_j = v_j + w_j$ with $v_j \in W^{1,p}(a,b)$, $w_j \in PC(a,b)$ and $v_j(a) = 0$. By the Poincarè inequality and the continuous imbedding of $W^{1,p}(a,b)$ into $L^\infty(a,b)$ we obtain that

$$||v_j||_{L^\infty(a,b)} \le c||v_j'||_{L^p(a,b)}. \tag{7.11}$$

Note that the condition $u_j^+ > u_j^-$ implies that w_j is increasing, so that

$$||w_j||_{L^\infty(a,b)} \le |u_a| + |u_b| + c||v_j'||_{L^p(a,b)}. \tag{7.12}$$

From the condition

$$\int_{(a,b)} f(u_j')\,dt + \sum_{S(u)} \vartheta(u_j^+ - u_j^-) - \int_{(a,b)} g u_j\,dt \le c$$

in particular we then get

$$\int_{(a,b)} |v_j'|^p\,dt - \int_{(a,b)} g v_j\,dt - \int_{(a,b)} g w_j\,dt \le c,$$

from which by (7.11) and (7.12) we deduce

$$\int_{(a,b)} |v_j'|^p\,dt - c\|v_j\|_{L^\infty(a,b)} - c\|w_j\|_{L^\infty(a,b)} \le c$$

and, from the inequalities above, eventually

$$\int_{(a,b)} |v_j'|^p\,dt \le c.$$

Hence, we may assume that v_j weakly converge in $W^{1,p}(a,b)$, and by (7.12) we obtain that (w_j) is a bounded sequence in $L^\infty(a,b)$. Hence (u_j) satisfies the assumptions of Theorem 7.3(i), so that we may assume that it converges to u in measure. Moreover, we may assume that w_j converges a.e. and in $L^1(a,b)$, so that we get that u is a minimum point for (7.10) by using Theorem 7.3(ii).

Comments on Chapter 7

'Free-discontinuity problems' is a more general terminology introduced by De Giorgi to denote variational problems on an open set $\Omega \subset \mathbf{R}^n$ where the unknown is a pair (u, K), where K is a closed set of measure 0 in Ω (typically, a union of closed hypersurfaces) and u is a (in general $\mathbf{R}^m$-valued) function defined on $\Omega \setminus K$. Free-discontinuity problems in an n-dimensional setting take into account energies of the general form

$$\int_{\Omega\setminus K} f(x,u,Du)\,dx + \int_K \varphi(x,u^+,u^-,\nu_K)\,d\mathcal{H}^{n-1}, \qquad (7.13)$$

with a *bulk* and an *interfacial* part. Here, $u^\pm$ denote the traces of u on both sides of K and ν_K is the normal to K. The prototype of such energies is the *Mumford–Shah functional* of Computer Vision

$$\int_{\Omega\setminus K} |Du|^2\,dx + \alpha\mathcal{H}^1(K),$$

in $\Omega \subset \mathbf{R}^2$ with scalar u (see Mumford and Shah (1989), Morel and Solimini (1995)). In this case K is a union of closed curves and $\mathcal{H}^1(K)$ is its total length.

In the one-dimensional case K is a finite set, the second integral in (7.13) is a sum, and we can take $\nu_K = 1$, so that the dependence on ν_K disappears.

Functional as in (7.6) are the energies of this form which are invariant under translations, and can also be seen as a particular case of functionals defined on measures. The complete characterization of Γ-convergence in the one-dimensional case for energies of the general form is given by Amar and Braides (1995). There it is shown that the 'principle' of separation between the integral and the segmentation parts of the energy for functionals of the general form does not hold, even if the coerciveness condition (7.7) is satisfied.

Special functions of bounded variation In the n-dimensional case free-discontinuity problems admit a weak formulation in the space of *special functions of bounded variation* $SBV(\Omega;\mathbf{R}^m)$ introduced by Ambrosio and De Giorgi, which are those functions of bounded variation whose distributional derivative can be written as the sum of a n-dimensional measure and a $n-1$-dimensional measure. The monograph of Ambrosio *et al.* (2000) contains a complete study of this space; a simplified introduction can be found in Braides (1998). The key property of these functions is that they are a closed class under coerciveness conditions analog to (7.7). Even though in higher dimensions it is not true that a special function of bounded variation on Ω can be decomposed as a sum of a Sobolev function and a piecewise-constant function (corresponding to a partition into sets of finite perimeter) the 'principle' of separation between the integral and the segmentation parts can be often transposed to the n-dimensional case. To this end, an interesting result by Cortesani and Toader (1999) shows that in a sense piecewise-affine (non-continuous) functions are 'strongly dense' (in the spirit of Remark 1.29) in the space of special functions of bounded variation (see Braides and Chiadò Piat (1996) for the same type of result for 'piecewise-Sobolev' functions).

If $u:\Omega\to\mathbf{R}^n$ then the energies in (7.13) may depend on the symmetric part of the gradient. In this case $SBV(\Omega;\mathbf{R}^n)$ is not suited as a framework for these energies, and the space of *special functions of bounded deformation* $SBD(\Omega)$ must be used (see Ambrosio *et al.* (1997). In this space it is possible to rephrase some issues of the *Griffith's theory of fracture* (see Griffith (1920)) in variational terms.

8

APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS

The treatment of free-discontinuity problems is, under some aspects (e.g., numerical approximation), rendered somewhat complex by the presence of two competing terms of different nature, the integral and the segmentation part. In this chapter we discuss the approximation of energies defined on piecewise-Sobolev functions by other types of functionals, which are under some aspects easier to handle. We will only treat the case when the 'target' energy is the Mumford–Shah functional

$$E(u) = \alpha \int_a^b |u'|^2 \, dt + \beta \#(S(u)) \tag{8.1}$$

with $\alpha, \beta > 0$, and we will give three answers to this question, using approximations with integral functionals, convolution energies and non-convex discrete energies, respectively.

The heuristic idea will be to treat discontinuities as degenerate gradients, or points where functions have a very steep slope. Note that a naïve approach would be to try an approximation by means of integral functionals, of the form

$$\int_a^b f_\varepsilon(u') \, dt \,, \tag{8.2}$$

defined in the Sobolev space $W^{1,2}(a, b)$. It is clear, though, that if an approximation worked by functionals of this form, then the functional E would also be the Γ-limit of their lower semicontinuous envelopes; that is, of the convex functionals

$$\int_a^b f_\varepsilon^{**}(u') \, dt \,, \tag{8.3}$$

and then should be convex, in contrast with the lack of convexity of E. We then have to resort to other types of energies.

8.1 The Ambrosio Tortorelli approximation

Following the ideas that lead to the approximation of segmentation energies, we can try to separately approximate the segmentation part of the energy E as in the gradient theory of phase transitions. We will prove that a possible approximating family that uses an auxiliary variable v is the following:

$$G_\varepsilon(u,v) = \alpha \int_a^b v^2|u'|^2\,dt + \frac{\beta}{2}\int_a^b \Big(\varepsilon|v'|^2 + \frac{1}{\varepsilon}(1-v)^2\Big)\,dt\,, \tag{8.4}$$

defined on functions $u, v \in W^{1,2}(a,b)$, which Γ-converges as $\varepsilon \to 0+$ with respect to the $(L^1(a,b))^2$-topology to the functional

$$G(u,v) = \begin{cases} E(u) & \text{if } v = 1 \text{ a.e. on } (a,b) \text{ and } u \in P\text{-}W^{1,2}(a,b) \\ +\infty & \text{otherwise,} \end{cases} \tag{8.5}$$

defined on $(L^1(a,b))^2$. Clearly, the functional G is equivalent to E as far as minimum problems are concerned.

The heuristic idea is that as $\varepsilon \to 0$ the term $\int_a^b \frac{1}{\varepsilon}(1-v)^2\,dt$ forces v to be 1 a.e., while the term $\int_a^b v^2|u'|^2\,dt$ forces v to be 0 on $S(u)$. The combined terms $\int_a^b(\varepsilon|v'|^2 + \frac{1}{\varepsilon}(1-v)^2)\,dt$ give a fixed contribution each time v passes from 0 to 1 as in the gradient theory of phase transitions. In this way, we get that the Γ-limit's domain is precisely (equivalent to) $P\text{-}W^{1,2}(a,b)$ and optimal $(u_\varepsilon, v_\varepsilon)$ are such that v_ε approach $1-\chi_{S(u)}$.

Theorem 8.1 *The functionals $G_\varepsilon : L^1(a,b) \times L^1(a,b) \to [0,+\infty]$ defined by* (8.4), *extended to $+\infty$ outside $W^{1,2}(a,b) \times W^{1,2}(a,b)$, Γ-converge as $\varepsilon \to 0^+$ to the functional $G : L^1(a,b) \times L^1(a,b) \to [0,+\infty]$, defined by* (8.5).

Proof For the sake of notation we define $G_\varepsilon(u,v,I)$ and $G(u,v,I)$ if $I \subset (a,b)$ as follows:

$$G_\varepsilon(u,v,I) = \begin{cases} \displaystyle\int_I \Big(v^2|u'|^2 + \frac{1}{\varepsilon}(1-v)^2 + \varepsilon|v'|^2\Big)\,dt & \text{if } u,v \in W^{1,2}(a,b) \\ +\infty & \text{otherwise,} \end{cases} \tag{8.6}$$

$$G(u,v,I) = \begin{cases} \alpha \displaystyle\int_I |u'|^2\,dt + \beta\,\#(S(u)\cap I) & \text{if } u \in P\text{-}W^{1,2}(a,b) \text{ and } v = 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \tag{8.7}$$

We check the lower semicontinuity inequality for the Γ-limit. Let $\varepsilon_j \to 0^+$, $u_j \to u$ and $v_j \to v$ in $L^1(a,b)$. Up to subsequences we can suppose that also $u_j \to u$ and $v_j \to v$ a.e., and that there exists the $\lim_j G_{\varepsilon_j}(u_j,v_j) < +\infty$.

It is clear that we must have $v = 1$ a.e., since otherwise $\int_a^b (1-v_j)^2\,dt$ does not tend to 0, and $G_{\varepsilon_j}(u_j,v_j) \to +\infty$. We note that we can apply Lemma 6.2 and Remark 6.3 with $Z = \{1\}$ and $W(s) = (1-s)^2$ to the sequence (v_j). We deduce that there exists a finite set S such that for every fixed open set I compactly contained in $(a,b)\setminus S$ we have definitively $1/2 < v_j < 3/2$ on I. For every such fixed I from the estimate

$$\frac{1}{2}\sup_j \int_I |u_j'|^2 \le \sup_j \int_a^b v_j^2|u_j'|^2\,dt < +\infty,$$

we deduce that $u \in W^{1,2}(I)$, $u_j \rightharpoonup u$ in $W^{1,2}(I)$, and, since $v_j \to 1$ in $L^2(I)$,

$$\int_I |u'|^2\,dt \le \liminf_j \int_I v_j^2 |u_j'|^2\,dt \le \liminf_j \int_a^b v_j^2|u_j'|^2\,dt. \tag{8.8}$$

Since this estimate is independent of I we deduce that $u \in P\text{-}W^{1,2}(a,b)$ and $S(u) \subset S$.

Let $t \in S(u)$. Then there exist t_j^1, t_j^2, s_j such that $t_j^1 < s_j < t_j^2$,

$$\lim_j t_j^1 = \lim_j t_j^2 = \lim_j s_j = t, \qquad \lim_j v_j(t_j^1) = \lim_j v_j(t_j^2) = 1$$

(by the convergence a.e. of v_j to 1) and $\lim_j v_j(s_j) = 0$ (otherwise, as above we would deduce that there exist a neighbourhood I of t such that $u \in W^{1,2}(I)$ and $t \notin S(u)$). As in (6.8), we then deduce that

$$\liminf_j \int_{t_j^1}^{s_j} \Big(\frac{1}{\varepsilon}(1-v_j)^2 + \varepsilon^2|v_j'|^2\Big)dt \ge \int_0^1 (1-s)\,ds = \frac{1}{2}$$

$$\liminf_j \int_{s_j}^{t_j^2} \Big(\frac{1}{\varepsilon}(1-v_j)^2 + \varepsilon^2|v_j'|^2\Big)dt \ge \int_0^1 (1-s)\,ds = \frac{1}{2}. \tag{8.9}$$

Repeating this reasoning for all $t \in S(u)$ and taking into account the arbitrariness of I in (8.8) we deduce the liminf inequality.

We now turn to the construction of a recovery sequence. It suffices to consider the case $(a,b) = (-1,1)$, $u \in P\text{-}W^{1,2}(-1,1)$, and $S(u) = \{0\}$. Choose $\xi_\varepsilon = o(\varepsilon)$, and let $u_\varepsilon \in W^{1,2}(-1,1)$ with $u_\varepsilon(t) = u(t)$ if $|t| > \xi_\varepsilon$. With fixed $\eta > 0$, let $T > 0$ and $v \in W^{1,2}(0,T)$ be such that

$$\int_0^T ((1-v)^2 + |v'|^2)\,dt \le 1+\eta,$$

$v(0) = 0$, $v(T) = 1$; we set

$$v_\varepsilon(t) = \begin{cases} 0 & \text{if } |t| \le \xi_\varepsilon \\ v\Big(\dfrac{|t|-\xi_\varepsilon}{\varepsilon}\Big) & \text{if } \xi_\varepsilon < |t| < \xi_\varepsilon + \varepsilon T \\ 1 & \text{if } |t| \ge \xi_\varepsilon + \varepsilon T\,. \end{cases}$$

(see Fig. 8.1) We then get

$$\begin{aligned} G_\varepsilon(u_\varepsilon, v_\varepsilon) &= \int_{-1}^1 \Big(\alpha v_\varepsilon^2|u'|^2 + \frac{\beta}{2\varepsilon}(1-v_\varepsilon)^2 + \frac{\beta}{2}\varepsilon|v_\varepsilon'|^2\Big)dt \\ &\le \int_{-1}^1 \Big(\alpha|u'|^2 + \frac{\beta}{2\varepsilon}(1-v_\varepsilon)^2 + \frac{\beta}{2}\varepsilon|v_\varepsilon'|^2\Big)dt \\ &\le \alpha\int_{-1}^1 |u'|^2\,dt + \beta + \beta\eta + 2\beta\frac{\xi_\varepsilon}{\varepsilon}, \end{aligned}$$

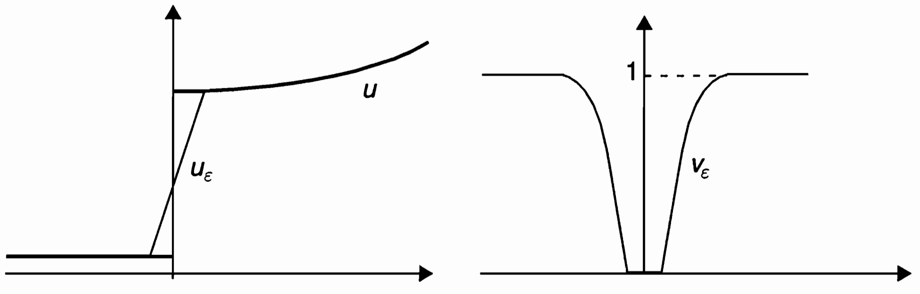

FIG. 8.1. Recovery sequences

so that

$$\limsup_{\varepsilon\to 0+} G_\varepsilon(u_\varepsilon, v_\varepsilon) \le \alpha \int_{-1}^{1} |u'|^2\,dt + \beta + \beta\eta\,.$$

By the arbitrariness of η we conclude the proof. □

Remark 8.2 (approximate minimum problems). The functionals G_ε are not coercive on $W^{1,2}(a,b)\times W^{1,2}(a,b)$ since for fixed v the first term may not satisfy a growth condition from below. We can slightly modify these functionals by adding a term of the form $k_\varepsilon \int_a^b |u'|^2\,dt$ with $k_\varepsilon > 0$, which clearly makes these functionals coercive. If $k_\varepsilon = o(\varepsilon)$ then the Γ-limit remains unchanged. The only thing to do is to check is the existence of recovery sequence (we can repeat the proof of the theorem and take $\xi_\varepsilon = \sqrt{k_\varepsilon \varepsilon}$ and u_ε affine in $(-\xi_\varepsilon, \xi_\varepsilon)$). In this case if $g \in L^2(a,b)$ then the problem

$$\min\Big\{G_\varepsilon(u,v) + k_\varepsilon \int_a^b |u'|^2\,dt + \int_a^b |u-g|^2\,dt : u, v \in W^{1,2}(a,b)\Big\}$$

admits a minimizing pair $(u_\varepsilon, v_\varepsilon)$ by following the direct methods.

8.2 Approximation by convolution problems

We go back for a moment to the considerations at the beginning of the chapter, where we have ruled out approximation by integral functionals. We will try and fix the 'convexity constraint' that prevents this type of energies to work.

We first examine energies of the form (8.2). Heuristically, we would like that a recovery sequence (u_ε) looked like the target function u outside an ε-neighbourhood of $S(u)$, and then have a steep slope (with derivative of order $(u^+ - u^-)/2\varepsilon$) inside this neighbourhood. Plugging such u_ε into the equality $\lim_{\varepsilon\to 0+} \int_a^b f_\varepsilon(u'_\varepsilon)\,dt = \alpha \int_a^b |u'|^2 dt + \beta\#(S(u))$, we get the conditions

$$\lim_{\varepsilon\to 0+} f_\varepsilon(z) = z^2, \qquad \lim_{\varepsilon\to 0+} 2\varepsilon\, f_\varepsilon\Big(\frac{w}{2\varepsilon}\Big) = \beta$$

for all $z, w \in \mathbf{R}$; a simple family (f_ε) that satisfies such conditions is obtained by a scaling procedure, choosing

$$f_\varepsilon(z) = \frac{1}{\varepsilon}\varphi(\varepsilon z^2), \text{ where } \lim_{s\to 0+}\frac{\varphi(s)}{s} = \alpha, \qquad \lim_{s\to+\infty}\varphi(s) = \frac{\beta}{2}. \tag{8.10}$$

For example, we can simply take $\varphi(s) = (\alpha s) \wedge (\beta/2)$, so that f_ε is as in Fig. 8.2.

Unfortunately, there is no *length scale* that forces the 'transition layer' between u^- and u^+ to be of size proportional to ε, and u_ε to be of the form above. A way to have such a length scale built in the energies is to substitute u' by some convolution with a mollifier of support of size of order ε.

8.2.1 *Convolution integral functionals*

Not all variational problems satisfy the convexity and subadditivity conditions for energies defined on Sobolev or piecewise-Sobolev functions examined up to this point. One way to get rid of these structure restrictions can be to consider non-local functionals of convolution type on the gradient; that is, energies of the form

$$F(u) = \int_{(a,b)} f(|u'|^p * \rho)\, dt, \tag{8.11}$$

defined on $W^{1,p}(a,b)$, where ρ is a positive mollifier and u' is extended outside (a,b) so as to meet the requirements of the problem.

Example 8.3 (non-local damage in elastic materials). We consider a bar described by a displacement u and suppose that the behaviour of the bar is purely (linearly) elastic for small vales of $|u'|$, while it obeys to a non-local damage law at a point x when the average value of $|u'|$ exceeds a certain constant L in a neighbourhood of x of a certain radius δ. We can model this behaviour by introducing a non-local energy; for example,

$$G(u) = \int_{(a,b)} |u'|^2\, dt - \int_{(a,b)} g\Big(\int_{x-\delta}^{x+\delta} |u'|^2\, dt\Big)\, dx,$$

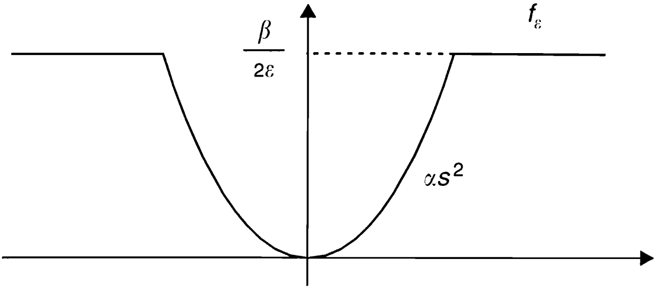

FIG. 8.2. The functions f_ε

where g is a non-decreasing function such that $g = 0$ on $(0, L)$, and u' is extended by 0 outside (a, b). The second integral is a convolution functional with $\rho = \chi_{(-\delta,\delta)}$.

The lower semicontinuity theorem for convolution functionals is particularly simple.

Theorem 8.4 *Let ρ be a positive function (with compact support) and f a non-decreasing lower semicontinuous function. Let F be defined by (8.11) on functions $u \in W^{1,p}_{\rm loc}(\mathbf{R})$ such that $u' = 0$ on $\mathbf{R} \setminus (a, b)$. Then F is lower semicontinuous with respect to the weak convergence in $W^{1,p}_{\rm loc}(\mathbf{R})$. If in addition $f \geq c|z|$ and $\rho > 0$ in a neighbourhood of 0 then F is also coercive with respect to the weak convergence in $W^{1,p}_{\rm loc}(\mathbf{R})$ on bounded subsets of $L^p_{\rm loc}(\mathbf{R})$.*

Proof We first prove the lower semicontinuity of F. Let $u_j \to u$ in $L^p_{\rm loc}(\mathbf{R})$. Then, for all fixed s we have

$$U(s) := \int_a^b \rho(s-t)|u'(t)|^p\,dt \leq \liminf_j \int_a^b \rho(s-t)|u_j'(t)|^p\,dt =: \liminf_j U_j(s).$$

Hence, by Fatou's Lemma and the monotonicity and lower semicontinuity of f, we get

$$\begin{aligned} F(u) &= \int_{(a,b)} f(U(s))\,ds \leq \int_{(a,b)} f\Big(\liminf_j U_j(s)\Big)\,ds \\ &\leq \int_{(a,b)} \liminf_j f(U_j(s))\,ds \leq \liminf_j \int_{(a,b)} f(U_j(s))\,ds = \liminf_j F(u_j). \end{aligned}$$

To prove the last statement, we may assume that $\rho \geq \chi_{(-\delta,\delta)}$. Hence, as $f(z) \geq c|z|$, we have

$$\begin{aligned} F(u) &\geq c\int_a^b \int_a^b \chi_{(-\delta,\delta)}(s-t)|u'(t)|^p\,dt\,ds \\ &\geq c\int_a^b \int_a^b \chi_{(-\delta,\delta)}(s-t)|u'(t)|^p\,dt\,ds \\ &= c\int_a^b \int_a^b \chi_{(t-\delta,t+\delta)}(s)\,ds|u'(t)|^p\,dt \geq c\delta \int_a^b |u'(t)|^p\,dt. \end{aligned}$$

This inequality gives the required coerciveness by Remark 2.26. □

8.2.2 *Limits of convolution functionals*

The convolution functionals considered above may lose their non-local character if we let the convolution kernels localize to a Dirac mass. In this case, we may obtain, as a Γ-limit, a functional defined on piecewise-Sobolev functions. We explicitly treat a model case only that leads to functionals of the Mumford–Shah type.

Theorem 8.5 *Let $\rho : \mathbf{R} \to [0, +\infty)$ be an even function, decreasing on $[0, +\infty)$, with* $\operatorname{spt}\rho = [-1, 1]$ *and* $\int_{\mathbf{R}} \rho\, dt = 1$. *Let $f(z) = |z| \wedge 1$ and let $p > 1$. For all $\varepsilon > 0$ let*

$$F_\varepsilon(u) = \frac{1}{\varepsilon} \int_{(a,b)} f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big) |u'(t)|^p\, dt\Big)\, dx$$

be defined on $W^{1,p}(a,b)$ with the convention that $u' = 0$ outside (a,b). Then the Γ-limit of F_ε with respect to the $L^1(a,b)$-convergence is finite on $P\text{-}W^{1,p}(a,b)$ and it equals

$$F(u) = \int_{(a,b)} |u'|^p\, dt + 2\#(S(u)).$$

Note that if we set $\rho_\varepsilon(s) = \frac{1}{\varepsilon}\rho\big(\frac{s}{\varepsilon}\big)$ and $f_\varepsilon(s) = \frac{1}{\varepsilon} f(\varepsilon s)$ then

$$F_\varepsilon(u) = \int_{(a,b)} f_\varepsilon(\rho_\varepsilon * |u'|^p)\, dt.$$

This form can be compared with (8.2) and (8.10).

Proof *Step 1* We first prove that if $u_j \to u$ in $L^1(a,b)$ and $\liminf_j F_{\varepsilon_j}(u_j) < +\infty$, then $u \in P\text{-}W^{1,p}(a,b)$. Let $\delta > 0$ and let $I_\delta = \{\rho > \delta\} =: [-\eta, \eta]$. Let

$$z_j^\delta(x) = \begin{cases} \displaystyle\int_{(a,b)\cap(x-\eta,x+\eta)} \rho\Big(\frac{x-t}{\varepsilon_j}\Big) |u_j'|^p\, dt & \text{if } x \in (a,b) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi_j(x) = \sum_{k \in \mathbf{Z}} f(z_j^\delta(2k\varepsilon_j\eta + x)).$$

We then have

$$F_{\varepsilon_j}(u_j) \geq \frac{1}{\varepsilon_j} \int_{\mathbf{R}} f(z_j^\delta(x))\, dx = \frac{1}{\varepsilon_j} \int_{-\varepsilon_j\eta}^{\varepsilon_j\eta} \sum_{k\in\mathbf{Z}} f(z_j^\delta(2k\varepsilon_j\eta + x))\, dx$$
$$= \frac{1}{\varepsilon_j} \int_{-\varepsilon_j\eta}^{\varepsilon_j\eta} \Phi_j(x)\, dx \geq 2\eta \Phi_j(x_j),$$

for some $x_j \in (-\varepsilon_j\eta, \varepsilon_j\eta)$. Note that the sum is performed only on the finite set I_j of k such that $x_j \pm \varepsilon_j\eta \in (a,b)$. Upon a small translation, we can assume $t_j = 0$.

Set

$$I_j^1 = \{k \in I_j :\ z_j^\delta(2k\varepsilon_j\eta + x) \leq 1\}, \qquad I_j^2 = I_j \setminus I_j^2.$$

We then have

$$F_{\varepsilon_j}(u_j) \geq \sum_{k \in I_j} 2\eta f(z_j^\delta(2k\varepsilon_j\eta + x))$$

$$= \sum_{k\in I_j^1} \int_{x-\varepsilon_j\eta}^{x+\varepsilon_j\eta} \rho\Big(\frac{2k\varepsilon_j\eta - t}{\varepsilon_j}\Big) |u_j'(t)|^p\,dt + 2\eta\,\#(I_j^2)$$

$$\geq 2\eta\delta \int_{(a,b)\setminus(I_j^2+[-\varepsilon_j\eta,\varepsilon_j\eta])} |u_j'|^p\,dt + 2\eta\,\#(I_j^2)$$

This inequality shows that $\#(I_j^2)$ is equibounded, so that we may assume it converges to a finite set S. Moreover, (u_j) converges to u weakly in $W^{1,p}((a,b)\setminus S)$; hence, $u \in P\text{-}W^{1,p}(a,b)$.

Step 2 By repeating the previous reasoning we obtain that, if $u_j \to u$, for all open intervals $I \subset (a,b)$ we have

$$\liminf_j \frac{1}{\varepsilon_j} \int_I f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big) |u_j'(t)|^p\,dt\Big)\,dx \geq 2\eta\delta \int_I |u_j'|^p\,dt + 2\eta\,\#(S(u)\cap I).$$

Step 3 Let $N \in \mathbf{N}$ and

$$\rho(x) = \sum_{m=-N+1}^{N} c_m \chi_{(\frac{m-1}{N},\frac{m}{N})}(x).$$

Let I be an interval, $I \subset\subset (a,b)$, where $u_j \rightharpoonup u$ weakly in $W^{1,p}(I)$. We then have (taking z_j constructed as in Step 1 with $1/N$ in place of $[-\eta,\eta]$),

$$\begin{aligned}
&\int_I f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big) |u_j'(t)|^p\,dt\Big)\,dx \\
&= \frac{1}{\varepsilon_j}\frac{1}{N} \sum_{m=-N+1}^{N} F_{\varepsilon_j}\Big(u_j\Big(\cdot + \frac{m}{N}\varepsilon_j\Big)\Big) \\
&= \frac{1}{N\varepsilon_j} \sum_{m=-N+1}^{N} \sum_{k\in\mathbf{Z}} \int_{-\varepsilon_j}^{\varepsilon_j} f\Big(z_j(2k\varepsilon_j + \frac{m}{N}\varepsilon_j + x)\Big)\,dx \\
&\geq \frac{1}{N} \sum_{m=-N+1}^{N} \sum_{k\in\mathbf{Z}} f\Big(z_j(2k\varepsilon_j + \frac{m}{N}\varepsilon_j)\Big) \\
&= \sum_{m=-N+1}^{N} \sum_{k\in I_j} \sum_{i=-N+1}^{N} \frac{c_i}{N} \int_{(2k-1)\varepsilon_j+\frac{m}{N}\varepsilon_j}^{(2k+1)\varepsilon_j+\frac{m}{N}\varepsilon_j} |u_j'|^p\,dt \\
&= \int_{-1}^{1} \rho(t)\,dt \int_I |u_j'|^p\,dt.
\end{aligned}$$

Step 4 Let now ρ be as in the statement of the theorem. By approximation from below with convolution kernels of the type as in the previous step, we obtain that

$$\liminf_j \int_I f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big) |u_j'(t)|^p\,dt\Big)\,dx \geq \int_I |u'|^p\,dt$$

for all intervals where $u_j \rightharpoonup u$ in $W^{1,p}(I)$.

Step 5 By letting $\eta \to 1$ in Step 2 and by the arbitrariness of I in Step 3, we obtain that

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \int_a^b |u'|^p\,dt + 2\#(S(u)).$$

Step 6 The proof of the limsup inequality when u is piecewise $W^{1,\infty}$ can be easily obtained by taking $u_\varepsilon = u$. In fact, set

$$U_\varepsilon(x) = \int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big)|u'(t)|^p\,dt$$

and $I_\varepsilon = \{x \in (a,b) :\ \mathrm{dist}\,(x, S(u)) > \varepsilon\}$. For ε small enough, we have $U_\varepsilon \leq 1$ on I_ε, so that

$$\begin{aligned}
F_\varepsilon(u) &= \frac{1}{\varepsilon}\int_{I_\varepsilon} f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big)|u'(t)|^p\,dt\Big)\,dx \\
&\quad + \frac{1}{\varepsilon}\int_{(a,b)\setminus I_\varepsilon} f\Big(\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big)|u'(t)|^p\,dt\Big)\,dx \\
&= \frac{1}{\varepsilon}\int_{I_\varepsilon} U^\varepsilon(x)\,dx + \frac{1}{\varepsilon}\int_{(a,b)\setminus I_\varepsilon} f(U^\varepsilon(x))\,dx \\
&\leq \frac{1}{\varepsilon}\int_{I_\varepsilon}\int_{(a,b)} \rho\Big(\frac{x-t}{\varepsilon}\Big)|u'(t)|^p\,dt\,dx + \frac{1}{\varepsilon}|(a,b)\setminus I_\varepsilon| \\
&\leq \int_{(a,b)} |u'(t)|^p\,dt + 2\#(S(u)).
\end{aligned}$$

The limsup inequality in the general case follows by approximation. □

Remark 8.6 In the case of a general ρ with compact support and $f(s) = (\alpha s) \wedge \beta$, the Γ-limit is

$$F(u) = c_{1,\rho}\alpha \int_a^b |u'|^p\,dt + c_{2,\rho}\beta\,\#(S(u)), \tag{8.12}$$

where $c_{1,\rho} = \int_{\mathbf{R}} \rho\,dt$ and $c_{2,\rho} = |\mathrm{spt}\,\rho|$.

Example 8.7 (fracture as a limit of non-local damage energies). We can apply the previous remark to a family of energies of the type in Example 8.3, as the 'non-local damage radius' goes to 0,

$$G_\varepsilon(u) = \int_{(a,b)} |u'|^2\,dt - \frac{1}{\varepsilon}\int_{(a,b)} g\Big(\varepsilon\int_{x-\varepsilon}^{x+\varepsilon} |u'|^2\,dt\Big)\,dx,$$

where $g(s) = (s-L)^+$. In fact, upon a vanishing error, we have

$$G_\varepsilon(u) = \frac{1}{\varepsilon}\int_{(a,b)} \varphi\Big(\varepsilon\int_{x-\varepsilon}^{x+\varepsilon} |u'|^2\,dt\Big)\,dx,$$

where $\varphi(s) = s \wedge L$, so that by (8.12) the Γ-limit equals

$$F(u) = 2\int_a^b |u'|^2\,dt + 2L\,\#(S(u)),$$

since $\rho = \chi_{(-1,1)}$.

8.3 Finite-difference approximation

A simpler way to introduce a length scale is by directly considering discrete energies with underlying lattice of step size ε. In the notation of discrete systems introduced in Chapter 4 we can take $\varepsilon = \lambda_n$ and

$$E_n(u) = \sum_{i=0}^{n-1} f\Big(\frac{(u_{i+1}-u_i)^2}{\lambda_n}\Big), \tag{8.13}$$

where again $f(z) = z \wedge 1$.

Theorem 8.8 *The functionals E_n Γ-converge on $P\text{-}W^{1,1}(a,b)$ to the functional*

$$F(u) = \int_a^b |u'|^2\,dt + \#(S(u)) \tag{8.14}$$

with respect to convergence in measure and convergence in $L^1(a,b)$.

Proof Let (u_n) be such that $\sup_n E_n(u_n) < +\infty$. Note that, if

$$I_n = \{i : (u_n(x_{i+1}^n) - u_n(x_i^n))^2 > \lambda_n\},$$

then $\#(I_n) \le \sup_n E_n(u_n)$. If we identify u_n with the function $\tilde u_n \in P\text{-}W^{1,\infty}(a,b)$ defined by $S(\tilde u_n) = I_n$, $\tilde u_n^+(x_i^n) - \tilde u_n^-(x_i^n) = u_n(x_{i+1}^n) - u_n(x_i^n)$,

$$\tilde u_n'(t) = \begin{cases} \frac{u_n(x_{i+1}^n)-u_n(x_i^n)}{\lambda_n} & \text{if } t \in (x_i^n, x_{i+1}^n),\ i \notin I_n \\ 0 & \text{if } t \in (x_i^n, x_{i+1}^n),\ i \in I_n, \end{cases}$$

then we easily have $E_n(u_n) = \displaystyle\int_a^b |\tilde u_n'|^2\,dt + \#(S(\tilde u_n))$.

If $u_n \to u$ then $\tilde u_n \to u$ in measure so that $u \in P\text{-}W^{1,2}(a,b)$ and by Theorem 7.3 $\liminf_n E_n(u_n) = \liminf_n F_n(\tilde u_n) = \liminf_n F(\tilde u_n) \ge F(u)$.

Conversely, a recovery sequence for a function in $P\text{-}W^{1,2}(a,b)$ is easily obtained by taking $u_n(x_i) = u(x_i)$ on I_n. □

Remark 8.9 The same proof as above shows that if we take E_n as in (4.11) with

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}\min\{\lambda_n c z^2, \alpha\} & \text{if } z \ge 0 \\ \frac{1}{\lambda_n}\min\{\lambda_n c z^2, \beta\} & \text{if } z \le 0, \end{cases} \tag{8.15}$$

then the limit is

$$F(u) = c\int_a^b |u'|^2\,dt + \alpha\#(\{t\in S(u):[u]>0\}) + \beta\#(\{t\in S(u):[u]<0\}) \tag{8.16}$$

on $P\text{-}W^{1,2}(a,b)$.

Comments on Chapter 8

The Ambrosio and Tortorelli (1990) approximation of free-discontinuity problems is directly inspired by the Modica and Mortola (1977) approximation of the perimeter, and, analogously to that problem, can be easily carried on to higher dimensions by the slicing procedure described in Chapter 15. More details on this method applied to approximations of free-discontinuity problems can be found in the lecture notes by Braides (1998). Approximation by convolution problems (in the general n-dimensional case) are dealt with by Cortesani (1998), who uses the localization methods outlined in Chapter 16 to generalize a result by Braides and Dal Maso (1997). The finite-difference approach was introduced by Chambolle (1992), and derives from the same 'weak membrane' model by Blake and Zisserman (1987) that had given rise to the Mumford–Shah functional (see also Morel and Solimini (1995)). A similar approach gives rise to a *finite-element approximation* (see Chambolle and Dal Maso (1999), and Bourdin and Chambolle (2000) for a numerical implementation).

Another approximation of the Mumford Shah functional that directly stems from the finite-difference approximation is that studied by Gobbino (1998) (and more in general by Gobbino and Mora (2001)) in an n-dimensional setting, where he considers functionals of the form

$$\frac{1}{\varepsilon}\int_\Omega\int_\Omega f\Big(\frac{(u(x)-u(y))^2}{\varepsilon}\Big)\rho_\varepsilon(x-y)\,dx\,dy,$$

with ρ_ε convolution kernels. The use of these kernels had been suggested by De Giorgi to overcome the lattice anisotropy that clearly results by simply repeating the finite-difference approach on lattices in more than one dimension. Also in this case the n-dimensional result is easily obtained from the one-dimensional result by 'slicing'. Other approximations, using a singular perturbation with second-order gradients, have been studied by Alicandro *et al.* (1998), Bouchitté *et al.* (2000), and Morini (2001) by considering functionals of the form

$$\frac{1}{\varepsilon}\int_a^b f_\varepsilon(\varepsilon|u'|^2)\,dt + \varepsilon^3\int_a^b |u''|^2\,dt.$$

In this case the limit segmentation energy density is described by an optimal-profile formula. It is interesting to note that if we take $f(z) = z\wedge 1$ then this energy density is $c\sqrt{|u^+ - u^-|}$ (with c explicitly computable). We refer to Braides (1998) for a survey on approximations.

9

MORE HOMOGENIZATION PROBLEMS

The scope of this chapter is to provide some examples in which we have interactions of two different types of asymptotic behaviour, one of which is of oscillating nature. We will see how in some cases the two types of convergence can be 'decoupled', while in other cases they cannot.

9.1 Oscillations and phase transitions

We examine the interaction between oscillations and the gradient theory of phase transition as a case when two scales are present at the same time. We consider the case of phase transitions in a periodic medium. In this case the perturbation term will take into account the inhomogeneity of the material. We can model such a situation by considering energies of the form

$$F_{\varepsilon,\delta}(u) = \int_{(a,b)} \Big(\frac{W(u)}{\varepsilon} + \varepsilon\varphi^2\Big(\frac{t}{\delta}\Big)|u'|^2\Big)\,dt \tag{9.1}$$

defined on $u \in W^{1,2}(a,b)$. The parameter ε takes into account the typical length scale of a transition layer, while the parameter δ represents the length scale of the inhomogeneities in the medium. The asymptotic behaviour of the functionals $F_{\varepsilon,\delta}$ as the two parameters approach 0 will exhibit an interplay between the averaging effect of homogenization due to the oscillations in the second term and the formation of a transition layer due to the balance of the two terms. The description of this asymptotic behaviour will turn out to be a good example of interaction between two different variational limits.

In order to avoid some technical difficulties we assume that $W, \varphi : \mathbf{R} \to [0, +\infty)$ are smooth, W vanishes only at 0 and 1, and satisfies a 2-growth condition, φ is 1-periodic and

$$0 < m = \min \varphi \le \max \varphi = M.$$

We set $c_W = 2\displaystyle\int_0^1 \sqrt{W(s)}\,ds$ as usual. Note that, by the inequalities

$$\int_{(a,b)} \Big(\frac{W(u)}{\varepsilon} + \varepsilon m^2|u'|^2\Big)\,dt \le F_{\varepsilon,\delta}(u) \le \int_{(a,b)} \Big(\frac{W(u)}{\varepsilon} + \varepsilon M^2|u'|^2\Big)\,dt$$

we immediately obtain that, however we choose $\delta = \delta(\varepsilon)$ the Γ-limit F_0 (if it exists) of $F_{\varepsilon,\delta(\varepsilon)}$ is finite only on functions $u \in PC(a,b)$ with $u \in \{0,1\}$ a.e. Moreover, for such an u we have the estimate

$$m\, c_W\, \#(S(u)) \le F_0(u) \le M c_W\, \#(S(u)). \tag{9.2}$$

We now describe how the exact value of F_0 depends on the behaviour of $\delta(\varepsilon)$ with respect to ε.

Oscillations on a slower scale than the transition layer We begin with the case

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = +\infty. \tag{9.3}$$

In this case we have a *separation of scales* effect: first we consider δ as fixed, and let ε tend to 0, obtaining

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta}(u) = c_W \sum_{t\in S(u)} \varphi\Big(\frac{t}{\delta}\Big).$$

By subsequently letting $\delta \to 0$ we then have (recall Exercise 5.6)

$$\Gamma\text{-}\lim_{\delta\to 0}\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta}\Big)(u) = m\, c_W \#(S(u))$$

for $u \in PC(a,b)$ with $u \in \{0,1\}$ a.e., which gives the form of F_0.

To check the validity of this guess it will suffice to exhibit a recovery sequence for $u = \chi_{(0,+\infty)}$, the liminf inequality being already proven by comparison in (9.2). With fixed $\eta > 0$ let $T > 0$ and $v \in W^{1,2}(-T,T)$ be such that $v(-T) = 0$, $v(T) = 1$ and

$$\int_{-T}^{T} \Big(W(v) + m^2|v'|^2\Big)\, dt \le m c_W + \eta.$$

Let $t_m \in [0,1]$ be such that $\varphi(t_m) = \min \varphi$. Define

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } t < \delta(\varepsilon)t_m - \varepsilon T \\ v((t - \delta(\varepsilon)t_m)/\varepsilon) & \text{if } \delta(\varepsilon)t_m - \varepsilon T \le t \le \delta(\varepsilon)t_m + \varepsilon T \\ 1 & \text{if } t > \delta(\varepsilon)t_m + \varepsilon T. \end{cases}$$

We then have $u_\varepsilon \to u$ and

$$F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon) = \int_{-T}^{T} \Big(W(v) + m^2|v'|^2\Big)\, dt + \int_{-T}^{T} \Big(\varphi^2\Big(t_m + \frac{t\varepsilon}{\delta(\varepsilon)}\Big) - \varphi^2(t_m)\Big)|v'|^2\, dt.$$

As the last term tends to 0 as $\varepsilon \to 0$, the proof is concluded.

Oscillations on a finer scale than the transition layer In the case

$$\lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = 0, \tag{9.4}$$

the separation of scales still takes place. We can apply the Homogenization Theorem first as $\delta \to 0$, keeping $\varepsilon > 0$ fixed and obtain

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta}(u) = \int_{(a,b)} \Big(\frac{W(u)}{\varepsilon} + \varepsilon c_\varphi^2 |u'|^2\Big)\, dt, \tag{9.5}$$

where $c_\varphi^{-2} = \int_0^1 \varphi^2(s)\, ds$. Note that the first term in the integral is a continuous contribution, so that it does not influence the computation of the Γ-limit. By subsequently letting $\varepsilon \to 0$ we have as a Γ-limit $F(u) = c_\varphi\, c_W \, \#(S(u))$.

To check the validity of this 'guess', first note that by (9.2) it suffices to verify the liminf inequality only when $u \in PC(a,b)$ and $u \in \{0,1\}$ a.e. To shorten the notation, we write $\delta = \delta(\varepsilon)$. First, consider the case $u = \chi_{(t_0,b)}$ with $a < t_0 < b$. Let $u_\varepsilon \to u$ a.e. then with fixed $\eta \in (0,1/2)$ we find (s_ε) and (t_ε) such that $s_\varepsilon < t_\varepsilon$, $s_\varepsilon \to t_0$, $t_\varepsilon \to t_0$, $u_\varepsilon(s_\varepsilon) = \eta$ and $u_\varepsilon(t_\varepsilon) = 1-\eta$. If $F_{\varepsilon,\delta}(u_\varepsilon) \le C$ we have the estimates

$$C \ge \int_{s_\varepsilon}^{t_\varepsilon} \frac{W(u_\varepsilon)}{\varepsilon}\, dt \ge \frac{m_\eta}{\varepsilon}(t_\varepsilon - s_\varepsilon), \qquad C \ge \varepsilon m \int_{s_\varepsilon}^{t_\varepsilon} |u_\varepsilon'|^2\, dt \ge \varepsilon m \frac{(1-2\eta)^2}{t_\varepsilon - s_\varepsilon},$$

where $m_\eta = \min\{W(s) : \eta \le s \le 1-\eta\}$. Hence, we may suppose that $\varepsilon T_1 \le t_\varepsilon - s_\varepsilon \le \varepsilon T_2$ with $T_1, T_2 > 0$ independent of η. Upon extraction of a subsequence, and a translation argument (details are left as an exercise) we may suppose that $s_\varepsilon = 0$ and $t_\varepsilon = \varepsilon T$. By performing the change of variable $\varepsilon\tau = t$ and setting $\xi_\varepsilon = \delta/\varepsilon$, we have

$$\int_{s_\varepsilon}^{t_\varepsilon} \Big(\frac{W(u_\varepsilon)}{\varepsilon} + \varepsilon \varphi^2\Big(\frac{t}{\delta}\Big)|u_\varepsilon'|^2\Big)\, dt = \int_0^T \Big(W(v_\varepsilon) + \varphi^2\Big(\frac{\tau}{\xi_\varepsilon}\Big)|v_\varepsilon'|^2\Big)\, d\tau,$$

where $v_\varepsilon(\tau) = u_\varepsilon(\varepsilon\tau)$. Hence, minimizing over all v with the same boundary conditions as v_ε, using the Homogenization Theorem taking into account that $\xi_\varepsilon \to 0$, and by the property of convergence of minimum problems, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} F_{\varepsilon,\delta}(u)\\
&\ge \lim_{\xi\to 0} \min\Big\{\int_0^T \Big(W(v) + \varphi^2\Big(\frac{\tau}{\xi}\Big)|v'|^2\Big)\, d\tau : v(0) = \eta,\ v(T) = 1-\eta\Big\}\\
&= \min\Big\{\int_0^T (W(v) + c_\varphi^2|v'|^2)\, d\tau : v(0) = \eta,\ v(T) = 1-\eta\Big\}\\
&\ge c_\varphi 2 \int_\eta^{1-\eta} \sqrt{W(s)}\, ds,
\end{aligned}$$

as in (6.8). By the arbitrariness of η we conclude the liminf inequality. In the case of an arbitrary u the same argument can be repeated at all $t_0 \in S(u)$.

The limsup inequality can be easily obtained by optimizing the estimates above. It suffices to consider the case of $u = \chi_{(0,+\infty)}$. In order to exhibit a recovery sequence, let $\eta > 0$, $T > 0$ be such that

$$\min\Big\{\int_{-T}^T \Big(W(v) + c_\varphi^2|v'|^2\Big)\, dt :\ v(0) = 0, v(T) = 1\Big\} \le c_\varphi c_W + \eta.$$

Again, note that by homogenization this minimum is the limit of

$$\min\Big\{\int_{-T}^{T}\Big(W(v)+\varphi^2\Big(\frac{t}{\delta(\varepsilon)/\varepsilon}\Big)|v'|^2\Big)\,dt:\ v(0)=0, v(T)=1\Big\}$$

as $\varepsilon\to 0$. Let v_ε be functions realizing the corresponding minima, and let

$$u_\varepsilon(t)=\begin{cases}0 & \text{if } t<-\varepsilon T\\ v_\varepsilon(t/\varepsilon) & \text{if } -\varepsilon T\le t\le \varepsilon T\\ 1 & \text{if } t>\varepsilon T.\end{cases}$$

Then $u_\varepsilon\to u$ and $\limsup_{\varepsilon\to 0+}F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon)\le c_\varphi c_W+\eta$. By the arbitrariness of η we conclude the proof.

Oscillations on the same scale of the transition layer Finally, if we have

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)}=K\in(0,+\infty),$$

then the separation of scales effect does not take place. We can reason as in the case when φ is a constant, showing that the Γ-limit is $F(u)=c_{W,K}\,\#(S(u))$, where

$$c_{W,K}=\min\Big\{\int_{-\infty}^{+\infty}(W(v)+\varphi(Ks)|v'|^2)\,ds:\ v(-\infty)=0,\ v(+\infty)=1\Big\}.$$

The interaction of the two limit processes results in an untangled contribution of both K and φ to the definition of the constant $c_{W,K}$. Details are left as an exercise.

9.2 Phase accumulation

In this section we include an example to show how we can obtain energies defined on piecewise-Sobolev functions by considering a variation of the singular-perturbation problem leading to phase transition energies.

Let $W:\mathbf{R}\to[0,+\infty)$ be a 1-periodic function such that $\{W=0\}=\mathbf{Z}$, and let F_ε be defined on $W^{1,2}(a,b)$ by

$$F_\varepsilon(u)=\int_{(a,b)}\Big(\frac{1}{\varepsilon}W\Big(\frac{u}{\varepsilon}\Big)+\varepsilon|u'|^2\Big)\,dt.$$

Note that at fixed $\varepsilon>0$ the set of 'phases' $\{s\in\mathbf{R}: W(s/\varepsilon)=0\}$ equals $\varepsilon\mathbf{Z}$, so that it gets dense as $\varepsilon\to 0$. As a consequence, the limit functional is not a phase-transition energy. We will show that the Γ-$\lim_{\varepsilon\to 0+}F_\varepsilon$ exists on P-$W^{1,1}(a,b)$, and it equals the functional F defined by

$$F(u)=c_W\Big(\int_{(a,b)}|u'|\,dt+\sum_{S(u)}|u^+-u^-|\Big),$$

where $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$ as usual. To prove this statement, first let $u_\varepsilon \to u$ in $L^1(a,b)$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. For all $s,t \in (a,b)$ we have

$$\int_s^t \Big(\frac{1}{\varepsilon}W\Big(\frac{u_\varepsilon}{\varepsilon}\Big) + \varepsilon|u_\varepsilon'|^2\Big)\,dt \geq 2\Big|\int_s^t \sqrt{W\Big(\frac{u_\varepsilon}{\varepsilon}\Big)}u_\varepsilon'\,dt\Big| = 2\Big|\int_{u(s)}^{u(t)} \sqrt{W\Big(\frac{s}{\varepsilon}\Big)}\,ds\Big|.$$

Note that we have

$$\lim_{\varepsilon\to 0}\Big|\int_{u(s)}^{u(t)} \sqrt{W\Big(\frac{s}{\varepsilon}\Big)}\,ds\Big| = |u(t)-u(s)|\int_0^1 W(s)\,ds. \tag{9.6}$$

We then obtain

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq c_W \sup\Big\{\sum_i |u(t_{i+1}) - u(t_i)| : \ N \in \mathbf{N},\ a < t_0 < \cdots < t_N < b\Big\}.$$

The last formula gives the variation of u on (a,b) and proves the liminf inequality (see Appendix A).

We check the limsup inequality for $u = z\chi_{(0,+\infty)}$, upon supposing $0 \in (a,b)$. With fixed $\eta > 0$ let $T > 0$ and $v_T \in W^{1,2}(0,T)$ satisfy $v_T(0) = 0$, $v_T(T) = 1$ and

$$\int_0^T (W(u) + |u'|^2)\,dt \leq c_W + \eta.$$

We then set

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ \varepsilon\Big[\frac{t}{\varepsilon^2 T}\Big] + \varepsilon v_T\Big(\frac{t}{\varepsilon^2} - T\Big[\frac{t}{\varepsilon^2 T}\Big]\Big) & \text{if } 0 \leq t \leq \varepsilon^2 T\Big[\frac{z}{\varepsilon}\Big] \\ \varepsilon\Big[\frac{z}{\varepsilon}\Big] & \text{if } t > \varepsilon^2 T\Big[\frac{z}{\varepsilon}\Big] \end{cases}$$

(see Fig. 9.1). We compute

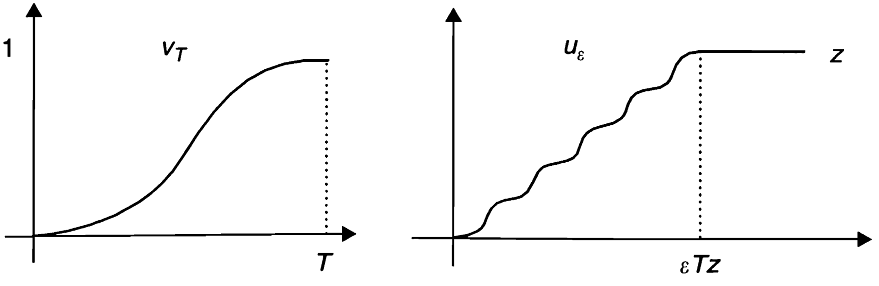

FIG. 9.1. Single optimal profile and recovery sequence for a jump

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon) &= \Big[\frac{z}{\varepsilon}\Big]\int_0^{\varepsilon^2 T}\Big(\frac{1}{\varepsilon}W\Big(\frac{u_\varepsilon}{\varepsilon}\Big)+\varepsilon|u_\varepsilon'|^2\Big)\,dt\\
&= \Big[\frac{z}{\varepsilon}\Big]\frac{1}{\varepsilon}\int_0^{\varepsilon^2 T}\Big(W\Big(v_T\Big(\frac{t}{\varepsilon^2}\Big)\Big)+\Big|v_T'\Big(\frac{t}{\varepsilon^2}\Big)\Big|^2\Big)\,dt\\
&= \Big[\frac{z}{\varepsilon}\Big]\varepsilon\int_0^{T}\Big(W(v_T)+|v_T'|^2\Big)\,dt \le \Big[\frac{z}{\varepsilon}\Big]\varepsilon(c_W+\eta),
\end{aligned}
$$

and the inequality is proved. By reasoning in the same way locally around jump points the same proof holds for $u \in PC(a,b)$. To check the limsup inequality for an arbitrary function in $P\text{-}W^{1,1}(a,b)$ it suffices to approximate it by a sequence (u_j) of piecewise-constant functions such that their variation tends to the variation of u, and use the lower semicontinuity of the Γ-limsup.

We note that more in general the Γ-limit above exists for all functions of bounded variation and it equals $c_W \operatorname{Var}(u)$. The proof of this fact is beyond the scope of this chapter.

9.3 Homogenization of free-discontinuity problems

We conclude this chapter by examining a simple homogenization problem on piecewise-Sobolev functions. Let f satisfy the hypotheses of Theorem 3.1 and let $\varphi : \mathbf{R} \to [1,2]$ be a lower semicontinuous function. We consider the energies

$$
F_\varepsilon(u) = \int_a^b f\Big(\frac{t}{\varepsilon}, u'\Big)dt + \sum_{t\in S(u)} \varphi\Big(\frac{t}{\varepsilon}\Big)
$$

defined on $P\text{-}W^{1,p}(a,b)$. We may interpret F_ε as the energy of a inhomogeneous brittle elastic bar subject to a displacement u. We show that also for the homogenization the 'principle' that the integral and jump part of the energy can be decoupled holds. The integral part is 'homogenized' in the same way as in the case of energies defined on Sobolev spaces, while the limit jump energy is simply obtained by introducing jumps where it is more convenient (i.e. where $\varphi(x/\varepsilon) = \min \varphi$).

Theorem 9.1 *The Γ-limit of F_ε as $\varepsilon \to 0+$ is given on $P\text{-}W^{1,p}$ by the functional*

$$
F_{\rm hom}(u) = \int_a^b f_{\rm hom}(u')\,dt + (\min\varphi)\,\#(S(u)),
$$

where $f_{\rm hom}$ is given by Theorem 3.1.

Proof The liminf inequality follows easily: let $u_\varepsilon \to u$ and write $u_\varepsilon = v_\varepsilon + w_\varepsilon$ with $v_\varepsilon \to v$ in $W^{1,p}(a,b)$ and $w_\varepsilon \to w$ in $PC(a,b)$. We then have

$$
\liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0+} F_\varepsilon(v_\varepsilon) + \sum_{t\in S(u_\varepsilon)} \varphi\Big(\frac{x}{\varepsilon}\Big)
$$

$$\geq \liminf_{\varepsilon\to 0+} F_\varepsilon(v_\varepsilon) + (\min \varphi)\,\#(S(u_\varepsilon)).$$

By the liminf inequality from Theorem 3.1 we get

$$\liminf_{\varepsilon\to 0+} F_\varepsilon(v_\varepsilon) \geq F_{\rm hom}(v) = \int_a^b f_{\rm hom}(v')\,dt.$$

Using the fact that $\liminf_{\varepsilon\to 0+} \#(S(u_\varepsilon)) \geq \#(S(u))$, we have the liminf inequality.

A recovery sequence is readily obtained: let $u = v+w$ with $v \in W^{1,p}(a,b)$ and $w \in PC(a,b)$, and let v_ε be a recovery sequence for Theorem 3.1. We then define $w_\varepsilon \in PC(a,b)$ as follows: let $S(w) = \{t_1,\dots,t_{N-1}\}$, and for all $j \in \{0,\dots,N\}$ let $t_j^\varepsilon \in (a,b)$ be such that $|t_j^\varepsilon - t_j| \leq \varepsilon$ and $\varphi(t_j^\varepsilon/\varepsilon) = \min\varphi$. Then we set $t_0^\varepsilon = t_0 = a$ and $t_N^\varepsilon = t_N = b$, and w_ε is defined on $(t_i^\varepsilon, t_{i+1}^\varepsilon)$ as the constant value of w on (t_i, t_{i+1}). It can be immediately checked that $u_\varepsilon = v_\varepsilon + w_\varepsilon$ is a recovery sequence for u. □

Comments on Chapter 9

In this chapter we have gathered the one-dimensional versions of some (much more complex) results that have been proved in a general n-dimensional setting. The interaction by homogenization and phase transitions has been studied by Ansini *et al.* (2002) by using the localization methods of Chapter 16; this is an example of an application of that method where the domain of the limit functional is different from the domains of the approximating functionals. Note that in this case the slicing method of Chapter 15 cannot be applied since recovery sequences oscillate in all directions and hence do not possess a uni-dimensional structure. The phase accumulation result is due to Modica and Mortola (1977) and can be extended by slicing. Finally, the homogenization of free-discontinuity problems can be found in Braides *et al.* (1996), where it is shown that the principle of separation between bulk and interfacial energies is compatible with homogenization even for vector-valued functions.

10

INTERACTION BETWEEN ELLIPTIC PROBLEMS AND PARTITION PROBLEMS

In this chapter we consider energies with integral and segmentation parts, that do not satisfy the coerciveness conditions of Corollary 7.4. In this case the two parts of the functional may interact, giving rise to some new quantitative compatibility conditions.

10.1 Quantitative conditions for lower semicontinuity

We begin by noticing that in general, for a functional of the form

$$F(u) = \int_{(a,b)} f(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-) \tag{10.1}$$

the only conditions that f is convex and lower semicontinuous and ϑ is subadditive and lower semicontinuous do not assure that F is lower semicontinuous on $P\text{-}W^{1,1}(a,b)$ with respect to the $L^1(a,b)$-convergence even though 'growth conditions of order 1' are satisfied, contrary to what happens both in $PC(a,b)$ and in $W^{1,1}(a,b)$. Other *compatibility conditions* must be added. To understand such conditions we first consider the case of a jump as a limit of smooth functions.

We can suppose that f is not identically equal to $+\infty$; we can thus suppose that $f(0) \in \mathbf{R}$. Take $z \neq 0$,

$$u(t) = \begin{cases} 0 & \text{if } x < 0 \\ z & \text{if } x \geq 0 \end{cases} \qquad \text{and} \qquad u_j(t) = \begin{cases} 0 & \text{if } x < 0 \\ jzt & \text{if } 0 \leq t < 1/j \\ z & \text{if } t \geq z/j \end{cases}$$

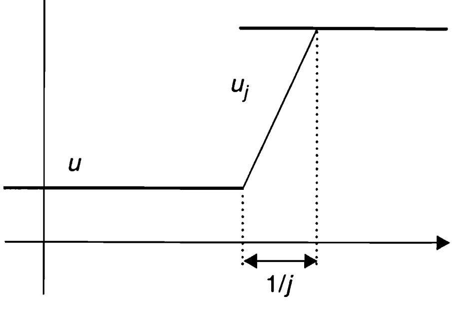

FIG. 10.1. Approximation of a jump

(see Fig. 10.1). Since $u_j \to u$, if F is lower semicontinuous then we have

$$\begin{aligned}(b-a)f(0)+\vartheta(z) &= F(u) \le \liminf_j F(u_j)\\ &= (b-a)f(0)+\liminf_j \frac{1}{j}f(zj);\end{aligned}$$

that is,

$$\vartheta(z) \le \liminf_j \frac{1}{j} f(zj). \tag{10.2}$$

On the other hand, we can approximate an affine function by piecewise-constant functions to obtain another compatibility condition. Take

$$u(t) = zt \qquad \text{and} \qquad u_j(t) = \left[\frac{t}{j}\right] zj$$

(see Fig. 10.2). Since $u_j \to u$, if F is lower semicontinuous we then obtain

$$(b-a)f(z) = F(u) \le \liminf_j F(u_j) = \liminf_j (b-a) j\vartheta\Big(\frac{z}{j}\Big);$$

that is,

$$f(z) \le \liminf_j j\,\vartheta\Big(\frac{z}{j}\Big). \tag{10.3}$$

In order to derive a quantitative criterion for lower semicontinuity from (10.2) and (10.3), it is convenient to introduce some functions related to f and ϑ. Before doing so, we note some properties of convex and of subadditive functions.

Remark 10.1 (i) If $f : \mathbf{R} \to [0,+\infty]$ is convex, then, from the monotonicity properties of the difference quotients, the limits

$$\lim_{t\to\pm\infty} \frac{f(t)}{t} \tag{10.4}$$

exist.

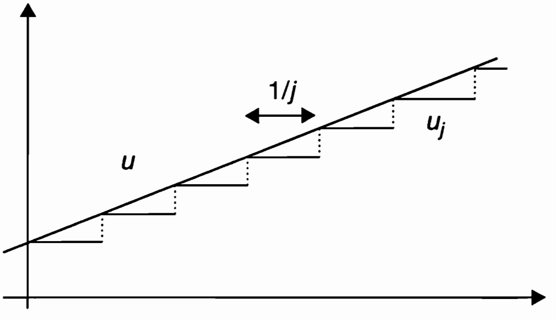

FIG. 10.2. Approximation of an affine function

(ii) We now want to prove the analogue of property (i) for subadditive functions. Let $\vartheta : \mathbf{R} \to [0,+\infty]$ be subadditive and lower semicontinuous. We want to prove the existence of the limits

$$\lim_{z\to 0^\pm} \frac{\vartheta(z)}{|z|}. \tag{10.5}$$

We deal with the case $z \to 0^+$, the other one being dealt with by a symmetric argument.

If $\liminf_{z\to 0^+} \vartheta(z)/z = +\infty$ there is nothing to prove. If ϑ is L-Lipschitz continuous then we have for fixed $0 < s < z$

$$\vartheta(z) \le \Big[\frac{z}{s}\Big]\vartheta(s) + \vartheta\Big(z - \Big[\frac{z}{s}\Big]s\Big) \le \frac{z}{s}\vartheta(s) + L\Big(z - \Big[\frac{z}{s}\Big]s\Big).$$

Dividing by z and letting first $s \to 0$ and then $z \to 0$ we obtain

$$\limsup_{z\to 0^+} \frac{\vartheta(z)}{z} \le \liminf_{s\to 0^+} \frac{\vartheta(s)}{s},$$

as desired.

If ϑ is subadditive and lower semicontinuous then it is the increasing limit of the family of the Lipschitz subadditive functions $T_\lambda\vartheta$ defined by (1.31). If $L = \liminf_{z\to 0^+} \vartheta(z)/z < +\infty$ then we also have $\lim_{z\to 0^+} \vartheta(z)/z \le L < +\infty$, which indeed implies that $T_\lambda\vartheta$ is Lipschitz continuous with constant L so that such is also ϑ (see Remark 5.15), and we can apply the reasonings above.

Definition 10.2 *Let $f : \mathbf{R} \to [0,+\infty]$ be convex. Then we define the* recession function *of f by*

$$f^\infty(z) = \lim_{t\to+\infty} \frac{f(tz)}{t}. \tag{10.6}$$

Let $\vartheta : \mathbf{R} \to [0,+\infty]$ be subadditive and lower semicontinuous. Then we define the recession function *of ϑ by*

$$\vartheta^0(z) = \lim_{t\to 0^+} \frac{\vartheta(tz)}{t}. \tag{10.7}$$

These functions are well defined by the previous remark.

Remark 10.3 The functions defined above are positively homogeneous of degree one; that is,

$$f^\infty(tz) = tf^\infty(z), \qquad \vartheta^0(tz) = t\vartheta^0(z)$$

if $t > 0$, so that they are determined on $\mathbf{R}\setminus\{0\}$ by the values $f^\infty(\pm 1)$ and $\vartheta^0(\pm 1)$, which can be interpreted as the 'slope' of f at $\pm\infty$ and of ϑ at $0\pm$, respectively. We always have $\vartheta^0(0) = 0$ while $f^\infty(0)$ is either 0 or $+\infty$ depending on whether $f(0) \ne +\infty$ or not.

With the notation introduced above, we can state a criterion of lower semicontinuity for the functional F in (10.1).

Proposition 10.4 (compatibility criterion). *Let F be given by* (10.1) *with f convex and lower semicontinuous and ϑ subadditive and lower semicontinuous. If F is lower semicontinuous with respect to the $L^1(a,b)$-convergence, then we have*

$$f^\infty(z) = \vartheta^0(z) \qquad \textit{for } z \neq 0. \tag{10.8}$$

Proof Let $z > 0$. By (10.2) and (10.6) we get

$$\vartheta(z) \leq \lim_j \frac{1}{j} f(zj) = \lim_{t \to +\infty} \frac{f(tz)}{t} = f^\infty(z) = z\, f^\infty(1).$$

Dividing by z and letting $z \to 0+$ we get $\vartheta^0(1) \leq f^\infty(1)$. By (10.3) and (10.7) we get

$$f(z) \leq \lim_j j\vartheta\Big(\frac{z}{j}\Big) = \lim_{t \to 0+} \frac{\vartheta(tz)}{t} = \vartheta^0(z) = z\,\vartheta^0(1). \tag{10.9}$$

Again, dividing by z and letting $z \to +\infty$ we obtain $f^\infty(1) \leq \vartheta^0(1)$, so that indeed $f^\infty(1) = \vartheta^0(1)$. A symmetry argument shows that also $f^\infty(-1) = \vartheta^0(-1)$ and proves the thesis. □

Remark 10.5 Note that if F is lower semicontinuous then in particular such are its restrictions to piecewise-constant functions and to Sobolev functions, respectively, from which we derive that f is lower semicontinuous and convex and ϑ is lower semicontinuous and subadditive.

10.2 Existence without lower semicontinuity

In this section, we give an example that shows that the compatibility conditions in Proposition 10.4 may not be necessary in order to prove the existence of minimizers. In order to simplify details we deal with Dirichlet boundary-value problems only.

Example 10.6 Let F be given by (10.1), and consider the minimum problem

$$\min\Big\{F(u) : u \in P\text{-}W^{1,1}(a,b),\ u(a) = 0,\ u(b) = d\Big\}.$$

We now prove that if f is convex and lower semicontinuous, and

$$\lim_{|z| \to +\infty} f(z) = +\infty, \tag{10.10}$$

and if ϑ is subadditive and lower semicontinuous, then this minimum problem admits a solution, even though the functional F may not be lower semicontinuous with respect to the $L^1(a,b)$-topology.

Let $u \in W^{1,1}(a,b)$ with $u(a) = 0$ and $u(b) = d$. We set

$$z = \frac{1}{b-a} \int_{(a,b)} u'\,dt, \qquad w = \sum_{S(u)} (u^+ - u^-). \tag{10.11}$$

Note that

$$(b-a)z + w = d. \tag{10.12}$$

Choose $t_0 \in (a,b)$ and set

$$v(t) = \begin{cases} z(t-a) & \text{if } t < t_0 \\ z(t-a) + w & \text{if } t > t_0. \end{cases} \tag{10.13}$$

We have $v(0) = 0$ and

$$F(v) = (b-a)f(z) + \vartheta(w) \le \int_{(a,b)} f(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-) = F(u)$$

by Jensen's inequality and the subadditivity inequality.

Hence, the minimum problem can be performed on functions of the form (10.12) and (10.13); that is, we have to solve

$$\min\Big\{(b-a)f(z) + \vartheta(w) : \ (b-a)z + w = d\Big\}.$$

The existence of a minimizing pair (w,z) is easily obtained by the lower semicontinuity of f and ϑ and by (10.10).

Note that if f is strictly convex and ϑ is strictly subadditive then the reasoning above shows that actually *all minimizers* are of the form (10.13).

10.3 Relaxation by interaction

Despite the fact that we may obtain solutions without lower semicontinuity, it may be useful to have a lower semicontinuity and relaxation result for functionals in $P\text{-}W^{1,1}(a,b)$. We first introduce the definition of inf-convolution of two functions, which will help us in describing the integrands of the relaxed functional.

Definition 10.7 *Let $g, h : \mathbf{R} \to [0,+\infty]$. We define the* inf-convolution of g and h *as the function $g\triangle h$ given by*

$$g\triangle h(z) = \inf\{g(z_1) + h(z_2) : \ z_1 + z_2 = z\} = \inf\{g(w) + h(z-w); \ w \in \mathbf{R}\} \tag{10.14}$$

for $z \in \mathbf{R}$.

The relaxation result is the following.

Theorem 10.8 (relaxation by interaction). *Let F be given by* (10.1) *with f convex and lower semicontinuous and ϑ subadditive and lower semicontinuous. Suppose moreover that*

$$\vartheta(z) \geq c(|z| \wedge 1) \qquad \textit{and} \qquad \lim_{|z|\to+\infty} \vartheta(z) = +\infty.$$

Then the lower semicontinuous envelope $\overline{F}$ of F with respect to the $L^1(a,b)$-convergence is given on P-$W^{1,1}(a,b)$ by the functional

$$\overline{F}(u) = \int_{(a,b)} (f \triangle \vartheta^0)(u')\,dt + \sum_{S(u)} (\vartheta \triangle f^\infty)(u^+ - u^-). \tag{10.15}$$

If $f^\infty = \vartheta^0$ then $F = \overline{F}$; that is, F is lower semicontinuous.

We postpone the proof after some comments about the theorem. This result states that the integrands of the lower-semicontinuous envelope of F are obtained by an interaction between the functions f and ϑ. To understand why this interaction is expressed as an inf-convolution consider the simple case of an affine function $u(t) = zt$. We may approximate it by 'mixing gradients and jumps', by considering functions of the form

$$u_j(t) = z_1 t + \left[\frac{t}{j}\right] z_2 j \tag{10.16}$$

(see Fig. 10.3). We have $u_j \to u$, so that $\overline{F}(u) \leq \lim_j F_j(u_j) = (b-a)(f(z_1) + \vartheta^0(z_2))$.

If we have a 'jump function' instead; for example,

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ z & \text{if } t \geq 0, \end{cases}$$

then we may choose

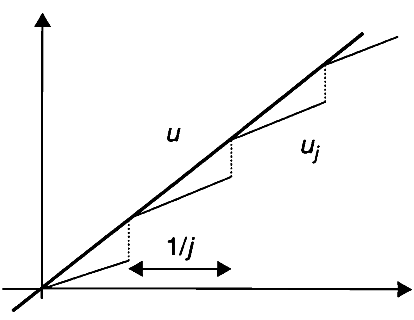

FIG. 10.3. Optimal approximation of a gradient by using jumps and gradients

$$u_j(t) = \begin{cases} 0 & \text{if } t < 0 \\ jt & \text{if } 0 \leq t < w/j \\ z & \text{if } t \geq w/j. \end{cases} \tag{10.17}$$

(see Fig. 10.4). We have $u_j \to u$, so that $\overline{F}(u) \leq \lim_j F_j(u_j) = (b-a)f(0) + (f^\infty(w) + \vartheta(z-w))$.

Indeed we will see that these sequences (u_j) are optimal.

Note that the theorem does not state that the domain of the relaxed functional is the space of piecewise-$W^{1,1}$ functions. Indeed it may be shown that its domain is the space of functions with bounded variation in (a,b). The proof of this fact, even though not difficult in the one-dimensional case, is beyond the scopes of this book. Note, moreover, that by Example 10.6 it is often sufficient to characterize the energies on $P\text{-}W^{1,1}(a,b)$.

To prove the liminf inequality along a sequence (u_j) we will write $u_j = w_j + v_j + \varphi_j$ where w_j is a sequence in $PC(a,b)$ to which we can apply Theorem 5.8 and (v_j) is a sequence in $W^{1,\infty}(a,b)$ to which we can apply Theorem 2.13. We will use the following lower-semicontinuity result, which can be easily checked, to deal with the remaining sequence (φ_j).

Proposition 10.9 *Let $\varphi \in P\text{-}W^{1,1}(a,b)$; then we have*

$$\int_{(a,b)} |\varphi'|\,dt + \sum_{S(\varphi)} |\varphi^+ - \varphi^-| \tag{10.18}$$

$$= \sup\Big\{ \int_{(a,b)} \psi'\varphi\,dt : \ \psi \in C^1([a,b]),\ \|\psi\|_\infty \leq 1,\ \psi(a) = \psi(b) = 0 \Big\};$$

in particular

$$\varphi \mapsto \int_{(a,b)} |\varphi'|\,dt + \sum_{S(\varphi)} |\varphi^+ - \varphi^-|$$

is lower semicontinuous with respect to the $L^1(a,b)$-convergence on $P\text{-}W^{1,1}(a,b)$.

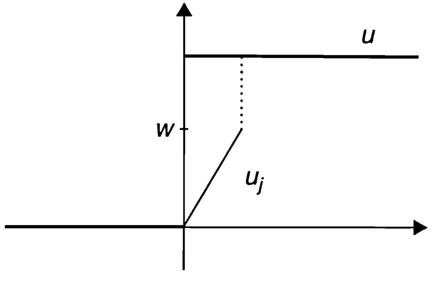

FIG. 10.4. Optimal approximation of a jump by using increasing slopes and a jump

Proof (Theorem 10.8) For the sake of simplicity we suppose that $f(z) = f(-z)$ and $\vartheta(z) = \vartheta(-z)$. Otherwise, we can write $u_j = u_{1,j} - u_{2,j}$ with $u'_{i,j} \geq 0$ and $u^+_{1,j} \geq u^-_{1,j}$, and deal with $u_{1,j}$ and $u_{2,j}$ separately.

Let $u_j, u \in W^{1,1}(a,b)$ and let $u_j \to u$ in $L^1(a,b)$. Upon extracting a subsequence we suppose that the limit $\lim_j F(u_j)$ exists and is finite, so that we are free to pass to further subsequences. Let $0 < K < \vartheta^0(1) \wedge f^\infty(1)$, and let $\rho \in (0, +\infty]$ be defined by $\rho = \sup\{\eta \in \mathbf{R} : \ K|s| \leq \vartheta(s) \text{ for all } |s| \leq \eta\}$. We define w_j as the function in $PC(a,b)$ satisfying $w_j(a+) = 0$, $S(w_j) = \{t \in S(u_j) : \ |u^+_j - u^-_j| > \rho\}$ and $w^+_j - w^-_j = u^+_j - u^-_j$. Note that, since $\vartheta(z) \to +\infty$ if $|z| \to +\infty$ then $\#(S(w_j)$ is equibounded as well as $\|w_j\|_\infty$. By Proposition 5.3 we can assume that the sequence (w_j) converges to $w \in PC(a,b)$ in measure, and then also in $L^1(a,b)$. Note that by Theorem 5.8 we have $F(w) \leq \liminf_j F(w_j)$.

Let $M = \inf\{z \in \mathbf{R} : \ f'(z+) \geq K\}$ and define $v_j \in W^{1,\infty}(a,b)$ by

$$v_j(t) = \int_a^t (-M \vee u'_j) \wedge M \, ds.$$

As $\sup_j \|v_j\|_{W^{1,\infty}(a,b)} < +\infty$ we can suppose that v_j converges weakly* to some $v \in W^{1,\infty}(a,b)$. Note that by Proposition 2.13 we have $F(v) \leq \liminf_j F(v_j)$.

Finally, we set $\varphi_j = u_j - v_j - w_j \in P\text{-}W^{1,1}(a,b)$, which converges in $L^1(a,b)$ to $\varphi = u - v - w$. Note that $S(\varphi_j) = S(u_j) \setminus S(w_j)$ and that $\varphi^+_j - \varphi^-_j = u^+_j - u^-_j$ on $S(\varphi_j)$. We will apply Proposition 10.9 to (φ_j). We can write

$$\begin{aligned} F(u_j) &= F(v_j) + F(w_j) \\ &\qquad + \int_{\{|u'_j| > M\}} (f(u'_j) - f(M)) \, dt + \sum_{S(u_j)\setminus S(w_j)} \vartheta(u^+_j - u^-_j) \\ &\geq F(v_j) + F(w_j) + K\Big(\int_{(a,b)} |\varphi'_j| \, dt + \sum_{S(\varphi_j)} |\varphi^+_j - \varphi^-_j|\Big). \end{aligned}$$

We then have

$$\begin{aligned} &\liminf{}_j F(u_j) \\ &\geq \int_{(a,b)} f(v') \, dt + \sum_{S(w)} \vartheta(w^+ - w^-) + K\Big(\int_{(a,b)} |\varphi'| \, dt + \sum_{S(\varphi_j)} |\varphi^+ - \varphi^-|\Big) \\ &\geq \int_{(a,b)} (f(v') + K|u' - v'|) \, dt \\ &\qquad + \sum_{S(u)} (\vartheta(w^+ - w^-) + K|(u^+ - u^-) - (w^+ - w^-)|) \\ &\geq \int_{(a,b)} (f \triangle K|\cdot|)(u') \, dt + \sum_{S(u)} (\vartheta \triangle K|\cdot|)(u^+ - u^-). \end{aligned}$$

Note that we have used the equalities $\varphi' = u' - v'$ and $\varphi^+ - \varphi^- = (u^+ - u^-) -$

$(w^+ - w^-)$ on $S(u)$. We eventually let K tend to $\vartheta^0(1) \wedge f^\infty(1)$ to obtain the liminf inequality.

In order to prove the lim sup inequality it suffices to use the constructions in (10.16) and (10.17) to obtain a recovery sequence for a piecewise-affine $u \in P\text{-}W^{1,1}(a,b)$, and then proceed by density. Details are left as an exercise. □

Remark 10.10 Suppose that $f(0) \neq +\infty$. Then it can be easily seen that

$$\vartheta \triangle f^\infty = \text{sub}\,(\vartheta \wedge f^\infty).$$

If $f(0) = 0$ then we have $f \triangle \vartheta^0 = (f \wedge \vartheta^0)^{**}$.

Example 10.11 Let f be convex and ϑ be subadditive. Suppose that $f^\infty(\pm 1) > \vartheta^0(\pm 1)$, respectively. In this case the lower-semicontinuous envelope leaves ϑ unchanged, while the relaxed energy density of the integral part $f \triangle \vartheta^0$ is given by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } T_- \leq z \leq T_+ \\ f(T_-) + \vartheta^0(-1)(z - T_-) & \text{if } z < T_- \\ f(T_+) + \vartheta^0(1)(z - T_+) & \text{if } z > T_+, \end{cases}$$

where (T_+, T_-) is defined as the interval where both right- and left-hand side derivatives of f belong to $[\vartheta^0(-1), \vartheta^0(1)]$ (see Fig. 10.5). In this case, since

$$\min\{\overline{F}(u) : \; u(a) = 0, \; u(b) = L\} = \min\{F(u) : \; u(a) = 0, \; u(b) = L\}$$

(the latter existing by Example 10.6) we deduce that if u is a minimizer for this problem, then $f(u') = \tilde{f}(u')$ a.e, and hence $u' \in [T_-, T_+]$ a.e.

Example 10.12 Let f be convex and ϑ be concave. Suppose that $f^\infty(\pm 1) < \vartheta^0(\pm 1)$, respectively. In this case the lower-semicontinuous envelope leaves f unchanged, while the relaxed energy density of the jump part $f^\infty \triangle \vartheta$ is given by

$$\tilde{\vartheta}(z) = \begin{cases} f^\infty(z) & \text{if } T_- \leq z \leq T_+ \\ \vartheta(z) & \text{otherwise,} \end{cases}$$

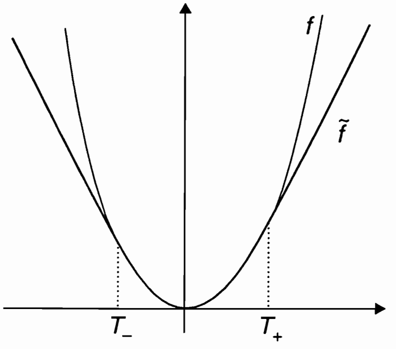

FIG. 10.5. The relaxed integral energy density

where $T_\pm$ are defined by the $\vartheta(T_\pm) = f^\infty(T_\pm)$ (see Fig. 10.6). The same conclusion holds if we directly suppose that $f^\infty \wedge \vartheta$ is subadditive. In this case, since

$$\min\{\overline{F}(u) :\ u(a) = 0,\ u(b) = L\} = \min\{F(u) :\ u(a) = 0,\ u(b) = L\}$$

(the latter again existing by Example 10.6) we deduce that if u is a minimizer for this problem, then $\vartheta(u^+ - u^-) = \tilde{\vartheta}(u^+ - u^-)$ on $S(u)$, and hence $u^+(t) - u^-(t) \notin [T_-, T_+]$ for all $t \in S(u)$.

10.4 Exercises

10.1 Let $p \geq 1$ and $\alpha \geq 0$. Compute the relaxed functionals of

$$F(u) = \int_{(a,b)^2} |u'|^p\, dt + \sum_{S(u)} |u^+ - u^-|^\alpha.$$

Comments on Chapter 10

The compatibility conditions in this chapter can be viewed as a particular case of those valid for functionals defined on measures or on functions of bounded variation. Functions of bounded variation in dimension one can be written as $u = v + w + \varphi$ where $v \in W^{1,1}(a,b)$, the distributional derivative of w can be written as a series of Dirac masses (which is the natural generalization of piecewise-constant functions) and φ is a continuous function whose derivative is singular with respect to the Lebesgue measure (the prototype of such a function is Cantor Vitali function, see e.g. Buttazzo *et al.* (1998)). Using the notation of Chapter 7 for u' and $u^\pm$, the simplest integral functionals on $BV(a,b)$ have the form

$$F(u) = \int_a^b f(u')\, dt + C \operatorname{Var} \varphi + \sum_{t \in S(u)} \vartheta(u^+ - u^-) \tag{10.19}$$

(with, for the sake of simplicity, f and ϑ even functions). The lower semicontinuity of F implies that $C = f^\infty(\pm 1) = \vartheta^0(\pm 1)$. All functionals in this chapter

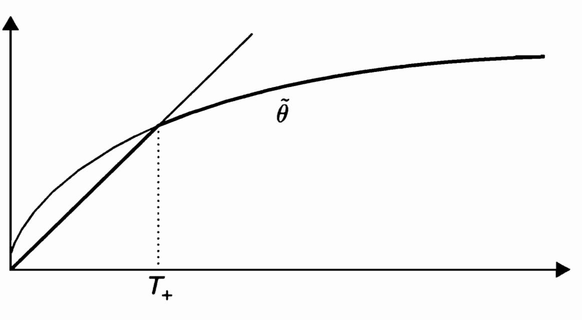

FIG. 10.6. The relaxed jump energy density

can be extended to $BV(a,b)$ as in (10.19); for the sake of simplicity we have only proved their representation on $P\text{-}W^{1,1}(a,b)$, also in view of Section 10.2, which states that in dimension one we frequently find piecewise-Sobolev solutions.

The integral representation of functionals defined on spaces of functions of bounded variation in higher dimensions and possibly depending on vector-valued functions (and the determination of the related compatibility conditions) is a very complex technical problem (see e.g. Braides and Chiadò Piat (1996), Bouchitté *et al.* (2001)). For a complete introduction to the structure of BV functions in any dimension we refer to Ambrosio *et al.* (2000).

Structured deformations Relaxation by interaction provides us the opportunity to make a connection with the theory of *structured deformations* in Continuum Mechanics by Del Piero and Owen (1993, 2000). In that context, upon rephrasing their original definitions, in place of functions $u \in P\text{-}W^{1,1}(a,b)$ we consider pairs (G,u) representing 'macroscopic displacements' and 'microdisarrangements', respectively. To understand the link with Theorem 10.8, we note that an energy on piecewise-Sobolev functions of the type (10.1) can be seen as defined on $W^{1,1}(a,b) \times PC(a,b)$ as

$$\mathcal{F}(v,w) = \int_a^b f(v')\,dt + \sum_{S(w)} \vartheta(w^+ - w^-).$$

Consider for the sake of simplicity the case that f is convex, ϑ is subadditive, $f^\infty(\pm 1) = +\infty$, and $\vartheta(\pm 1) = L < +\infty$. In this case, it can be easily seen (using Theorem 10.8 separately in v and w) that the relaxed functional of $\mathcal{F}$ can be written on $W^{1,1}(a,b) \times P\text{-}W^{1,1}(a,b)$ as

$$\overline{\mathcal{F}}(v,w) = \int_a^b (f(v') + L|w'|)\,dt + \sum_{S(w)} \vartheta(w^+ - w^-).$$

If we rewrite this process in terms of $u = v + w$ and $G = v'$ the relaxed functional is

$$\overline{\mathcal{F}}(G,u) = \int_a^b (f(G) + L|u' - G|)\,dt + \sum_{S(u)} \vartheta(u^+ - u^-).$$

In this formulation the first integral takes separately into account the contributions of the overall deformation that are due to 'macroscopic displacements' and 'microdisarrangements' respectively. We refer to Del Piero and Owen (1993, 2000) for exact definitions in a more general setting and for the mechanical interpretation of structured deformations, and to Choksi and Fonseca (1997) for a characterization of general bulk and interfacial energy densities for structured deformations.

11

DISCRETE SYSTEMS AND FREE-DISCONTINUITY PROBLEMS

In this chapter we treat limits of discrete systems outside the Sobolev setting of Chapter 4. We will see that those system may give as a limit functionals defined on piecewise-Sobolev functions with interacting integral and segmentation parts as those in the previous chapter. The definition of the integrands of the two parts will highlight a *scaling effect*. For the sake of simplicity we deal with nearest-neighbour interactions only so that this effect is not coupled with oscillations on the lattice scale.

We will consider the limit of energies defined on discrete systems of n points as n tends to $+\infty$ giving rise to free-discontinuity problems. In the notation of Chapter 4 we will deal with sequences (E_n) with $E_n : \mathcal{A}_n(0,L) \to [0,+\infty]$ of the form

$$E_n(u) = \sum_{i=1}^{n} f_n(u_i - u_{i-1}), \tag{11.1}$$

where $\mathcal{A}_n(0,L)$ is the set of discrete functions defined on $I_n = \{iL/n : i = 0,\dots,n\}$. With fixed n we denote $x_i = iL/n$ and $u_i = u(x_i)$.

As in Chapter 4, the first step is to identify each $\mathcal{A}_n(0,L)$ with a subspace of a common space of functions defined on $(0,L)$. Since the functions in the limit problems may have discontinuities, we face the choice between identifying each $u \in \mathcal{A}_n(0,L)$ with a piecewise-constant or a piecewise-affine interpolation. In the first case we may define the extension $\tilde{u} : (0,L) \to \mathbf{R}$ by setting

$$\tilde{u}(s) = u_i \qquad \text{if } s \in \Big(x_i - \frac{L}{2n}, x_i + \frac{L}{2n}\Big) \cap (0,L). \tag{11.2}$$

If $\mathcal{A}_n(0,L)$ is identified with those $u \in PC(0,L)$ such that $S(u) \subset (I_n + \frac{L}{2n}) \cap (0,L)$, the functional E_n may be viewed as a functional defined on $PC(0,L)$ by

$$E_n(u) = \begin{cases} \displaystyle\sum_{S(u)} f_n(u^+ - u^-) & \text{if } u \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{11.3}$$

Note that the functional directly defined by $\widetilde{F}_n(u) = \sum_{S(u)} f_n(u^+ - u^-)$ on the whole $PC(0,L)$ may not be lower semicontinuous, and its relaxation can be dramatically different. For example, if f_n is smooth, convex and $f_n(0) = 0$ then the lower-semicontinuous envelope of $\widetilde{F}_n$ is identically 0.

On the other hand, as in Chapter 4, we may choose to extend u by $\tilde{u} : (0, L) \to \mathbf{R}$ defined as

$$\tilde{u}(s) = u_{i-1} + \frac{s - x_{i-1}}{x_i - x_{i-1}}(u_i - u_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{11.4}$$

In this case, $\mathcal{A}_n(0, L)$ is identified with those continuous $u \in W^{1,\infty}(0, L)$ such that u is affine on each (x_{i-1}, x_i). With this identification, E_n may be viewed as a functional defined on $W^{1,1}(0, L)$ by

$$E_n(u) = \begin{cases} \displaystyle\int_0^L \frac{n}{L} f_n\Big(\frac{L}{n}u'\Big)\, dt & \text{if } u \in \mathcal{A}_n(0, L) \\ +\infty & \text{otherwise.} \end{cases} \tag{11.5}$$

Again, note that the functional defined by $\widetilde{F}_n(u) = \int_0^L \frac{n}{L} f_n(\frac{L}{n}u')\, dt$ on the whole $W^{1,1}(0, L)$ may be not lower semicontinuous, and its relaxation can be quite different. For example, if f_n is bounded from above, then the lower-semicontinuous envelope of $\widetilde{F}_n$ is identically the constant $c = \inf f_n$.

In the previous chapter we have introduced a class of energies with interacting integral and segmentation parts. We may combine the two choices of identification above to extend E_n to a functional of that form. In this process it will be crucial to have a criterion to discriminate between discrete interaction that must be considered as interpolations of integrals and those which have to be regarded as jumps.

11.1 Interpolation with piecewise-Sobolev functions

For each $n \in \mathbf{N}$ we introduce two 'thresholds' $T^n_\pm$ with

$$-\infty \le T^n_- \le 0 \le T^n_+ \le +\infty.$$

When the difference quotient of a discrete function lies in the interval between the two thresholds we will interpret it as a gradient, while external values are seen as corresponding to jumps. Note that the case $T^n_\pm = \pm\infty$ corresponds to the interpolation with piecewise-affine functions while the case $T^n_\pm = 0$ gives the piecewise-constant interpolations.

If $u \in \mathcal{A}_n(0, L)$, we define $\tilde{u} \in P\text{-}W^{1,\infty}(0, L)$ on each (x_{i-1}, x_i) as follows (see Fig. 11.1): in view of (11.5) it is convenient to define $\lambda_n = L/n$ (the 'discretization step length') and

(a) if $\dfrac{u_i - u_{i-1}}{\lambda_n} < T^n_-$ then

$$\tilde{u}(s) = \begin{cases} u_{i-1} + T^n_-(s - x_{i-1}) & \text{if } s \in \Big(x_{i-1}, x_{i-1} + \dfrac{\lambda_n}{2}\Big) \\ u_i + T^n_-(s - x_i) & \text{if } s \in \Big(x_{i-1} + \dfrac{\lambda_n}{2}, x_i\Big). \end{cases} \tag{11.6}$$

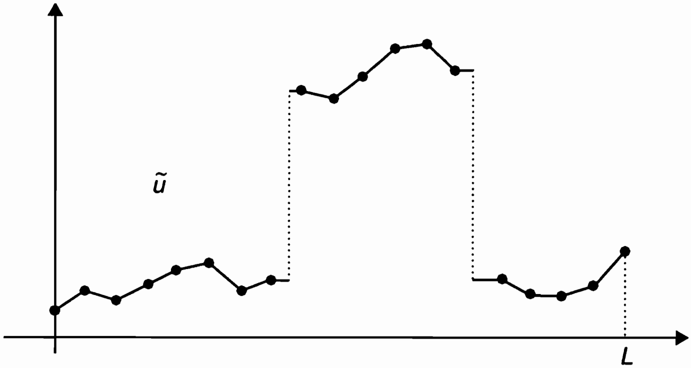

FIG. 11.1. Mixed interpolation of a discrete function

In this case $\tilde{u}$ has a discontinuity at $(x_{i-1}+x_i)/2$ of size $u_i - u_{i-1} - \lambda_n T_-^n$;

(b) if $\dfrac{u_i - u_{i-1}}{\lambda_n} \in [T_-^n, T_+^n]$ then

$$\tilde{u}(s) = u_{i-1} + \frac{s - x_{i-1}}{x_i - x_{i-1}}(u_i - u_{i-1}); \tag{11.7}$$

(c) if $\dfrac{u_i - u_{i-1}}{\lambda_n} > T_+^n$ then

$$\tilde{u}(s) = \begin{cases} u_{i-1} + T_+^n(s - x_{i-1}) & \text{if } s \in \left(x_{i-1}, x_{i-1} + \dfrac{\lambda_n}{2}\right) \\ u_i + T_+^n(s - x_i) & \text{if } s \in \left(x_{i-1} + \dfrac{\lambda_n}{2}, x_i\right). \end{cases} \tag{11.8}$$

In this case $\tilde{u}$ has a discontinuity at $(x_{i-1}+x_i)/2$ of size $u_i - u_{i-1} - \lambda_n T_+^n$;

We then have $\tilde{u} \in P\text{-}W^{1,\infty}(0,L)$, $\tilde{u}' \in [T_-^n, T_+^n]$ a.e. and

$$S(\tilde{u}) = \left\{\frac{x_i + x_{i-1}}{2} : 1 \le i \le n,\ u_i - u_{i-1} < \lambda_n T_-^n \text{ or } u_i - u_{i-1} > \lambda_n T_+^n\right\}. \tag{11.9}$$

Remark 11.1 Note that if $\lim_n \lambda_n T_\pm^n = 0$ then the convergence of the piecewise-constant interpolations of u_n in $L^1(0,L)$ implies the convergence in $L^1(0,L)$ of both the piecewise-affine interpolations of u_n defined in (11.4) and the 'mixed-type' defined above.

In fact, if $\tilde{u}_n^1$ and $\tilde{u}_n^2$ denote the piecewise-constant interpolation of u_n and the piecewise-affine interpolation of u_n, respectively, then we have

$$\int_{x_{i-1}}^{x_i} |\tilde{u}_n^1 - \tilde{u}_n^2|\, dx = \frac{\lambda_n}{2} |u_n(x_i) - u_n(x_{i-1})|,$$

so that

$$\int_0^L |\tilde{u}_n^1 - \tilde{u}_n^2|\,dx = \frac{\lambda_n}{2}\sum_{i=1}^n |u_n(x_i) - u_n(x_{i-1})|$$
$$= \frac{1}{2}\int_0^{b-\lambda_n} |u_n(s) - u_n(s-\lambda_n)|\,dt, \qquad (11.10)$$

which tends to 0 as $n \to +\infty$. If $\tilde{u}_n$ denotes the 'mixed-type' interpolations then the same conclusion applies since either $\tilde{u}_n^1 \le \tilde{u}_n \le \tilde{u}_n^2$ or $\tilde{u}_n^2 \le \tilde{u}_n \le \tilde{u}_n^1$. Moreover, we have

$$\int_{x_{i-1}}^{x_i} |\tilde{u}_n^1 - \tilde{u}_n|\,dx = -\frac{\lambda_n}{2}T_-^n, \quad 0, \text{ and } \quad \frac{\lambda_n}{2}T_+^n,$$

in the three cases (a), (b) and (c), respectively. In particular,

$$\int_{x_{i-1}}^{x_i} |\tilde{u}_n^1 - \tilde{u}_n|\,dx \le \frac{\lambda_n}{2}(T_+^n - T_-^n)\#(S(\tilde{u}_n)). \qquad (11.11)$$

11.2 Equivalent energies on piecewise-Sobolev functions

We now consider E_n given on $\mathcal{A}_n(0,L)$ by

$$E_n(u) = \sum_{i=1}^n \lambda_n \psi_n\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big). \qquad (11.12)$$

In the notation above it corresponds to defining $\psi_n(z) = \frac{1}{\lambda_n} f_n(\lambda_n z)$. Note that in this way we simply have $\int_0^L \psi_n(u')\,dt$ in (11.5).

We fix thresholds $T_\pm^n$ as in the previous section. By taking the construction of the mixed-type interpolation $\tilde{u}$ into account we obtain

$$E_n(u) = F_n(\tilde{u}) := \int_0^L \psi_n(\tilde{u}')\,dt + \sum_{S(\tilde{u})} g_n(\tilde{u}^+ - \tilde{u}^-), \qquad (11.13)$$

where

$$g_n(z) = \begin{cases} \lambda_n\Big(\psi_n\Big(\dfrac{z}{\lambda_n} + T_-^n\Big) - \psi_n(T_-^n)\Big) & \text{if } z < 0 \\ \lambda_n\Big(\psi_n\Big(\frac{z}{\lambda_n} + T_+^n\Big) - \psi_n(T_+^n)\Big) & \text{if } z > 0. \end{cases} \qquad (11.14)$$

Note that the segmentation part and the integral part are obtained from ψ_n by different scaling arguments.

The equality in (11.12) holds for $u \in \mathcal{A}_n(0,L)$, which is now identified with the subspace of $L^1(0,L)$ of the mixed-type interpolations defined above. To be more precise we should then indicate the dependence on $T_\pm^n$ in this identification, but we drop it for the sake of simplicity.

Using (11.12) the estimate of the Γ-liminf of E_n can be translated into the estimate of Γ-limits of functionals F_n of the form considered in the previous chapter. Note that the form of F_n depends on the choice of $T^n_\pm$, so that the sharpest lower bound will be obtained by optimizing the role of these thresholds.

A model case for this problem are energy densities $\psi_n : \mathbf{R} \to [0, +\infty)$ of convex/concave type (see Fig. 11.2), for which a possible choice for $T^m_\pm$ is given by the inflection points. We suppose that $-\infty \leq T^n_- \leq 0 \leq T^n_+ \leq +\infty$ exist satisfying

- the restriction of ψ_n to (T^n_-, T^n_+) is convex; (11.15)
- the restrictions of ψ_n to $(-\infty, T^n_-)$ and to $(T^n_+, +\infty)$ are concave. (11.16)

For the sake of simplicity, we additionally assume that each ψ_n is continuous.

Remark 11.2 In what follows the concavity in (11.16) can be replaced by subadditivity. We state our result in terms of convexity and concavity, so that it is easier to check whether given ψ_n fit into our hypotheses.

Note that g_n is subadditive and that $F_n(\tilde{u})$ as defined in (11.12) does not depend on the extension of ψ_n outside (T^n_-, T^n_+), which we may choose convex on the whole $\mathbf{R}$. Hence, F_n can be viewed as a lower-semicontinuous functional on $P\text{-}W^{1,1}(0, L)$, under compatibility conditions on ψ_n and g_n.

11.3 Softening and fracture problems as limits of discrete models

We now consider a limit process that can be interpreted as a discrete approximation of mechanical problems with softening and possible fracture. In order to briefly explain the behaviour of the limit continuum model, we consider the problem of a bar subject to a forced displacement d at its ends. For small values of the boundary datum the displacement u' is uniform inside the bar (elastic solution), until its value reaches a critical size σ_0. At this point, the bar reaches a 'softening' regime, where the dependence of u' on d is decreasing, reaching possibly the value 0 for large values of d (fracture) (see Fig. 11.3). This behaviour will be interpreted as a limit of systems of 'molecular interactions' given by energies E_n. The mechanical analysis of this phenomenon is beyond the scopes of this

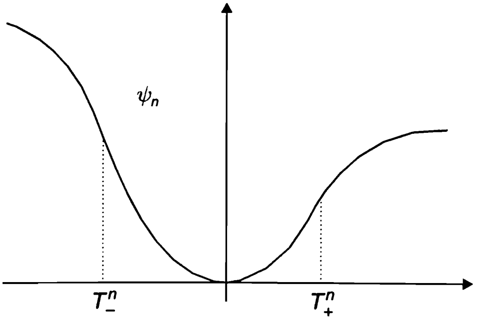

FIG. 11.2. The shape of the discrete energy density

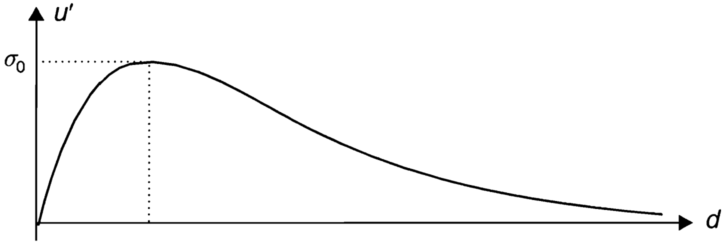

FIG. 11.3. Dependence of strain on total displacement

book. We only highlight the presence of different scalings giving rise to the limit energies. For the sake of simplicity we will treat a symmetric case only.

We now consider the limit of energies E_n with respect to the convergence in $L^1(0,L)$. We make some simplifying hypotheses, which will make our convergence result clearer. We suppose that $T^n_\pm = \pm T$ for some $T > 0$, and that there exist f and ϑ (convex the first, concave the second) such that

$$\psi_n = f \text{ on } [-T,T] \qquad \text{and} \qquad g_n = \vartheta \tag{11.17}$$

for all $n \in \mathbf{N}$, where g_n is defined by (11.14). Rewriting ψ_n in terms of f and ϑ, our hypotheses turn into

$$\psi_n(z) = \begin{cases} \dfrac{1}{\lambda_n}\vartheta(\lambda_n(z+T)) & \text{if } z < -T \\ f(z) & \text{if } |z| \le T \\ \dfrac{1}{\lambda_n}\vartheta(\lambda_n(z-T)) & \text{if } z > -T. \end{cases} \tag{11.18}$$

Furthermore, we suppose that ψ_n is C^1, which implies that $\varphi^0(\pm 1) = f'(\pm T)$.

Remark 11.3 In all that follows we could replace the conditions above by asymptotic conditions: $\lim_n T^n_\pm = \pm T$ for some $T > 0$, $\lim_n \psi_n = f$ on $[-T,T]$ and $\lim_n g_n = \vartheta$. Note that these conditions are compact, under some local equi-boundedness assumptions

Remark 11.4 Let $u_n \to u \in P\text{-}W^{1,1}(0,L)$ with $\sup_n E_n(u_n) < +\infty$. If ϑ satisfies the hypotheses of Theorem 10.8 then we have

$$\begin{aligned} \liminf{}_n E_n(u_n) &= \liminf{}_n F_n(\tilde{u}_n) \\ &\ge \int_0^L (f \triangle \vartheta^0)(u')\,dt + \sum_{S(u)} (\vartheta \triangle f^\infty)(u^+ - u^-) \end{aligned}$$

$$= \int_0^L (f\triangle\vartheta^0)(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-) \tag{11.19}$$

Note that the last expression does not depend on the choice of f outside $[-T,T]$.

Theorem 11.5 *Let ψ_n, f and ϑ satisfy the conditions above and let ϑ satisfy the hypotheses of Theorem* 10.8. *Then the Γ-limit of E_n with respect to the L^1-convergence is given by*

$$F(u) = \int_0^L (f\triangle\vartheta^0)(u')\,dt + \sum_{S(u)} \vartheta(u^+ - u^-) \tag{11.20}$$

on $P\text{-}W^{1,1}(0,L)$.

Proof The liminf inequality is given by the previous remark. A recovery sequence for a function in $P\text{-}W^{1,\infty}(0,L)$ is easily obtained by taking $u_n(x_i) = u(x_i)$ on I_n, and then for a function in $P\text{-}W^{1,1}(0,L)$ by approximation. Details are left as an exercise. □

Remark 11.6 If f is strictly convex and ϑ is strictly subadditive (which simply means it is not affine in a neighbourhood of 0, but may be definitively constant), the 'softening behaviour' of the limit energy when subject to Dirichlet boundary conditions is easily derived. In fact, by Example 10.11 and the fact that $\vartheta^0(\pm 1) = f'(\pm T)$ we obtain

$$(f\triangle\vartheta^0)(z) = \begin{cases} f(-T) + \vartheta^0(-1)(z+T) & \text{if } z < -T \\ f(z) & \text{if } -T \le z \le T \\ f(T) + \vartheta^0(1)(z-T) & \text{if } z > T, \end{cases}$$

while proceeding as in Example 10.6 we deduce that the solution the problem

$$\min\{F(u) : u(0) = 0,\ u(L) = d\} \tag{11.21}$$

is either the linear solution $u(t) = (d/L)t$, or a piecewise-affine function with $u' = z$ and one jump of size w minimizing $Lf(z) + \vartheta(w)$ under the condition $zL + w = d$; i.e.

$$f'(z) = \vartheta'(d - Lz). \tag{11.22}$$

The deduction of a graph as in Fig. 11.3 from this relation is left as an exercise; note that its shape depends on L (see also Exercises 11.1 and 11.2).

11.4 Fracture as a phase transition

We now deal with sequences of energies that are obtained as a scaling of a fixed potential of Lennard Jones type. In this case a development by Γ-convergence is needed to completely explain the effect of the passage from a discrete to a continuum setting. The first-order Γ-limit can be interpreted as a fracture term obtained as a higher-order perturbation.

Let $J : \mathbf{R} \to [0, +\infty]$ satisfy the following conditions: $J = +\infty$ on $(-\infty, 0]$, J is continuous on $[0, +\infty)$, there exists $C > 1$ such that J is convex on $(0, C]$ with minimum in 1 and concave on $[C, +\infty)$, and there exists the limit $J(+\infty) \in \mathbf{R}$ (see Fig. 11.4). We study functionals E_n with $\psi_n(z) = J(z/\lambda_n)$; i.e,

$$E_n(u) = \sum_{i=1}^{n} \lambda_n J\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big)$$

Theorem 11.7 *Under the hypotheses above the functionals defined on $\mathcal{A}_n(0, L)$ by E_n Γ-converge to the functional F defined by*

$$F(u) = \int_{(0,L)} J_0(u')\, dt$$

on P-$W^{1,1}(0, L)$, with $J_0(z) = J(z \wedge 1) = J^{**}(z)$, and the functionals

$$E_n^{(1)}(u) = \frac{1}{\lambda_n}(E_n(u) - \min F))$$

Γ-converge to the functional $E^{(1)}$ given by

$$E^{(1)}(u) = \begin{cases} (J(+\infty) - J(1))\, \#(S(u)) & \text{if } u \text{ is piecewise affinet on } (0, L) \\ & u' = 1 \text{ and } u^+ > u^- \text{ on } S(u) \\ +\infty & \text{otherwise} \end{cases}$$

on P-$W^{1,1}(0, L)$. This functional is the first-order Γ-limit of (E_n).

Proof The existence of the 'zero-order' Γ-limit F and its representation follow immediately from Theorem 11.5 (with $T_-^n = -\infty$) by a comparison argument.

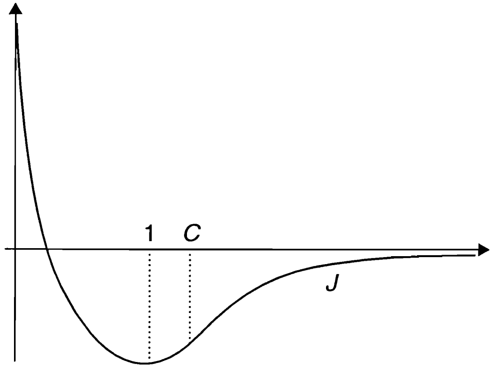

FIG. 11.4. The potential J

Note that $\min F = J(1) = \min J$ so that we have

$$E_n^{(1)}(u) = \sum_{i=1}^{n} \Big(J\Big(\frac{u(x_i) - u(x_{i-1})}{\lambda_n}\Big) - J(1) \Big).$$

We check now the first-order Γ-limit. We first give an estimate from below by comparison (see Fig. 11.5). Let $\gamma > 0$ be such that

$$\begin{cases} J(1) + \gamma(z-1)^2 \le J(z) & \text{if } z \le 1, \\ J(1) + \min\{\gamma(z-1)^2, J(+\infty) - \gamma\} \le J(z) & \text{if } z > 1. \end{cases}$$

We then have

$$E_n^{(1)}(u) \ge \sum_{i=1}^{n} \psi_n \Big(\frac{u(x_i) - u(x_{i-1})}{\lambda_n}\Big),$$

whenever n is large enough and ψ_n is given by (8.15), where $\alpha = J(+\infty) - \gamma - J(1)$, and c and β are arbitrary. Upon changing variables and considering $v(t) = u(t) - t$, we can apply Remark 8.9 to estimate from below the Γ-limit by

$$\begin{aligned} F(u) = c \int_{(0,L)} |u' - 1|^2 \, dt + (J(+\infty) - \gamma - J(1)) \#(\{t \in S(u) : [u] > 0\}) \\ + \beta \#(\{t \in S(u) : [u] < 0\}). \end{aligned} \tag{11.23}$$

Since c, β and γ are arbitrary positive numbers we obtain the desired estimate from below.

To complete the proof it suffices to exhibit a recovery sequence for such a u. Let u_n be defined simply by $u_n(x_i) = u(x_i)$. It suffices to consider the case of a single jump: $S(u) = \{x_0\}$, with $u(x_0+) = z$, $u(x_0-) = 0$. In this case we trivially have

$$\lim_n E^{(1)}(u_n) = \lim_n \Big(J\Big(\frac{z}{\lambda_n}\Big) - J(1) \Big) = J(+\infty) - J(1),$$

and the proof is concluded. □

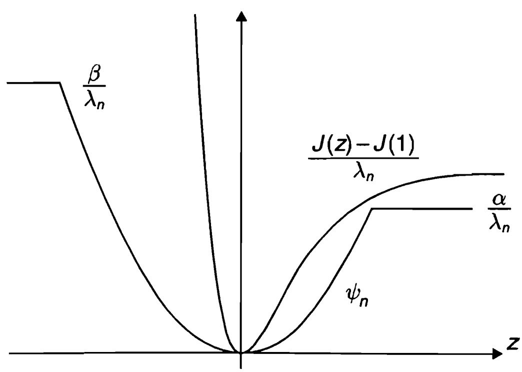

FIG. 11.5. Estimate from below of the scaled potential of J

11.5 Malik Perona approximation of free-discontinuity problems

We can make a variation of the approximation of free-discontinuity problems given in Section 7 by considering energies with smoother integrands with logarithmic growth. We can consider

$$E_n(u) = \sum_{i=0}^{n-1} \frac{1}{\log \lambda_n} \log\Big(1 + |\log \lambda_n| \frac{(u_{i+1} - u_i)^2}{\lambda_n}\Big) \tag{11.24}$$

(*Malik Perona approximation*). Note that if we write

$$E_n = \sum_{i=0}^{n-1} \lambda_n f_n\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big), \text{ i.e., } f_n(z) = \frac{1}{\lambda_n |\log \lambda_n|} \log(1 + \lambda_n |\log \lambda_n| z^2), \tag{11.25}$$

we have

$$\lim_n f_n(z) = z^2, \qquad \lim_n \lambda_n f_n\Big(\frac{w}{\lambda_n}\Big) = 1 \tag{11.26}$$

if $w \neq 0$, which explains the choice of the scaling.

To obtain a lower bound we can apply the procedure above, considering

$$T_n = \frac{1}{\lambda_n \sqrt{|\log \lambda_n|}}, \qquad \tilde f_n = \begin{cases} \Big(f_{n|[-T_n, T_n]}\Big)^{**} & \text{on } [-T_n, T_n] \\ f_n & \text{elsewhere.} \end{cases}$$

Note that in this case we do not take as $\pm T_n$ the inflection points of f_n.

By construction, $\tilde f_n \leq f_n$, $\tilde f_n$ is convex on $[-T_n, T_n]$ and is concave on $(-\infty, -T_n]$ and on $[T_n, +\infty)$. Furthermore, $\tilde f_n$ still satisfies (11.26), from which we obtain that the domain of the Γ-lim inf is the set of piecewise-Sobolev functions and the lower inequality

$$\Gamma\text{-}\lim_n F_n(u) \geq \int_0^L |u'|^2\, dt + \#(S(u)).$$

By (11.26) it is immediately checked that we can choose as a recovery sequence (the discretization of) the target function u itself.

11.6 Exercises

11.1 Analyse the dependence on L of the gradient of the solution of (11.21) when taking $f(z) = z^2$ and $\vartheta(w) = \begin{cases} 2w - w^2 & \text{if } w < 1 \\ 1 & \text{otherwise,} \end{cases}$ with $T = 1$ in Remark 11.6.

Hint: denote by z the (constant) gradient of the solution to (11.21). With fixed $d > 0$ either we have $z = d/L$ (elastic solution) or z is given by (11.22) (softening), or $z = 0$ (fracture). In the second case we simply obtain (for $L \neq 1$) $z = (d-1)/(L-1)$. By singling out the energetically-convenient solution we then highlight a dependence on L which is given in Fig. 11.6.

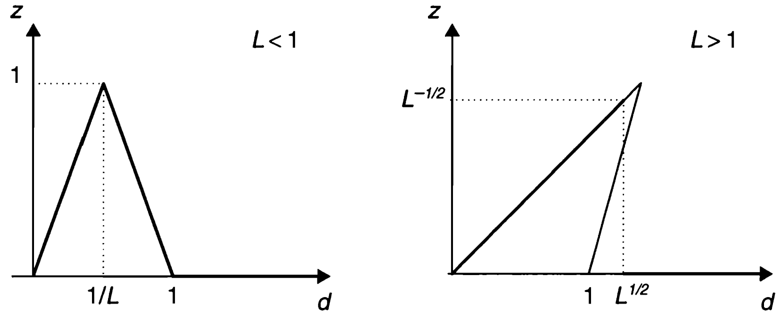

FIG. 11.6. Dependence of the softening behaviour on the length L

11.2 Take $f(z) = z^2/2$ and $\vartheta(w) = w/(1+w)$ in Exercise 11.1.

Comments on Chapter 11

The separation of scales giving rise to the 'mixed-type' interpolations of discrete functions for arbitrary discrete problems is remarked by Braides *et al.* (1999), where a general compactness and representation result is given, together with an interpretation in terms of softening and fracture. The approach to fracture as a phase transition is due to Truskinovsky (1996). All hypotheses on the discrete energy densities can be removed by combining discretization, convexity and subadditivity arguments (see Braides and Gelli 2002).

For an analysis of softening and fracture phenomena we refer to Carpinteri (1989). For references on the Malik Perona model in Computer Vision see Morel and Solimini (1995) and Perona and Malik (1987). An interesting approximation of functionals in the framework of Griffith's theory of fracture by discrete energies is provided by Alicandro *et al.* (2000). In that case, non-central interactions (i.e. involving more than two points of the lattice) must be used to recover all linear elastic bulk energies.

12

*SOME COMMENTS ON VECTORIAL PROBLEMS

In this section and the following ones we treat some examples of Γ-convergence in a more-than-one-dimensional framework. All the problems we have treated hitherto can be extended to higher dimensions in many ways to problems whose technical complexity is much greater. Treating in depth the technical details of those problems does not fall within the scopes of this book; hence, we will feel free to refer to other monographs when needed.

From the viewpoint of Γ-convergence, the arguments used in one-dimensional problems are a good guideline to the general case, even though there is no common rule that always allows to reduce to a one-dimensional analysis. At times problems are meaningful only in a space of sufficiently high dimension so that no one-dimensional counterpart is readily available; as an example we can think of limits of Dirichlet problems in perforated domains (see Chapter 13). In this case, though, the one-dimensional analysis turns out very useful, since a crucial point is a scaling and optimal-transition argument that is not far from the arguments described in the study of one-dimensional phase transitions. There are also situations where the study can be completely reduced to dimension one, whenever recovery sequences possess an essentially one-dimensional structure. This is the case of the gradient theory of phase transitions in higher dimensions, where the computation of the Γ-limit is reduced to the study of the asymptotic behaviour of one-dimensional sections (the method is described in Chapter 15). Some other types of problems can be equally formulated in any space dimension but one must be careful to understand which of the reasoning can be transported from the one-dimensional framework. This is the case of integral functionals defined on vector-valued functions, in which case the key point is to understand until what extent convexity arguments can be repeated.

We begin with some remarks on multiple integrals depending on (possibly) vector-valued functions defined on some Sobolev space. This is a case when a complete theory, analogous to that illustrated for the one-dimensional case has been developed. We will omit the proofs of the facts that are by now considered standard in this rich subject and can be found, for example, in the books by Buttazzo (1989), Dacorogna (1989), Braides and Defranceschi (1998), and Fonseca and Leoni (2002).

In this chapter we want to note that many one-dimensional reasonings can be repeated, but the way they have to be used to describe the much richer structure of those energies must be carefully analysed. In particular we will see that
— lower semicontinuity with respect to weak convergence is again completely described by looking at the behaviour of integrals on sequences of oscillating

functions, but in the genuinely vector case this does not lead to the convexity of the integrands (but to a new notion, called *quasiconvexity*);
— convexity is still a sufficient condition for lower semicontinuity since the same argument as in the one-dimensional case still works. We can find other sufficient 'convexity conditions' that are intermediate between convexity and quasiconvexity (as that called *polyconvexity*);
— in order to describe homogenization problems the principle to 'optimize among oscillating functions' still holds, but in the vector case we cannot limit the analysis to oscillations with the same period as that of the integrand. This phenomenon is highlighted by the fact that we recover some of the formulas for the limit integrand as in the one-dimensional case, but not all. In particular the formula on the periodicity cell does not hold;
— by general Γ-convergence considerations, the class of functionals with convex integrands is closed by homogenization. Other intermediate conditions between convexity and quasiconvexity, as polyconvexity, even though still ensuring lower semicontinuity, do not give closed classes;
— even if restricting to convex integrands or to quadratic forms depending on scalar-valued functions the characterization of the homogenized coefficients is much more difficult than in the one-dimensional case. We will see for example that all quadratic forms with constant coefficients can be obtained as limits of integrals of the form $\int a(x/\varepsilon)|Du|^2\,dx$. This shows that an easy characterization by some weak limit of the coefficients as in the one-dimensional case is not possible.

12.1 Lower semicontinuity conditions

In this chapter and the following ones we will deal with energies defined on Sobolev spaces in more than one dimension. We refer the interested reader to Adams (1975) for the general theory of these spaces (see also Ziemer 1989). We begin by considering functionals of the form

$$F(u) = \int_\Omega f(Du)\,dx, \tag{12.1}$$

where Ω is a bounded open subset of $\mathbf{R}^N$, $u \in W^{1,p}(\Omega;\mathbf{R}^M) := (W^{1,p}(\Omega))^M$, with $N, M \in \mathbf{N}$, and Du denotes the matrix of all partial derivatives $D_i u_j$ of $u = (u_1, \dots, u_N)$. For the sake of simplicity we assume that f is sufficiently smooth and satisfies a growth condition of order $p > 1$:

$$|A|^p \le f(A) \le C(1 + |A|^p), \tag{12.2}$$

where now the argument of f is a $M \times N$ matrix (we write $A \in \mathbf{M}^{M\times N}$). In this way, there is no problem in the definition of F. Note that F is coercive on bounded sets with respect to the $L^p(\Omega;\mathbf{R}^M)$ metric, so that lower-semicontinuity conditions for this functional may be stated equivalently with respect to the weak $W^{1,p}$ or the strong L^p convergence.

12.1.1 *Quasiconvexity*

As for the one-dimensional case, we may test the lower semicontinuity inequality

$$F(u) \le \liminf_{\varepsilon\to 0^+} F(u_\varepsilon) \tag{12.3}$$

when $u(x) = Ax$ and $u_\varepsilon(x) = u(x)+\varepsilon\varphi(\frac{x}{\varepsilon})$, with $\varphi \in W^{1,p}_{\rm loc}(\mathbf{R}^N;\mathbf{R}^M)$ periodic in each coordinate direction with period one. In this way (noting that the argument of Example 2.24 can be repeated) we are led to the *necessary condition* on the integrand f

$$f(A) \le \int_{(0,1)^N} f(A + D\varphi(y))\,dy \tag{12.4}$$

for all matrices A and one-periodic φ in $W^{1,p}_{\rm loc}(\mathbf{R}^N;\mathbf{R}^M)$. In particular inequality (12.4) holds for all $\varphi \in C^\infty_0(\mathbf{R}^N;\mathbf{R}^M)$; this condition is called the *quasiconvexity* of f. It can be proved that quasiconvexity is also a sufficient condition for lower semicontinuity, as in the following result.

Theorem 12.1 *Let* $1 < p < \infty$ *and* $f : \mathbf{M}^{M\times N} \to \mathbf{R}$ *be a function satisfying* (12.2). *Then the functional* F *in* (12.1) *is (sequentially) weakly l.s.c. on* $W^{1,p}(\Omega;\mathbf{R}^M)$ *if and only if* f *is quasiconvex, or equivalently if and only if* f *satisfies* (12.4) *for all one-periodic* φ *in* $W^{1,p}_{\rm loc}(\mathbf{R}^N;\mathbf{R}^M)$

We only give a hint of the proof. To check that quasiconvexity implies (12.3) for arbitrary $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbf{R}^M)$ we can proceed as follows:
— note that by a density argument (valid by (12.2)) we can suppose that all u_ε are smooth;
— if u is affine note that quasiconvexity implies (12.3) if $u_\varepsilon - u$ has compact support (by a scaling argument we can always suppose that $\Omega \subset (0,1)^N$);
— if u is affine and $u_\varepsilon \rightharpoonup u$ (in particular $u_\varepsilon \to u$ strongly in L^p) by a careful cut-off argument near the boundary of Ω, we can modify u_ε without changing the limit of $F(u_\varepsilon)$ so that $u_\varepsilon - u$ has compact support, and the previous step applies;
— if u is piecewise affine, we can repeat the previous argument on each set where u is affine;
— we can approximate every u by piecewise-affine functions.
We note that these arguments are not different from those used in dimension one. The technical point here is the cut-off argument near $\partial\Omega$, as we cannot directly follow for example the argument of Proposition 2.37 (we suggest as an exercise to check were it fails in the vector case).

Equivalent definitions of the quasiconvexity of f are that we have

$$\begin{aligned} f(A) &= \min\Big\{\int_{(0,1)^N} f(A + D\varphi(y))\,dy : \varphi \in W^{1,p}_{\rm loc}(\mathbf{R}^N;\mathbf{R}^M) \text{ 1-periodic}\Big\} \\ &= \min\Big\{\int_{(0,1)^N} f(A + D\varphi(y))\,dy : \varphi \in W^{1,p}_0((0,1)^N;\mathbf{R}^M)\Big\}. \end{aligned} \tag{12.5}$$

Note that in the case $N = 1$ or $M = 1$ quasiconvexity reduces to convexity. The quasiconvexity of a function f is not a very transparent condition, and also very difficult to check, as it involves a computation on the whole $W_0^{1,p}((0,1)^N;\mathbf{R}^M)$. The next paragraph provides some more handy sufficient conditions.

12.1.2 *Convexity and polyconvexity*

As in the one-dimensional case, *convexity is a sufficient condition* for lower semicontinuity. This fact can be proven as in Proposition 2.13 (and is left as an exercise). An intermediate condition between convexity and quasiconvexity, which is of great importance in problems of non-linear elasticity and that provides some non-trivial examples of quasiconvex integrands, is polyconvexity

Definition 12.2 (polyconvexity). *A function $f : \mathbf{M}^{M\times N} \to \mathbf{R}$ is* polyconvex *if there exists a convex function $g : \mathbf{R}^{\tau(N,M)} \to \mathbf{R}$ such that*

$$f(A) = g(\mathcal{M}(A)) \qquad \text{for all } A \in \mathbf{M}^{M\times N}, \tag{12.6}$$

where $\mathcal{M}(A)$ represents the ordered vector of all the minors *of order* $1, 2, \ldots, N \wedge M$ *of A, and $\tau(N,M) = \sum_{k=1}^{N\wedge M} \binom{M}{k}\binom{N}{k}$.*

Example 12.3 If $N = 1$ or $M = 1$, then we have $\mathcal{M}(A) = A$, and polyconvexity is the same as convexity. If $N = M = 2$ and A_{ij} are the entries of A then we have

$$\mathcal{M}(A) = (A, \det A) = (A_{11}, A_{12}, A_{21}, A_{22}, A_{11}A_{22} - A_{12}A_{21}),$$

and f is polyconvex if and only if there exists a convex $g : \mathbf{R}^5 \to \mathbf{R}$ such that

$$f(A) = g(A_{11}, A_{12}, A_{21}, A_{22}, A_{11}A_{22} - A_{12}A_{21}) = g(A, \det A).$$

For example the function $f(A) = |A|^p + |\det A|$ $(1 \le p < \infty)$ is polyconvex.

Polyconvexity is a sufficient condition for the weak lower semicontinuity of multiple integrals under more general growth conditions than (12.2), as stated in the following theorem.

Theorem 12.4 *Let $f : \mathbf{M}^{M\times N} \to \mathbf{R}$ be a polyconvex function, such that a positive convex g exists satisfying* (12.6). *Then the functional F in* (12.1) *is weakly l.s.c. on $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^M)$ with $p \ge N \wedge M$.*

We do not prove the theorem but only give a hint in the case $N = M = 2$. First we observe that if $u = (u^1, u^2) \in W^{1,2}$ then we can write

$$\det Du = \operatorname{div}(u^1 D_2 u^2, -u^1 D_1 u^2) \tag{12.7}$$

in the sense of distributions. In fact if $u \in C^2(\Omega;\mathbf{R}^2)$ then the equality holds pointwise, and is easily extended to arbitrary $u \in W^{1,2}$ by a density argument. From this we obtain that if $u_j \rightharpoonup u$ in $W^{1,2}$ then $\det Du_j \to \det Du$ in the sense of distributions, and hence also $(Du_j, \det Du_j) \to (Du, \det Du)$. Now the lower semicontinuity of $\int_\Omega g(v)\,dx$ with respect to the convergence in the sense of distributions can be easily proved (see Exercise 2.5), so that

$$\begin{aligned} F(u) &= \int_\Omega g(Du, \det Du)\,dx \\ &\leq \liminf_j \int_\Omega g(Du_j, \det Du_j)\,dx = \liminf_j F(u_j). \end{aligned}$$

12.2 Homogenization and convexity conditions

The classes of 'homogeneous integrals' (i.e. with integrands not depending on x) as in (12.1) and (12.2) with convex, polyconvex, and quasiconvex integrands, respectively, are closed with respect to Γ-convergence, and in those classes Γ-convergence reduces to the pointwise convergence of the integrands. The situation is more complex for non-homogeneous integrals. We now briefly give some remarks on the Γ-convergence of integrals of the form

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx,$$

where $f : \mathbf{R}^N \times \mathbf{M}^{M\times N} \to \mathbf{R}$ is a Borel function.

If f satisfies a growth condition of order $p > 1$ analogue to (12.2) then a homogenization theorem as in the one-dimensional case can be proved, showing the Γ-convergence to a functional of the form

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du)\,dx,$$

where $f_{\rm hom}$ is given by the *asymptotic homogenization formula*

$$f_{\rm hom}(A) = \lim_{T\to+\infty} \inf\Big\{\frac{1}{T^N}\int_{(0,T)^N} f(y, Du(y)+A)\,dy : u \in W_0^{1,p}((0,T)^N; \mathbf{R}^M)\Big\}, \tag{12.8}$$

or equivalently by

$$f_{\rm hom}(A) = \inf_{j\in\mathbf{N}} \inf\Big\{\frac{1}{j^N}\int_{(0,j)^N} f(y, Du(y)+A)\,dy : u \in \mathrm{W}^{1,p}_{\rm loc}(\mathbf{R}^N; \mathbf{R}^M)\ j\text{-periodic}\Big\} \tag{12.9}$$

for all $A \in \mathbf{M}^{M\times N}$. These formulas are easily derived from the quasiconvexity of $f_{\rm hom}$ (recall that quasiconvexity is a necessary condition for lower semicontinuity and hence automatically satisfied by $f_{\rm hom}$), and hence by expressing $f_{\rm hom}(A)$ as the minimum in (12.5) and using the property of convergence of minima, as in the one-dimensional case. The proof of this homogenization result can be obtained by following the proof of Theorem 3.1 with some technical subtleties that are beyond the scope of this book. For details we refer to Braides and Defranceschi (1998).

Remark 12.5 (non-validity of the cell-problem formula) . If f is *convex* then we can repeat the reasoning by convex combination of Theorem 3.2 that shows that $f_{\rm hom}$ is also given by a single minimum problem on the periodicity cell

$$f_{\rm hom}(A) = \inf\Big\{\int_{(0,1)^N} f(y, Du + A)\, dy : u \in W^{1,p}_{\rm loc}(\mathbf{R}^N;\mathbf{R}^M) \text{ 1-periodic}\Big\}. \tag{12.10}$$

An example by Müller (1987) shows that there exist functions f such that $f(x,\cdot)$ is *polyconvex* at all x such that (12.10) does not hold; that is, contrary to the convex case examining only oscillations at the same scale of the periodicity does not provide a complete description of the homogenization process.

12.2.1 *Instability of polyconvexity*

We now show that polyconvexity is not a stable property by Γ-convergence, and homogenization in particular. We apply the homogenization procedure to a 'composite' function constructed using two other functions f_1, f_2 with different behaviours. To construct these functions let

$$f_0(A) = |A_{11} - A_{22}| + |A_{12}| + |A_{21}| + (|A_{11}| - 1)^+$$

(t^+ is the positive part of t),

$$f_1(A) = (f_0(A))^p, \qquad f_2(A) = (f_0(A))^2 + (f_0(A))^p + |1 - \det A|$$

with $1 < p < 2$. Note that f_1 is convex and satisfies a growth condition of order p, with $1 < p < 2$, and $f_1(I) = f_1(-I) = 0$ (and hence $f_1(tI) = 0$ for $-1 \le t \le 1$ by convexity), while $f_2 : \mathbf{M}^{2\times 2} \to [0,+\infty)$ is polyconvex and satisfies a growth condition of order 2, and $f_2(A) = 0$ if and only if $A \in \{I, -I\}$. Moreover, there exists a constant $\gamma > 0$ such that we have

$$f_i(A) \ge \gamma(|A_{11} - A_{22}|^p + |A_{21} + A_{12}|^p) \tag{12.11}$$

for $i = 1, 2$.

We consider the function $f : \mathbf{R}^2 \times \mathbf{M}^{2\times 2} \to [0,+\infty)$ defined on $[0,1)^2 \times \mathbf{M}^{2\times 2} \to [0,+\infty)$ by

$$f(x,A) = \begin{cases} f_1(A) & \text{if } x \in [0,1)^2 \setminus \left[\frac14,\frac34\right]^2 \\ f_2(A) & \text{if } x \in \left[\frac14,\frac34\right]^2 \end{cases} \tag{12.12}$$

(see Fig. 12.1), and extended by periodicity to the whole $\mathbf{R}^2$. Note that $f(x,\cdot)$ is polyconvex for all x. We denote

$$E = \left[\tfrac14,\tfrac34\right]^2 + \mathbf{Z}^2,$$

so that $\frac1j E = \{x \in \mathbf{R}^2 :\ f(jx,\cdot) = f_2\}$ for all $j > 0$.

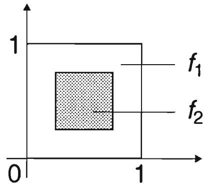

FIG. 12.1. Cell structure

Even though f does not satisfy a growth condition of order p it can be seen that the homogenization process can be carried on since the region E where the growth condition is not satisfied is composed by well-separated isolated sets (see Braides and Defranceschi 1998). Hence, we can consider the homogenized energy density $f_{\hom}$, which can be described by (12.9). We now check that *the function* $f_{\hom}$ *is not polyconvex.*

Note that by the growth conditions from below on f_1 and f_2 we immediately obtain that

$$|A|^p - c \leq f_{\hom}(A) \tag{12.13}$$

for all $A \in \mathbf{M}^{2\times 2}$. We also get an estimate from above, by taking $u(x) = (x - Ax)\varphi(jx)$ as a test function in (12.9), where $\varphi \in W_0^{1,\infty}((0,1)^2)$ is such that $\varphi = 1$ on $[\frac{1}{4}, \frac{3}{4}]^2$, $0 \leq \varphi \leq 1$ and $|D\varphi| \leq 4$. We then obtain

$$f_{\hom}(A) \leq c_1(1 + |A|^p) \tag{12.14}$$

for all $A \in \mathbf{M}^{2\times 2}$, so that $f_{\hom}$ satisfies a growth condition of order $p < 2$. This implies that if it were polyconvex, then it should be *convex*. In fact if it is not convex then it must be a convex function of the determinant, and hence satisfy a growth condition of order at least 2. In order to show that this is not the case, first we remark that $f_{\hom}(I) = f_{\hom}(-I) = 0$; this can be immediately checked by taking $u \equiv 0$ in (12.9). The proof is concluded if we show that $f_{\hom}(0) > 0$, which contradicts the convexity condition. Since we have

$$f_{\hom}(0) = \inf\Big\{\liminf_j \int_{(0,1)^2} f(jy, Du_j)\,dy : \\ u_j \in W_0^{1,2}((0,1)^2;\mathbf{R}^2),\ u_j \to 0 \text{ in } L^p((0,1)^2;\mathbf{R}^2)\Big\}$$

for all $\delta > 0$, it suffices to show that the latter infimum is strictly positive. We argue by contradiction: consider a sequence $(u_j) = (u_j^1, u_j^2)$ in $W_0^{1,2}((0,1)^2;\mathbf{R}^2)$ converging to 0 in $L^p((0,1)^2;\mathbf{R}^2)$ such that

$$0 = \lim_j \int_{(0,1)^2} f(jx, Du_j)\,dx. \tag{12.15}$$

By (12.11) we then have

$$\lim_j \int_{(0,1)^2} (|D_1 u_j^1 - D_2 u_j^2|^p + |D_1 u_j^2 + D_2 u_j^1|^p)\,dx = 0. \tag{12.16}$$

We can write

$$\Delta u_j^1 = D_1(D_1 u_j^1 - D_2 u_j^2) + D_2(D_2 u_j^1 + D_1 u_j^2)$$
$$\Delta u_j^2 = D_1(D_2 u_j^1 + D_1 u_j^2) - D_2(D_1 u_j^1 - D_2 u_j^2)$$

so that by (12.16) we have that $\Delta u_j^k \to 0$ in $W^{-1,p}((0,1)^2)$ $(k = 1, 2)$. Using the L^p estimates for the Laplace operator (see e.g. Stein (1970)), we obtain that Du_j converges to 0 strongly in $L^p((0,1)^2; \mathbf{M}^{2\times 2})$. In particular this implies that the measure of the set $\{x \in (0,1)^2 : |Du_j| \geq 1\}$ converges to 0. Notice now that since f_2 is continuous and vanishes only at I and $-I$, we have $c_3 := \inf\{f_2(A) : |A| \leq 1\} > 0$, so that

$$\begin{aligned}\lim_j \int_{(0,1)^2} f(jx, Du_j)\, dx &\geq \lim_j \int_{(0,1)^2 \cap \frac{1}{j}E} f_2(Du_j)\, dx \\ &\geq \lim_j \int_{(0,1)^2 \cap \frac{1}{j}E \cap \{|Du_j|<1\}} f_2(Du_j)\, dx \\ &\geq \lim_j c_3 \Big|(0,1)^2 \cap \frac{1}{j}E \cap \{|Du_j| < 1\}\Big| = \frac{1}{4}c_3 > 0,\end{aligned}$$

and we reach a contradiction.

12.2.2 *Density of isotropic quadratic forms*

We have remarked that quadratic forms are closed by homogenization (see Exercise 1.7). We now show that in the more-than-one-dimensional case the coefficients of the limit quadratic form cannot be deduced from the behaviour of the corresponding oscillating coefficients. In particular, we show that all scalar quadratic functionals with constant coefficients can be obtained as Γ-limit of *isotropic* quadratic functionals.

For the sake of notational simplicity we will perform the proof in the two-dimensional case only. Let $(\overline{a}_{ij})_{i,j=1,2}$ be a 'target' symmetric positive definite matrix. We want to construct one-periodic coefficients a such that

$$\int_\Omega \sum_{i,j=1}^{2} \overline{a}_{ij} D_i u D_j u\, dx = \Gamma\text{-}\lim_{\varepsilon\to 0+} \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|Du|^2\, dx.$$

Upon a rotation of the axes we can suppose that the matrix $(\overline{a}_{ij})_{i,j=1,2}$ is diagonal; that is, $\overline{a}_{12} = \overline{a}_{21} = 0$. We then look for coefficients a of the form $a = a_1(x_1)a_2(x_2)$. Consider the functionals

$$F_\varepsilon(u) = \int_\Omega a_1\Big(\frac{x_1}{\varepsilon}\Big) a_2\Big(\frac{x_2}{\varepsilon}\Big)(|D_1 u|^2 + |D_2 u|^2)\, dx.$$

Their Γ-limit is described by the function $f_{\rm hom}$ satisfying the cell-problem formula (12.10) for $z = (z_1, z_2) \in \mathbf{R}^2$:

$$
\begin{aligned}
f_{\rm hom}(z) &= \inf\Big\{\int_{(0,1)^2} a(y)|z+Du|^2\,dy : u\in W^{1,2}_{\rm loc}(\mathbf{R}^2)\text{ 1-periodic}\Big\}\\
&= \inf\Big\{\int_0^1 a_2(t)\int_0^1 a_1(s)|z_1+D_1u(s,t)|^2\,ds\,dt\\
&\qquad + \int_0^1 a_1(s)\int_0^1 a_2(t)|z_2+D_2u(s,t)|^2 dtds : u\in W^{1,2}_{\rm loc}(\mathbf{R}^2)\text{ 1-periodic}\Big\}\\
&\geq \int_0^1 a_2(t)\inf\Big\{\int_0^1 a_1(s)|z_1+v'|^2\,ds : v\in W^{1,2}_{\rm loc}(\mathbf{R})\text{ 1-periodic}\Big\}\,dt\\
&\qquad + \int_0^1 a_1(s)\inf\Big\{\int_0^1 a_2(t)|z_2+v'|^2\,ds : v\in W^{1,2}_{\rm loc}(\mathbf{R})\text{ 1-periodic}\Big\}\,ds\\
&= \int_0^1 a_2(t)\,dt\Big(\int_0^1\frac{1}{a_1(s)}\,ds\Big)^{-1}z_1^2 + \int_0^1 a_1(s)\,ds\Big(\int_0^1\frac{1}{a_2(t)}\,ds\Big)^{-1}z_2^2,
\end{aligned}
$$

where we have used Fubini's Theorem and the one-dimensional homogenization result in Remark 2.36. Conversely, if u_i is a minimum point for

$$
\inf\Big\{\int_0^1 a_i(s)|z_i+v'|^2\,ds : v\in W^{1,2}_{\rm loc}(\mathbf{R})\text{ 1-periodic}\Big\}
$$

then we may use $u(x_1,x_2) = u_1(x_1)+u_2(x_2)$ as a test function to estimate $f_{\rm hom}(z)$ and check that indeed

$$
f_{\rm hom}(z) = \int_0^1 a_2(t)\,dt\Big(\int_0^1\frac{1}{a_1(s)}\,ds\Big)^{-1}z_1^2 + \int_0^1 a_1(s)\,ds\Big(\int_0^1\frac{1}{a_2(t)}\,ds\Big)^{-1}z_2^2.
$$

Now, coefficients a_i such that $f_{\rm hom}(z) = \overline{a}_{11}z_1^2+\overline{a}_{22}z_2^2$ can be easily constructed. This construction is left as an exercise (hint: choose piecewise-constant a_i).

Comments on Chapter 12

The notion of quasiconvexity is due to Morrey (1952) while that of polyconvexity to Ball (1977), who applies it to obtain existence theorems in non-linear elasticity. It must be remarked that in Theorem 12.4 we can consider f taking also the value $+\infty$, which allows to include constraints of the form $\det Du>0$. A fundamental result for the lower semicontinuity of integrals of the general form $\int_\Omega f(x,u,Du)\,dx$ can be found in Acerbi and Fusco (1984), whose method has been also extended to cover vector free-discontinuity problems. A general reference is the book by Dacorogna (1989).

The argument in Theorem 2.29 leading to convexity as a necessary condition for semicontinuity in dimension one can be repeated, but gives the convexity of f only on rank-one lines (i.e. on sets of the form $\{A+tB : t\in\mathbf{R}\}$ with B of rank one). This notion is called *rank-one convexity*. In general it is not sufficient for lower semicontinuity (and hence it does not imply quasiconvexity) as shown by a famous example of Šverák (1992). If F is given by (12.1) and (12.2) with f

non quasiconvex, then we can characterize the lower-semicontinuous envelope of F as $\mathrm{sc}\,F(u) = \int_\Omega Qf(Du)$, where Qf is the *quasiconvex envelope* of f given by

$$Qf(A) = \inf\Big\{\int_{(0,1)^N} f(A + D\varphi)\,dx : \varphi \in W_0^{1,p}((0,1)^N; \mathbf{R}^M)\Big\}$$

(we refer to Dacorogna (1989) or Braides and Defranceschi (1998) for details and generalizations). We note that again this formula may be derived from the one-dimensional considerations, once we understand that we have to use an integral characterization, and not a pointwise one, of convexity conditions.

The homogenization theorem in the vector-valued setting was proved by Müller (1987) with a direct proof and independently by Braides (1985) using the localization method (see Chapter 16). The paper by Müller also contains the fundamental example referred to in Remark 12.5. The proof of the instability of polyconvexity is a generalization of a method introduced by Šverák (1991) to give non trivial examples of quasiconvex functions (note that the final $f_{\rm hom}$ is an example of quasiconvex and not polyconvex function). The last example shows in particular that in the higher-dimensional case even the determination of the Γ-limit of a mixture of two isotropic quadratic energies (and more in general of two or more energies of a given form) is a non-trivial issue, contrary to the one-dimensional case, where the limit coefficients are determined by the local proportion of the two energies. For a treatment of these topics (mainly by methods other than Γ-convergence) we refer to the monograph by Milton (2002) and the references therein.

13

*DIRICHLET PROBLEMS IN PERFORATED DOMAINS

We now treat a problem which has no direct one-dimensional counterpart but to whose solution many arguments used in dimension one will be of help. The computation of the Γ-limit in this chapter will highlight a well-known result on the asymptotic behaviour of Dirichlet problems in perforated domains. The problems we deal with exhibit the appearance of a 'strange' extra term as the period of the perforation tends to 0. In order to explain this phenomenon we have to introduce some notation. Let Ω be a bounded open set in $\mathbf{R}^n$, $n \geq 3$ and for all $\delta > 0$ let Ω_δ be the *periodically perforated domain*

$$\Omega_\delta = \Omega \setminus \bigcup_{i \in \mathbf{Z}^n} \overline{B_i^\delta},$$

where B_i^δ denotes the open ball of centre $x_i^\delta = i\delta$ and radius $\delta^{n/(n-2)}$; that is, Ω_δ is obtained by removing a periodic collections of balls from a fixed open set Ω (see Fig. 0.8). Let $\phi \in L^2(\Omega)$ be fixed, and let $u_\delta \in W_0^{1,2}(\Omega)$ be the solution of the problem

$$\begin{cases} -\Delta u_\delta = \phi \\ u_\delta \in W_0^{1,2}(\Omega_\delta), \end{cases}$$

extended to 0 outside Ω_δ. Then, as $\delta \to 0$, the sequence u_δ converges weakly in $W_0^{1,2}(\Omega)$ to the function u which solves the problem

$$\begin{cases} -\Delta u + Cu = \phi \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where C denotes the *capacity of the unit ball* in $\mathbf{R}^n$:

$$C = \inf\Big\{ \int_{\mathbf{R}^n} |D\zeta|^2 dx : \zeta \in W^{1,2}(\mathbf{R}^n),\ \zeta = 1 \text{ on } B_1(0) \Big\}. \tag{13.1}$$

Summarizing, if we solve a sequence of elliptic boundary problems with Dirichlet boundary conditions on a finely-perforated domain, we may obtain as the limit of those solutions, a function solving another type of problem with an extra lower-order term. The reasons why the 'critical radii' must be of the size $\delta^{n/(n-2)}$ and why the capacity of the unit ball comes into play will be clear from the treatment of the Γ-limit.

The result described above can be easily translated in a equivalent variational form and set in the framework of Γ-convergence, since u_δ is the solution of the minimum problem

$$\min\Big\{\int_\Omega |Dv|^2\,dx - 2\int_\Omega \phi v\,dx : v\in W_0^{1,2}(\Omega),\ v=0 \text{ on } \Omega\setminus\Omega_\delta\Big\},$$

and the limit function u solves

$$\min\Big\{\int_\Omega (|Dv|^2 + C|v|^2 - 2\phi v)\,dx : v\in W_0^{1,2}(\Omega)\Big\}.$$

In this way, the convergence of the functions u_δ is explained as a consequence of the Γ-convergence of the corresponding functionals.

The heuristic idea is that, with fixed $v\in W_0^{1,2}(\Omega)$, we may estimate the limit of the energies corresponding to a sequence $v_\delta\in W_0^{1,2}(\Omega_\delta)$ converging to v by separating the contribution of the integral of $|Dv_\delta|^2$ 'close' to the perforation and that 'far' from the perforation. Since the Lebesgue measure of the perforation tends to 0 we may estimate the second term simply by $\int_\Omega |Dv|^2\,dx$, using the lower semicontinuity of the L^2-norm of the gradient. To estimate the term due to the perforation, we remark that we may suppose that $v_\delta\to v$ strongly in $L^2(\Omega)$, so that close to every ball B_i^δ the function v_δ must pass from the value 0 (on B_i^δ) to an average value comparable with the local average value of v (say, on a cube Q_i^δ of centre x_i^δ and side length δ). Optimizing this transition, a 'capacitary' computation shows that with our choice of the radii the contribution of the integral of $|Dv_\delta|^2$ 'close' to B_i^δ is at least $C\int_{Q_i^\delta}|v|^2\,dx$, so that a lower estimate for the Γ-limit is obtained by summing up all these contributions. The upper estimate is obtained by showing that these reasonings are sharp.

The crucial point in this argument is the separation of the contributions, which has to be made precise. We will do this by a 'joining lemma for perforated domains', which, loosely speaking, allows us to restrict our attention to families of functions (v_δ), converging to a function v, which equal a constant (which is comparable to $v(x_i^\delta)$) on suitable annuli surrounding B_i^δ. In this way the contribution of such functions can be decoupled in a part concentrated on such annuli (which gives a term of capacitary type) and a part on the complementary set, which is easily treated.

The proof of this lemma is interesting as it uses an argument which is often repeated in problems in the Calculus of Variations, that is used, for example, for the treatment of boundary valued (as in the proof of Theorem 12.1). We first perform a 'cut-off construction' to modify an arbitrary function so that it takes a constant value on a given annulus. Then the choice of the 'most convenient' annulus is done by carefully estimating the energy of the functions given by that construction on different annuli and choosing one with small energy.

13.1 Statement of the Γ-convergence result

In all that follows $n\geq 3$ is fixed and Ω is a bounded open subset of $\mathbf{R}^n$. For all $\delta>0$ we consider the lattice $\delta\mathbf{Z}^n$ whose points will be denoted by $x_i^\delta=\delta i$ ($i\in\mathbf{Z}^n$). Moreover, for all $i\in\mathbf{Z}^n$ $B_i^\delta = B_{\delta^{n/(n-2)}}(x_i^\delta)$ denotes the ball of centre x_i^δ and radius $\delta^{n/(n-2)}$. The convergence result is the following.

Theorem 13.1 *Let Ω be a bounded open subset of* $\mathbf{R}^n$ *with* $|\partial\Omega| = 0$. *Then the functionals* $F_\delta : W^{1,2}(\Omega) \to [0, +\infty]$ *defined by*

$$F_\delta(u) = \begin{cases} \int_\Omega |Du|^2\,dx & \text{if } u = 0 \text{ a.e. on } \bigcup_{i\in\mathbf{Z}^n} B_i^\delta \cap \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{13.2}$$

Γ-converge as $\delta \to 0$ with respect to the $L^2(\Omega)$ convergence to the functional $F : W^{1,2}(\Omega) \to [0, +\infty)$ *defined by*

$$F(u) = \int_\Omega |Du|^2\,dx + C\int_\Omega |u|^2\,dx, \tag{13.3}$$

with C the capacity of the unit ball defined in (13.1).

The proof of the theorem will be obtained in the rest of the chapter. We immediately deduce the limit behaviour of minimum problems.

Corollary 13.2 (convergence of minimum problems). *For all $\phi \in L^2(\Omega)$ the minimum values*

$$m_\delta = \inf\Big\{F_\delta(u) + \int_\Omega \phi u\,dx : \ u \in W_0^{1,2}(\Omega)\Big\}$$

converge to

$$m = \min\Big\{F(u) + \int_\Omega \phi u\,dx : \ u \in W_0^{1,2}(\Omega)\Big\}.$$

Moreover, if u_δ is such that $F_\delta(u_\delta) + \int_\Omega \phi u_\delta\,dx = m_\delta + o(1)$ as $\delta \to 0$, then it admits a subsequence weakly converging in $W_0^{1,2}(\Omega)$ to the solution of the problem defining m.

Proof By a cut-off argument near $\partial\Omega$ (here we can repeat, e.g. the proof of Proposition 2.37) if $u \in W_0^{1,2}(\Omega)$ then the recovery sequences in the definition of Γ-convergence can be taken in $W_0^{1,2}(\Omega)$ as well, while by the boundedness of the L^2 norm of the gradients we have $u_j \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$. This fact, together with the continuity of $G(u) = \int_\Omega \phi u\,dx$, implies that the functionals

$$\Phi_\delta(u) = \begin{cases} F_\delta(u) + G(u) & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty \end{cases}$$

Γ-converge to

$$\Phi_0(u) = \begin{cases} F(u) + G(u) & \text{if } u \in W_0^{1,2}(\Omega) \\ +\infty \end{cases}$$

on $W^{1,2}(\Omega)$. We can then apply Theorem 1.21 with $K = \{u \in W_0^{1,2}(\Omega) : \|Du\|_{L^2(\Omega)} \le c\}$ for a suitable $c > 0$. □

13.2 A joining lemma on varying domains

In this section we prove the technical result which allows to modify sequences of functions near the sets B_i^δ. We fix a sequence (δ_j) of positive numbers converging to 0. Note that in this section and the following ones sometimes we simply write δ in place of δ_j not to overburden notation.

Lemma 13.3 *Let (u_j) converge weakly to u in $W^{1,2}(\Omega)$, and let*

$$Z_j = \{i \in \mathbf{Z}^n : \ \mathrm{dist}\,(x_i^\delta, \mathbf{R}^n \setminus \Omega) > \delta_j\}. \tag{13.4}$$

Let $k \in \mathbf{N}$ be fixed. Let (ρ_j) be a sequence of positive numbers with $\rho_j < \delta_j/2$. For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \ 2^{-k_i-1}\rho_j < |x - x_i^\delta| < 2^{-k_i}\rho_j\Big\}, \tag{13.5}$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j\, dx \quad (\textit{the } \text{mean value } \textit{of } u_j \ \textit{on } C_i^j), \tag{13.6}$$

and

$$\rho_j^i = \tfrac{3}{4} 2^{-k_i}\rho_j \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{13.7}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $W^{1,2}(\Omega)$ such that

$$w_j = u_j \ \textit{on } \Omega \setminus \bigcup_{i \in Z_j} C_i^j \tag{13.8}$$

$$w_j(x) = u_j^i \ \textit{if } |x - x_i^\delta| = \rho_j^i \tag{13.9}$$

and

$$\int_\Omega \Big| |Dw_j|^2 - |Du_j|^2 \Big|\, dx \le c\frac{1}{k}. \tag{13.10}$$

Moreover, if $\rho_j = o(\delta_j)$ and the sequence $(|Du_j|^2)$ is equi-integrable (in particular if $u_j = u$), then we can choose $k_i = 0$ for all $i \in Z_j$.

Proof For all $j \in \mathbf{N}$, $i \in Z_j$ and $h \in \{0, ..., k-1\}$ let

$$C_{i,h}^j = \Big\{x \in \Omega : \ 2^{-h-1}\rho_j < |x - x_i^\delta| < 2^{-h}\rho_j\Big\},$$

and let

$$u_j^{i,h} = |C_{i,h}^j|^{-1} \int_{C_{i,h}^j} u_j\, dx, \qquad \text{and} \qquad \rho_j^{i,h} = \tfrac{3}{4} 2^{-h}\rho_j.$$

Consider a function $\phi = \phi_{i,h}^j \in C_0^\infty(C_{i,h}^j)$ such that $\phi = 1$ on $\partial B_{\rho_j^{i,h}}(x_i^\delta)$ and $|D\phi| \le c/2^{-h}\rho_j = c/\rho_j^{i,h}$. Let $w_j^{i,h}$ be defined on $C_{i,h}^j$ by

$$w_j^{i,h} = u_j^{i,h}\phi + (1-\phi)u_j \ \text{on } C_{i,h}^j,$$

with $\phi = \phi^j_{i,h}$ as above. We then have

$$\int_{C^j_{i,h}} |Dw^{i,h}_j|^2\,dx = \int_{C^j_{i,h}} |D\phi(u^{i,h}_j - u_j) + (1-\phi)Du_j)|^2\,dx$$
$$\leq c\int_{C^j_{i,h}} (|D\phi|^2|u_j - u^{i,h}_j|^2 + |Du_j|^2)\,dx.$$

By the Poincaré inequality and its scaling properties (see Appendix) we have

$$\int_{C^j_{i,h}} |u_j - u^{i,h}_j|^2\,dx \leq c(\rho^{i,h}_j)^2 \int_{C^j_{i,h}} |Du_j|^2\,dx, \tag{13.11}$$

so that, recalling that $|D\phi| \leq c/\rho^{i,h}_j$,

$$\int_{C^j_{i,h}} |Dw^{i,h}_j|^2\,dx \leq c\int_{C^j_{i,h}} |Du_j|^2\,dx.$$

Since by summing up in h we trivially have

$$\sum_{h=0}^{k-1}\int_{C^j_{i,h}} |Du_j|^2\,dx \leq \int_{B_{\rho_j}(x^\delta_i)} |Du_j|^2\,dx,$$

there exists $k_i \in \{0,\dots,k-1\}$ such that

$$\int_{C^j_{i,k_i}} |Du_j|^2\,dx \leq \frac{1}{k}\int_{B_{\rho_j}(x^\delta_i)} |Du_j|^2\,dx, \tag{13.12}$$

There follows that

$$\int_{C^j_{i,k_i}} |Dw^{i,k_i}_j|^2\,dx \leq \frac{c}{k}\int_{B_{\rho_j}(x^\delta_i)} |Du_j|^2\,dx. \tag{13.13}$$

By (13.12) and (13.13) we get

$$\int_{C^j_{i,k_i}} ||Du_j|^2 - |Dw^{i,k_i}_j|^2|\,dx \leq \int_{C^j_{i,k_i}} (|Du_j|^2 + |Dw^{i,k_i}_j|^2)\,dx$$
$$\leq \frac{c}{k}\int_{B_{\rho_j}(x^\delta_i)} |Du_j|^2\,dx.$$

Note that if $(|Du_j|^2)$ is equi-integrable and $\rho_j = o(\delta_j)$ then we do not need to use this argument, and may simply choose $k_i = 0$ for all $i \in Z_j$.

With this choice of k_i for all $i \in Z_j$, conditions (13.8)—(13.10) are satisfied by choosing $h = k_i$ in the definitions above, that is, with $C^j_i = C^j_{i,k_i}$, $u^i_j = u^{i,k_i}_j$

$\rho^i_j = \rho^{i,k_i}_j$ and w_j defined by (13.8) and $w_j = u^i_j\phi + (1-\phi)u_j$ on C^j_i, with $\phi = \phi^j_{i,k_i}$.

Finally, we prove the convergence of w_j to u in $L^2(\Omega)$. By (13.11)

$$\begin{aligned}\int_\Omega |w_j - u|^2\,dx &= \int_{\Omega\setminus\bigcup_{i\in Z_j} C^j_i} |u_j - u|^2\,dx \\ &\quad + \int_{\bigcup_{i\in Z_j} C^j_i} |u^i_j\phi^j_{i,k_i} + (1-\phi^j_{i,k_i})u_j - u|^2\,dx \\ &\le \int_{\Omega\setminus\bigcup_{i\in Z_j} C^j_i} |u_j - u|^2\,dx \\ &\quad + c\sum_{i\in Z_j}\int_{C^j_i} |u_j - u^i_j|^2\,dx + c\int_{\bigcup_{i\in Z_j} C^j_i} |u_j - u|^2\,dx \\ &\le c\int_\Omega |u_j - u|^2\,dx + c\rho_j^2\sum_{i\in Z_j}\int_{C^j_i} |Du_j|^2\,dx \\ &\le c\int_\Omega |u_j - u|^2\,dx + c\rho_j^2\sup_j\int_\Omega |Du_j|^2\,dx.\end{aligned}$$

Hence passing to the limit as j tends to $+\infty$ we get the desired convergence. In particular, since (w_j) is bounded in $W^{1,2}(\Omega)$, we get that (w_j) weakly converges to u in $W^{1,2}(\Omega)$. □

The next proposition provides a 'discretization' of $\int_\Omega |u|^2\,dx$.

Proposition 13.4 *Let (u_j) be a sequence weakly converging to u in $W^{1,2}(\Omega)$, let (C^j_i) $(i \in Z_j)$ be a collection of annuli of the form* (13.5) *for an arbitrary choice of k_i, let u^i_j be defined by* (13.6), *and let ψ_j be defined by*

$$Q^\delta_i = x^\delta_i + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n, \qquad \psi_j = \sum_{i\in Z_j} |u^i_j|^2\chi_{Q^\delta_i}. \tag{13.14}$$

Then we have

$$\lim\nolimits_j \int_\Omega |\psi_j - |u|^2|\,dx = 0. \tag{13.15}$$

Proof By Hölder's and Poincaré's inequalities, we have

$$\int_{Q^\delta_i} |u^i_j - u_j|\,dx \le \delta_j^{n/2}\Big(\int_{Q^\delta_i} |u^i_j - u_j|^2\,dx\Big)^{1/2} \le \delta_j^{n/2}c\delta_j\Big(\int_{Q^\delta_i} |Du_j|^2\,dx\Big)^{1/2},$$

so that

$$\sum_{i\in Z_j}\int_{Q^\delta_i} |u^i_j - u_j|\,dx \le c\delta_j\Big(\int_\Omega |Du_j|^2\,dx\Big)^{1/2},$$

which proves the limit in (13.15), after using a triangular inequality. □

13.3 Proof of the lim inf inequality

Let $u \in W^{1,2}(\Omega)$ and let $u_j \to u$ in $L^2(\Omega)$ be such that $\sup_j F_j(u_j) < +\infty$. Note that $u_j \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$. We can use a sequence (w_j) constructed as in Lemma 13.3 to estimate the liminf inequality for (F_j).

We fix $k, N \in \mathbf{N}$ with $N > 2^k$, and define w_j as in Lemma 13.3 with

$$\rho_j = N\delta_j^{n/(n-2)}. \tag{13.16}$$

Note that with this choice of ρ_j we always have $w_j = u_j = 0$ on B_i^δ. Let $E_j = E_j^{k,N}$ be given by

$$E_j = \bigcup_{i \in Z_j} B_i^j, \qquad \text{where} \qquad B_i^j = B_{\rho_j^i}(x_i^\delta)$$

for all $i \in Z_j$ (Z_j given by (13.4) and ρ_j^i by (13.7)). We first deal with the contribution of the part of Du_j outside the set E_j.

Proposition 13.5 *We have*

$$\liminf_j \int_{\Omega \setminus E_j} |Du_j|^2\, dx \geq \int_\Omega |Du|^2\, dx - \frac{c}{k} \tag{13.17}$$

Proof Let

$$v_j(x) = \begin{cases} u_j^i & \text{if } x \in B_i^j \\ w_j(x) & \text{if } x \in \Omega \setminus E_j. \end{cases}$$

Note that by Lemma 13.3 (v_j) is bounded in $W^{1,2}(\Omega)$ and that $\lim_j |\{x \in \Omega : u_j(x) \neq v_j(x)\}| = 0$. We deduce that $v_j \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$ so that

$$\begin{aligned} \liminf_j \int_{\Omega \setminus E_j} |Du_j|^2\, dx + \frac{c}{k} &\geq \liminf_j \int_{\Omega \setminus E_j} |Dw_j|^2\, dx \\ &= \liminf_j \int_\Omega |Dv_j|^2\, dx \geq \int_\Omega |Du|^2\, dx, \end{aligned}$$

the last inequality following from the weak lower semicontinuity of the norm. □

We now turn to the estimate of the contribution on E_j. With fixed $j \in \mathbf{N}$ and $i \in Z_j$, let

$$\zeta(y) = w_j\left(x_i^\delta + \delta_j^{n/(n-2)} y\right)$$

be defined on $B_{\frac{3}{4}2^{-k_i}N}(0)$, and extended to u_j^i outside this ball. Note that

$$\zeta - u_j^i \in W_0^{1,2}(B_N(0)) \qquad \text{and} \qquad \zeta = 0 \text{ on } B_1(0). \tag{13.18}$$

By a change of variables we obtain

$$\int_{B_i^j} |Dw_j|^2\,dx = \delta_j^n \int_{B_N(0)} |D\zeta|^2\,dx \geq \delta_j^n C|u_j^i|^2. \tag{13.19}$$

Hence, to give the estimate on E_j we have to compute the limit

$$\liminf_j \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 = \liminf_j \int_\Omega \psi_j\,dx, \tag{13.20}$$

where ψ_j is defined as in (13.14).

Proposition 13.6 *We have*

$$\Gamma\text{-}\liminf_j F_j(u) \geq \int_\Omega |Du|^2\,dx + C\int_\Omega |u|^2\,dx$$

for all $u \in W^{1,2}(\Omega)$.

Proof Let $u_j \to u$ in $L^2(\Omega)$. We can assume, upon possibly passing to a subsequence, that there exists the limit

$$\lim_j F_j(u_j) < +\infty,$$

so that $u_j \rightharpoonup u$ in $W^{1,2}(\Omega)$. From Lemma 13.3, (13.20), and Proposition 13.4, we get that

$$\liminf_j \int_{E_j} |Du_j|^2\,dx + \frac{c}{k} \geq \liminf_j \sum_{i\in Z_j} \delta_j^n C|u_j^i|^2 \geq C\int_\Omega |u|^2\,dx. \tag{13.21}$$

Summing up (13.21) and (13.17) and by the arbitrariness of k, we then obtain

$$\liminf_j F_j(u_j) \geq \int_\Omega |Du|^2\,dx + C\int_\Omega |u|^2\,dx \tag{13.22}$$

as desired. □

13.4 Proof of the lim sup inequality

The limsup inequality is obtained by suitably modifying the target function u close to the perforation.

Proposition 13.7 *If* $|\partial\Omega| = 0$ *then we have*

$$\Gamma\text{-}\limsup_j F_j(u) \leq \int_\Omega |Du|^2\,dx + C\int_\Omega |u|^2\,dx$$

for all $u \in W^{1,2}(\Omega)$.

Proof Let $u \in W^{1,2}(\Omega)$. With fixed $N \in \mathbf{N}$, by Lemma 13.3 applied with $u_j = u$,

$$\rho_j = \frac{4}{3} N \delta_j^{n/(n-2)},$$

and taking the equi-integrability condition into account we obtain functions v_j (the w_j of the lemma) which equal a constant u_i^j (the average of u on C_i^j) on $\partial B_{\rho'_j}(x_i^\delta)$ for all $i \in Z_j$, where

$$\rho'_j = N \delta_j^{n/(n-2)}.$$

We first assume that in addition $u \in L^\infty(\Omega)$. Let $\eta > 0$ be fixed. We now modify the sequence (v_j) to obtain functions $u_j \in W^{1,2}(\Omega)$ such that

$$u_j = v_j \text{ on } \Omega \setminus \bigcup_{i \in \mathbf{Z}^n} B_{\rho'_j}(x_i^\delta), \qquad u_j = 0 \text{ on } \Omega \cap \bigcup_{i \in \mathbf{Z}^n} B_i^\delta$$

and

$$\limsup_j \int_{\Omega \cap \bigcup_{i \in \mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} |Du_j|^2 \, dx \le C \int_\Omega |u|^2 \, dx + \eta |\Omega|. \tag{13.23}$$

The sequence (u_j) will then be a recovery sequence for the limsup inequality. In fact, clearly $u_j \to u$ in $L^2(\Omega)$ since $\lim_j |\{u_j \neq v_j\}| = 0$ and (u_j) is bounded in $W^{1,2}(\Omega)$, and

$$\begin{aligned}
\limsup_j \int_\Omega |Du_j|^2 \, dx &\le \limsup_j \int_{\Omega \setminus \bigcup_{i \in \mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} |Dv_j|^2 \, dx \\
&\quad + \limsup_j \int_{\Omega \cap \bigcup_{i \in \mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} |Du_j|^2 \, dx \\
&\le \lim_j \int_\Omega |Dv_j|^2 \, dx + C \int_\Omega |u|^2 \, dx + \eta |\Omega| \\
&= \int_\Omega |Du|^2 \, dx + C \int_\Omega |u|^2 \, dx + \eta |\Omega|.
\end{aligned} \tag{13.24}$$

We now define u_j on each $B_{\rho'_j}(x_i^\delta) \cap \Omega$. We separately treat the cases $i \in Z_j$ and $i \in \mathbf{Z}^n \setminus Z_j$. For all $N > 1$ let C_N be defined by

$$C_N = \inf\Big\{ \int_{\mathbf{R}^n} |D\zeta|^2 dx : \zeta \in W_0^{1,2}(B_N(0)), \ \zeta = 1 \text{ on } B_1(0) \Big\} \tag{13.25}$$

(the *capacity of* $B_1(0)$ *with respect to* $B_N(0)$). It can be easily checked that C_N converges decreasingly to C. We can choose N such that

$$C \geq C_N - \frac{\eta}{3\|u\|^2_{L^\infty(\Omega)}}. \tag{13.26}$$

Let $\zeta^i_j \in v^i_j + W^{1,2}_0(B_N(0))$ be such that $\zeta^i_j = 0$ on $B_1(0)$ and

$$\int_{B_N(0)} |D\zeta^i_j|^2\, dx \leq C_N |v^i_j|^2 + \frac{\eta}{3} \leq C|v^i_j|^2 + \eta, \tag{13.27}$$

the last inequality being a consequence of (13.26) and (13.25), taking into account that $|u^j_i| \leq \|u\|_{L^\infty(\Omega)}$. We define u_j on $B_{\rho'_j}(x^\delta_i)$ by

$$u_j(x) = \zeta^i_j\Big((x - x^\delta_i)\delta_j^{-n/(n-2)}\Big).$$

By a change of variables we then have

$$\int_{B_{\rho'_j}(x^\delta_i)} |Du_j|^2\, dx = \delta^n_j \int_{B_N(0)} |D\zeta^i_j|^2\, dx \leq \delta^n_j |u^j_i|^2 + \delta^n_j \eta. \tag{13.28}$$

If $i \notin Z_j$ it is not possible to use the construction above since $B_{\rho'_j}(x^\delta_i)$ might intersect $\partial\Omega$. We then consider $\zeta \in W^{1,2}(B_N(0))$ such that $\zeta - 1 \in W^{1,2}_0(B_N(0))$, $0 \leq \zeta \leq 1$ and $\zeta = 0$ on $B_1(0)$, and simply define

$$u_j(x) = u(x)\, \zeta\Big((x - x^\delta_i)\delta_j^{-n/(n-2)}\Big)$$

on $B_{\rho'_j}(x^\delta_i) \cap \Omega$. We then have

$$\begin{aligned}
&\int_{B_{\rho'_j}(x^\delta_i)\cap\Omega} |Du_j|^2\, dx \\
&\leq \int_{B_{\rho'_j}(x^\delta_i)\cap\Omega} \Big(|Du|^2 + \delta_j^{-2n/(n-2)} \Big|D\zeta\Big((x - x^\delta_i)\delta_j^{-n/(n-2)}\Big)\Big|^2 |u|^2\Big)\, dx \\
&\leq c\delta^n_j \Big(\|u\|^2_{L^\infty(\Omega)} \int_{B_N(0)} |D\zeta|^2\, dx\Big) + c\int_{B_{\rho'_j}(x^\delta_i)\cap\Omega} |Du|^2\, dx.
\end{aligned} \tag{13.29}$$

Let

$$E'_j = \bigcup_{i\in \mathbf{Z}^n\setminus Z_j} B_{\rho'_j}(x^\delta_i)\cap\Omega \quad \text{and} \quad \Omega'_j = \bigcup\{Q^\delta_i : i \in \mathbf{Z}^n \setminus Z_j,\ Q^\delta_i \cap \Omega \neq \emptyset\}.$$

Then (13.29) above implies that

$$\int_{E'_j} |Du_j|^2\, dx \leq c\Big(|\Omega'_j| + \int_{E'_j} |Du|^2\, dx\Big) = o(1), \tag{13.30}$$

by the fact that $\lim_j |\Omega'_j| = |\partial\Omega| = 0$.

Taking (13.28) and (13.30) into account, we have

$$\limsup_j \int_{\Omega\cap\bigcup_{i\in\mathbf{Z}^n} B_{\rho'_j}(x_i^\delta)} |Du_j|^2\,dx \le C\limsup_j \sum_{i\in Z_j} \delta_j^n |u_i^j|^2 + \eta|\Omega|,$$

so that (13.23) is proved by Proposition 13.4.

Finally, for arbitrary $u \in W^{1,2}(\Omega)$, simply note that it can be approximated by a sequence of functions $u_k \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ with respect to the strong convergence of $W^{1,2}(\Omega)$. □

Comments on Chapter 13

The asymptotic behaviour of Dirichlet problems in varying domains is a very much studied subject, also with methods different from Γ-convergence. If no assumption is made on the geometry of the perforation, in the limit we obtain what is called a *relaxed Dirichlet problem*; that is, related to a Γ-limit of the form

$$\int_\Omega |Du|^2\,dx + \int_\Omega |u|^2\,d\mu,$$

where μ is a measure possibly taking also the value $+\infty$ and that is 0 on set with capacity 0. If the perforation is periodic as in the example then $\mu = c\,dx$. Results of this type, not stated in terms of Γ-convergence, date back to Marchenko and Khrushlov (1974), and in closer terms to Cioranescu and Murat (1982). These problems have been treated within the framework of the localization methods of Γ-convergence by Dal Maso and Mosco (1987), Buttazzo *et al.* (1987), and many other authors. We refer to the review article by Dal Maso (1997) for more bibliographical information.

The approach we present here is taken from Ansini and Braides (2002); it is based on minimization considerations only and can be easily extended to nonlinear vector problems. The main lemma is inspired by the method introduced by De Giorgi (1975) to treat boundary-value problems within the theory of Γ-convergence.

14

*DIMENSION-REDUCTION PROBLEMS

In this chapter we briefly describe an approach to the derivation of 'lower-dimensional' theories from 'full-dimensional' energies (see Example 0.3). For the sake of simplicity we prove a convergence result for the scalar convex case of quadratic growth and the passage from n to $n-1$ dimensions only. Important differences arise when dealing with non-convex energies defined on vector-valued functions, for which we provide an example.

14.1 Convex energies

Let ω be a bounded open subset of $\mathbf{R}^{n-1}$, and let $f : \mathbf{R}^n \to \mathbf{R}$ be a strictly-convex function with quadratic growth; that is,

$$c_1|z|^2 - c_2 \le f(z) \le c_3(1+|z|^2) \qquad \text{for all } z \in \mathbf{R}^n.$$

For all $\varepsilon > 0$ we consider the energy

$$E_\varepsilon(u) = \int_{\omega\times(0,\varepsilon)} f(Du)\,dx$$

defined on $W^{1,2}(\omega \times (0,\varepsilon))$. The problem we have in mind is the following: *Do these energies defined on thin n-dimensional domains 'converge' to an energy defined on an $n-1$-dimensional domain?*

To provide a (meaningful) answer to this question we have first to scale the energies E_ε, otherwise the limit energy is trivially 0, and perform a change of

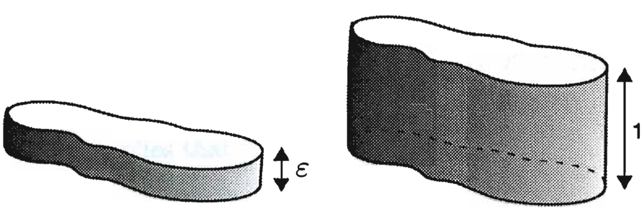

FIG. 14.1. Thin and scaled domains

variables, so that the competing functions are defined in the same fixed domain (see Fig. 14.1). We are then let to study the energy

$$F_\varepsilon(u) = \int_{\omega\times(0,1)} f\Big(D_1u, D_2u, \ldots, D_{n-1}u, \frac{1}{\varepsilon}D_nu\Big)\,dx \tag{14.1}$$

defined on $W^{1,2}(\omega \times (0,1))$. In this way we have

$$F_\varepsilon(u) = \frac{1}{\varepsilon}E_\varepsilon(\tilde u), \qquad \text{where} \qquad \tilde u(x) = u\Big(x_1, \ldots, x_{n-1}, \frac{x_n}{\varepsilon}\Big). \tag{14.2}$$

In order to shorter notation we set $\Omega = \omega \times (0,1)$ and $\hat x = (x_1, \ldots, x_{n-1})$, $x = (\hat x, x_n)$, $\widehat{D}u = (D_1u, \ldots, D_{n-1}u)$, $Du = (\widehat{D}u, D_nu)$.

Compactness and dimension-reduction

Let $\varepsilon_j \to 0$ and let (u_j) be such that $\sup_j(\|u_j\|_{L^2(\Omega)} + F_{\varepsilon_j}(u_j)) < +\infty$. By the estimates

$$F_{\varepsilon_j}(u_j) + c_2|\Omega| \geq c_1\int_\Omega |\widehat{D}u_j|^2\,dx + \frac{1}{\varepsilon_j^2}c_1\int_\Omega |D_nu_j|^2\,dx, \tag{14.3}$$

we first deduce that

$$\text{(i) } \sup_j \int_\Omega |Du_j|^2\,dx < +\infty, \qquad \text{(ii) } \sup_j \frac{1}{\varepsilon_j^2}\int_\Omega |D_nu_j|^2\,dx < +\infty, \tag{14.4}$$

From the boundedness in $L^2(\Omega)$ and (14.4)(i), we get that, upon extracting a subsequence, (u_j) weakly converges to a function u in $W^{1,2}(\Omega)$. Next, from (14.4)(ii) and the lower semicontinuity of the norm, we obtain that

$$\int_\Omega |D_nu|^2\,dx \leq \liminf_j \int_\Omega |D_nu_j|^2\,dx = 0,$$

so that $D_nu = 0$ a.e.; that is, u *is independent of* x_n. Hence, there exists $w \in W^{1,2}(\omega)$ such that $u(x) = w(\hat x)$. In this sense, *the domain of the limit energy is* $n-1$ *dimensional.*

Lower bound

Since the limit is finite only on functions independent of x_n, a lower bound is trivially obtained by minimizing the effect of the derivative in that direction: for $z \in \mathbf{R}^{n-1}$ set

$$\overline{f}(z) = \min\{f(z,b) : \ b \in \mathbf{R}\}.$$

We obviously have

$$F_\varepsilon(u) \geq \int_\Omega \overline{f}(\widehat{D}u)\,dx. \tag{14.5}$$

Note that $\overline{f}$ is of quadratic growth and it is convex on $\mathbf{R}^{n-1}$: let $z_1, z_2 \in \mathbf{R}^{n-1}$ and $t \in (0,1)$. If $\overline{b}$ is such that $\overline{f}(tz_1 + (1-t)z_2) = f(tz_1 + (1-t)z_2, \overline{b})$, then we get

$$\overline{f}(tz_1 + (1-t)z_2) \geq tf(z_1, \overline{b}) + (1-t)f(z_2, \overline{b}) \geq t\overline{f}(z_1) + (1-t)\overline{f}(z_2).$$

In particular, we deduce that the functional defined on the right-hand side of (14.5) is weakly lower semicontinuous in $W^{1,2}(\Omega)$. Hence, if $u_j \to u$ in $L^2(\Omega)$ and $F_{\varepsilon_j}(u_j)$ are equibounded, then by the compactness argument above $u_j \rightharpoonup u$ in $W^{1,2}(\Omega)$ and

$$\liminf_j F_{\varepsilon_j}(u_j) \geq \liminf_j \int_\Omega \overline{f}(\widehat{D}u_j)\,dx \geq \int_\Omega \overline{f}(\widehat{D}u)\,dx. \tag{14.6}$$

Upper bound

We now show that the lower bound is reached. We fix $w \in W^{1,2}(\omega)$ and set $u(x) = w(\hat{x})$. Let $b(x) \in L^2(\omega)$ be defined by $\overline{f}(\widehat{D}w(\hat{x})) = f(\widehat{D}w(\hat{x}), b(\hat{x}))$. This function is well defined by the strict convexity of f. A requirement for the recovery sequence (u_ε) is that

$$\widehat{D}u_\varepsilon = \widehat{D}w + o(1), \qquad D_n u_\varepsilon = \varepsilon b,$$

so that a 'natural' candidate for a recovery sequence is

$$u_\varepsilon(x) = w(\hat{x}) + \varepsilon x_n\, b(\hat{x}).$$

unfortunately, b need not be differentiable, so we have to use in addition an approximation argument and construct an 'approximate' recovery sequence.

With fixed $\eta > 0$ choose $b_\eta \in W^{1,2}(\omega)$ such that $\|b - b_\eta\|_{L^2} \leq \eta$. We then define

$$u_\varepsilon^\eta(x) = w(\hat{x}) + \varepsilon x_n\, b_\eta(\hat{x}),$$

so that

$$\widehat{D}u_\varepsilon^\eta(x) = \widehat{D}w(\hat{x}) + \varepsilon x_n\, \widehat{D}b_\eta(\hat{x}), \qquad D_n u_\varepsilon^\eta = \varepsilon b_\eta(\hat{x}).$$

We then have, using the convexity of f and Hölder's inequality,

$$\begin{aligned}
F_\varepsilon(u_\varepsilon^\eta) &= \int_\Omega f(\widehat{D}w(\hat{x}) + \varepsilon x_n\, \widehat{D}b_\eta(\hat{x}), b_\eta(\hat{x}))\,dx \\
&= \int_\Omega \overline{f}(\widehat{D}u)\,dx \\
&\quad + \int_\Omega \Big(f(\widehat{D}w(\hat{x}) + \varepsilon x_n\, \widehat{D}b_\eta(\hat{x}), b_\eta(\hat{x})) - f(\widehat{D}w(\hat{x})), b(\hat{x})\Big)\,dx \\
&\leq \int_\Omega \overline{f}(\widehat{D}u)\,dx \\
&\quad + c\int_\Omega (1 + |\widehat{D}w| + |\varepsilon x_n \widehat{D}b_\eta| + |b| + |b_\eta|)(|\varepsilon x_n \widehat{D}b_\eta| + |b_\eta - b|)\,dx \\
&\leq \int_\Omega \overline{f}(\widehat{D}u)\,dx + c\Big(\int_\Omega (1 + |\widehat{D}w|^2 + |\varepsilon x_n \widehat{D}b_\eta|^2 + |b|^2 + |b_\eta|^2)\,dx\Big)^{1/2}
\end{aligned}$$

$$\times\Big(\int_\Omega (|\varepsilon x_n \widehat{D} b_\eta|^2 + |b_\eta - b|^2)\,dx\Big)^{1/2}. \tag{14.7}$$

By taking the limit as $\varepsilon \to 0^+$ we then get

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon^\eta) \le \int_\Omega \overline{f}(\widehat{D}u)\,dx + c\Big(\int_\Omega (1+|\widehat{D}w|^2 + |b|^2 + |b_\eta|^2)\,dx\Big)^{1/2} \|b_\eta - b\|_{L^2},$$

which gives the upper bound by the arbitrariness of η.

We have proved the following result.

Theorem 14.1 (dimension reduction). *Let f be as above. Then we have*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\omega\times(0,\varepsilon)} f(Du)\,dx = \int_\omega \overline{f}(\widehat{D}u)d\hat{x},$$

where $\overline{f}(z) = \min\{f(z,b) : b \in \mathbf{R}\}$, upon identifying $\omega\times(0,\varepsilon)$ with $\omega\times(0,1)$ by scaling in the nth variable and $W^{1,2}(\omega)$ with the functions in $\omega\times(0,1)$ independent of the nth variable, as described above.

14.2 Non-convex vector-valued problems

Theorem 14.1 shows that in the convex case dimension reduction amounts to eliminating the dependence of the derivative with respect to the 'small' direction by means of a minimization procedure. This process is consistent with the lower semicontinuity requirements since it maintains convexity. It can be seen that the same statement holds when u is vector valued if we still suppose f convex (the verification is left as an exercise). This is not the case for other conditions which are sufficient for the lower semicontinuity of functionals depending on vector-valued functions. We now describe a simple example in the case $n = 2$.

Consider the polyconvex function $f : \mathbf{M}^{2\times 2} \to \mathbf{R}$ defined by

$$f(A) = \det A + (A_{12})^2 + (A_{22})^2 + (A_{11}^2 + A_{21}^2 - 1)^+$$

(t^+ is the positive part of t). If we compute

$$\overline{f}(z) = \min\{f(z,b) : b \in \mathbf{R}^2\},$$

where in this notation we have identified the vector b with the second column of a 2×2 matrix whose first column is z, we easily get

$$\overline{f}(z) = \begin{cases} -\frac{1}{4}|z|^2 & \text{if } |z| \le 1 \\ \frac{3}{4}|z|^2 - 1 & \text{if } |z| > 1, \end{cases}$$

which is not convex. Hence, the corresponding functional is nor lower semicontinuous and cannot describe the dimension-reduction process.

Comments on Chapter 14

The remark that quasiconvexity is not maintained by dimension-reduction is due to Le Dret and Raoult (1995). A characterization of the limit energy in a general vector setting is given by

$$\int_\omega Q\overline{f}(\widehat{D}u)\, d\hat{x},$$

where Q denotes the operation of quasiconvexification in the low-dimensional space (the proof of this formula is a suggested exercise). An approach using the localization methods of Chapter 16 is given in Braides *et al.* (2000) and it is applied to the description of thin films with varying profile; as a consequence of the representation result therein an alternative asymptotic formula can be given for $Q\overline{f}(A)$, which interprets the necessity of quasiconvexification as a homogenization phenomenon. More applications of this approach can be found in Shu (2000).

Other dimension-reduction problems within the framework of Γ-convergence have been studied, for example, by Acerbi *et al.* (1991), Anzellotti *et al.* (1994), Fonseca and Francfort (1998). Different scalings give rise to higher-order functionals with bending terms (see Friesecke *et al.* (2002)). For an introduction to shell theory in linear elasticity see Ciarlet (1998).

15

*THE 'SLICING' METHOD

In this section we describe a fruitful method to recover the lower semicontinuity inequality for Γ-limits through the study of one-dimensional problems by a 'sectioning' argument. The main idea of this method is the following. Let F_ε be a sequence of functionals defined on a space of functions with domain a fixed open set $\Omega \subset \mathbf{R}^n$. Then we may examine the behaviour of F_ε on one-dimensional sections as follows: for each $\xi \in S^{n-1}$ we consider the hyperplane

$$\Pi_\xi := \{z \in \mathbf{R}^n : \langle z, \xi \rangle = 0\} \tag{15.1}$$

passing through 0 and orthogonal to ξ. For each $y \in \Pi_\xi$ we then obtain the one-dimensional set (see Fig. 15.1)

$$\Omega_{\xi,y} := \{t \in \mathbf{R} : y + t\xi \in \Omega\}, \tag{15.2}$$

and for all u defined on Ω we define the one-dimensional function

$$u_{\xi,y}(t) = u(y + t\xi) \tag{15.3}$$

defined on $\Omega_{\xi,y}$. We may then give a lower bound for the Γ-liminf of F_ε by looking at the limit of some functionals 'induced by F_ε' on the one-dimensional sections.

We will treat in detail the case of the gradient theory of phase transitions only. In this case the functionals we start with are

$$F_\varepsilon(u) = \int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \varepsilon |Du|^2\Big)\, dx, \tag{15.4}$$

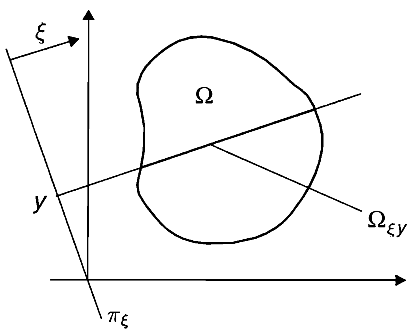

FIG. 15.1. A section of the domain Ω

which we can rewrite, using Fubini's Theorem as

$$F_\varepsilon(u) = \int_{\Pi_\xi} \int_{\Omega_{\xi,y}} \Big(\frac{1}{\varepsilon} W(u(y+t\xi)) + \varepsilon |Du(y+t\xi)|^2\Big)\, dt\, dy. \tag{15.5}$$

The idea of the 'slicing' method is to use the Γ-limit of the one-dimensional functionals

$$v \mapsto \int_{\Omega_{\xi,y}} \Big(\frac{1}{\varepsilon} W(v(t)) + \varepsilon |v'(t)|^2\Big)\, dt$$

and the trivial inequality (from (15.5))

$$F_\varepsilon(u) \geq \int_{\Pi_\xi} \int_{\Omega_{\xi,y}} \Big(\frac{1}{\varepsilon} W(u_{\xi,y}(t)) + \varepsilon |u'_{\xi,y}(t)|^2\Big)\, dt\; dy, \tag{15.6}$$

to give a lower bound for the Γ-liminf of F_ε, by using Fatou's Lemma and (locally) optimizing the choice of ξ.

We stress the fact that the method does not only apply to this problem, but can be fruitfully used to prove, for example, lower semicontinuity and Γ-convergence results in the framework of free-discontinuity problems.

15.1 A lower inequality by the slicing method

We now pass to the more detailed treatment of the example. We have seen in Chapter 6 how the limit of perturbed non-convex functionals gives rise to an energy on piecewise-constant functions. In the special case when the non-convex energy density possesses exactly two minimum points, say 0 and 1, the domain of the limit energy is given exactly by the characteristic functions of intervals. In this section we will give an analogous result in more than one dimension. The analogue of the class of all finite unions of intervals will be the class of the sets whose boundary has finite surface measure. This concept has to be given in a sufficiently-weak form that leads to the definition of *set of finite perimeter*, which is recalled in Appendix A together with some technical facts that will be used in this section. For the reader not acquainted with this notion, as long as the description of the slicing method is concerned, it is sufficient to think of those sets as sets E with (sufficiently-)smooth boundary, so that a normal to the boundary of E is defined up to a negligible set with respect to the $(n-1)$-dimensional surface measure.

We now give an estimate for the Γ-liminf of the family of functionals

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega W(u)\, dx + \varepsilon \int_\Omega |Du|^2\, dx & \text{if } u \in W^{1,2}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{15.7}$$

defined on $L^1(\Omega)$. We suppose that $W : \mathbf{R} \to [0, +\infty)$ is a C^1 function such that $W(z) = 0$ if and only if $z \in \{0, 1\}$ and satisfying a 2-growth condition. We will prove the following estimate (see Appendix A for notation).

Proposition 15.1 *Under the hypotheses above, we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u) \geq c_W\, P(u)\,, \tag{15.8}$$

where $P : L^1(\Omega) \to [0,+\infty]$ *is the* perimeter functional *defined by*

$$P(u) = \begin{cases} \mathcal{H}^{n-1}(\partial^* E) & \text{if } u = \chi_E \text{ with } E \text{ of finite perimeter} \\ +\infty & \text{otherwise,} \end{cases} \tag{15.9}$$

and $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$.

Proof This proposition will be proven by localizing the estimate (15.6) and by the characterization of sets of finite perimeter by their one-dimensional sections (Theorem A.16), to obtain the functional P by optimizing the role of ξ. The procedure can be summarized in the following steps, which we state in a more general form, to highlight the generality of the method.

1. *We 'localize' the functional* F_ε *highlighting its dependence on the set of integration.*

This is done by defining functionals $F_\varepsilon(\cdot, A)$ for all open subsets $A \subset \Omega$:

$$F_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A W(u)\,dx + \varepsilon\int_A |Du|^2\,dx & \text{if } u \in W^{1,2}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{15.10}$$

2. *For all* $\xi \in S^{n-1}$ *and for all* $y \in \Pi_\xi$, *we find functionals* $F_\varepsilon^{\xi,y}(v,I)$, *defined for* $I \subset \mathbf{R}$ *and* $v \in L^1(I)$, *such that setting*

$$F_\varepsilon^\xi(u,A) = \int_{\Pi_\xi} F_\varepsilon^{\xi,y}(u_{\xi,y}, A_{\xi,y})\,d\mathcal{H}^{n-1}(y) \tag{15.11}$$

we have $F_\varepsilon(u,A) \geq F_\varepsilon^\xi(u,A)$.

In the specific case we choose

$$F_\varepsilon^{\xi,y}(v,I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I W(v)\,dt + \varepsilon\int_I |v'|^2\,dt & \text{if } v \in W^{1,2}(I) \\ +\infty & \text{otherwise} \end{cases} \tag{15.12}$$

(independent of y). We then have, by Fubini's Theorem,

$$F_\varepsilon^\xi(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A W(u)\,dx + \varepsilon\int_A |\langle \xi, Du\rangle|^2\,dx & \text{if } u_{\xi,y} \in W^{1,2}(A_{\xi,y}) \text{ for a.e. } y \\ +\infty & \text{otherwise.} \end{cases} \tag{15.13}$$

3. *We compute the* Γ-$\liminf_{\varepsilon\to 0} F_\varepsilon^{\xi,y}(v,I) = F^{\xi,y}(v,I)$ *and define*

$$F^\xi(u,A)=\int_{\Pi_\xi} F^{\xi,y}(u_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)\,. \tag{15.14}$$

By Theorem 6.4 we have

$$F^{\xi,y}(v,I)=\begin{cases} c_W\#(S(v)) & \text{if } v\in PC(I) \text{ and } v\in\{0,1\} \text{ a.e. on } I,\\ +\infty & \text{otherwise.}\end{cases} \tag{15.15}$$

We define F^ξ as in (15.14). Note that $F^\xi(u,A)$ is finite if and only if $u\in\{0,1\}$ a.e. in A, $u_{\xi,y}\in PC(A_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.a. $y\in\Pi_\xi$, and

$$\int_{\Pi_\xi}\#(S(u_{\xi,y}))d\mathcal{H}^{n-1}(y)<+\infty\,. \tag{15.16}$$

4. *Apply Fatou's Lemma.*

If $u_\varepsilon\to u$, we have

$$\begin{aligned}\liminf_{\varepsilon\to0+}F_\varepsilon(u_\varepsilon,A)&\geq\liminf_{\varepsilon\to0+}F_\varepsilon^\xi(u_\varepsilon,A)=\liminf_{\varepsilon\to0+}\int_{\Pi_\xi}F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)\\ &\geq\int_{\Pi_\xi}\liminf_{\varepsilon\to0+}F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)\\ &\geq\int_{\Pi_\xi}F^{\xi,y}(u_{\xi,y},A_{\xi,y})\,d\mathcal{H}^{n-1}(y)=F^\xi(u,A)\,.\end{aligned}$$

Hence, we deduce that $\Gamma\text{-}\liminf_{\varepsilon\to0+}F_\varepsilon(u,A)\geq F^\xi(u,A)$ for all $\xi\in S^{n-1}$;

5. *Describe the domain of the limit.*

By (15.16) and Theorem A.16(b) we deduce that the Γ-lower limit $F'(u,A)=\Gamma\text{-}\liminf_{\varepsilon\to0+}F_\varepsilon(u,A)$ is finite only if u is the characteristic function of a set of finite perimeter. Moreover, $F'(u,A)\geq c\mathcal{H}^{n-1}(S(u))$ for some $c>0$.

6. *Obtain a direction-dependent estimate.*

If u is the characteristic function of a set of finite perimeter, from Theorem A.16(a) we have

$$F^\xi(u,A)=c_W\int_{A\cap S(u)}|\langle\xi,\nu_u\rangle|d\mathcal{H}^{n-1}(y)\,. \tag{15.17}$$

Hence

$$F'(u,A)\geq c_W\int_{A\cap S(u)}|\langle\xi,\nu_u\rangle|d\mathcal{H}^{n-1}(y). \tag{15.18}$$

7. *Optimize the lower estimate.*

This is a slightly technical point which implies that inequality (15.18), which we can read as

$$F'(u,A) \geq \sup_{\xi \in S^{n-1}} c_W \int_{A \cap S(u)} |\langle \xi, \nu_u \rangle| d\mathcal{H}^{n-1}(y) \tag{15.19}$$

can be optimized locally (i.e. the supremum can be moved inside the integral), so that the right-hand side turns out to be simply $c_W\, P(u)$. To this end we will use a simple but interesting lemma in measure theory (see Lemma 15.2).

Since all F_ε are local, then if u is the characteristic function of a set of finite perimeter the set function $\mu(A) = F'(u,A)$ is superadditive on disjoint open sets. From Lemma 15.2 applied with the measure $\lambda = \mathcal{H}^{n-1} \llcorner S(u)$ given by $\lambda(B) = \mathcal{H}^{n-1}(B \cap S(u))$, and $\psi_i(x) = \chi_{S(u)}|\langle \xi_i, \nu_u \rangle|$, where (ξ_i) is a dense sequence in S^{n-1}, we obtain that

$$F'(u,A) \geq c_W \int_{S(u)\cap A} \sup_i \{|\langle \xi_i, \nu \rangle|\}\, d\mathcal{H}^{n-1} = c_W \mathcal{H}^{n-1}(S(u) \cap A), \tag{15.20}$$

which concludes the proof. □

We conclude the section stating and proving the result used in Step 7 above.

Lemma 15.2 (supremum of a family of measures). *Let μ be a function defined on the family of open subsets of Ω, which is super-additive on open sets with disjoint compact closures (i.e. $\mu(A \cup B) \geq \mu(A) + \mu(B)$ if $\overline{A} \cap \overline{B} = \emptyset$, $\overline{A} \cup \overline{B} \subset\subset \Omega$), let λ be a positive measure on Ω, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i\, d\lambda$ for all open sets A and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi\, d\lambda$ for all open sets A.*

Proof We have, by the regularity of the measures $\psi_i \lambda$,

$$\begin{aligned}
\int_A \psi\, d\lambda &= \sup\Big\{ \sum_{i=1}^k \int_{B_i} \psi_i\, d\lambda : (B_i) \text{ Borel partition of } A,\ k \in \mathbb{N} \Big\} \\
&= \sup\Big\{ \sum_{i=1}^k \int_{K_i} \psi_i\, d\lambda : (K_i) \text{ disjoint compact subsets of } A,\ k \in \mathbb{N} \Big\} \\
&= \sup\Big\{ \sum_{i=1}^k \int_{A_i} \psi_i\, d\lambda : (A_i) \text{ disjoint open subsets of } A,\ k \in \mathbb{N} \Big\} \leq \mu(A),
\end{aligned}$$

that is, the thesis. □

15.2 An upper inequality by density

Given a family of functionals $F_\varepsilon : L^1(\Omega) \to [0, +\infty]$ we use the notation

$$F''(u) := \Gamma\text{-}\limsup_{\varepsilon \to 0^+} F_\varepsilon(u)$$

for its Γ-limsup. While it is usually difficult to directly prove a meaningful upper inequality for F'' on the whole $L^1(\Omega)$, a recovery sequence can often be easily constructed if the target function u has some special structure. In order to give an upper estimate of F'' by some functional $F: L^1(\Omega) \to [0,+\infty]$ it is therefore useful to proceed as follows.

Step 1. Define a subset $\mathcal{D}$ of $L^1(\Omega)$, dense in $\{F(u) < +\infty\}$ (the domain of F), such that for each $u \in L^1(\Omega)$ such that $F(u) < +\infty$ we can find a sequence $(u_j) \subset \mathcal{D}$ such that $u_j \to u$ in $L^1(\Omega)$, and $F(u) = \lim_j F(u_j)$;

Step 2. Prove that we have $F''(u) \le F(u)$ for each $u \in \mathcal{D}$.

As noted in Remark 1.29 from Steps 1 and 2 we may conclude that $F'' \le F$ on $L^1(\Omega)$.

In our case the domain of F is the family of all (characteristic functions of) sets of finite perimeter and the family $\mathcal{D}$ in Step 1 is given by sets which are the restriction to Ω of sets with C^∞ boundary (see Proposition A.17). It remains to prove Step 2; that is, to construct a recovery sequence for characteristic functions of smooth sets. In this case we follow the one-dimensional *ansatz*, and construct a recovery sequence which follows the behaviour of the one-dimensional 'optimal profiles' on tubular neighbourhoods of ∂E.

Proposition 15.3 *We have $F''(u) \le F(u)$ for each $u \in \mathcal{D}$.*

Proof Let $u = \chi_E \in \mathcal{D}$. Since ∂E is of class C^∞ up to the boundary of Ω, for $\eta > 0$ sufficiently small the projection $\pi : \{x \in \Omega : \operatorname{dist}(x, \partial E) < \eta\} \to \partial E$ is well defined. Let v be a minimizer of the problem

$$c_W = \min\Big\{ \int_{-\infty}^{+\infty} (W(u) + |u'|^2)\, dt : \ u(-\infty) = 0, \ u(+\infty) = 1 \Big\}.$$

With fixed $\eta \in (0,1)$, we set $v^\eta = 0 \vee (((1+2\eta)v - \eta) \wedge 1)$. Note that

$$c_W^\eta := \int_{-\infty}^{+\infty} \Big(W(v^\eta) + |(v^\eta)'|^2\Big)\, dt \longrightarrow c_W, \qquad \text{as } \eta \to 0.$$

We define $d(x) = \operatorname{dist}(x, \Omega \setminus E) - \operatorname{dist}(x, E)$, the *signed distance function* to ∂E, which is a 1-Lipschitz function, and

$$u_\varepsilon(x) = \begin{cases} v^\eta\Big(\frac{d(x)}{\varepsilon}\Big) & \text{if } |d(x)| \le T\varepsilon, \\ 0 & \text{otherwise in } \Omega \setminus E \\ 1 & \text{otherwise in } E, \end{cases}$$

where $T > 0$ is large enough as to have $\operatorname{spt}(v^\eta)' \subset [-T, T]$. We also set $\nu(x) = (x - \pi(x))/|x - \pi(x)|$. Note that this is a good definition if ε is small enough, so as to have a good definition of π on $\{|d| \le \varepsilon T\}$, and that $\nu(x)$ coincides with a normal to E at $\pi(x)$.

If Ω' is any open set with $\Omega \subset\subset \Omega'$, we now can estimate

$$\int_\Omega \Big(\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon |Du_\varepsilon|^2\Big)\, dx$$

$$
\begin{aligned}
&\leq \int_{\{|d|\leq T\varepsilon\}\cap\Omega} \Big(\frac{1}{\varepsilon}W\Big(v^\eta\Big(\frac{d(x)}{\varepsilon}\Big)\Big) + \frac{1}{\varepsilon}\Big|Dv^\eta\Big(\frac{d(x)}{\varepsilon}\Big)\Big|^2\Big)\,dx \\
&\leq \int_{-T\varepsilon}^{T\varepsilon}\int_{\{d(x)=t\}} \frac{1}{\varepsilon}\Big(W\Big(v^\eta\Big(\frac{t}{\varepsilon}\Big)\Big) + \Big|Dv^\eta\Big(\frac{t}{\varepsilon}\Big)\Big|^2\Big)d\mathcal{H}^{n-1}(x)\,dt \\
&\leq \int_{\partial E\cap\Omega'}\int_{-\varepsilon T}^{-\varepsilon T} \frac{1}{\varepsilon}\Big(W(v^\eta(t)) + |Dv^\eta(t)|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y) + o(1) \\
&\leq c_W\mathcal{H}^{n-1}(\partial E\cap\Omega') + o(1)\,,
\end{aligned}
$$

where $y = x + t\nu$. We have used the coarea formula (A.18) and the fact that $|Dd| = 1$ a.e. By the arbitrariness of Ω' we get

$$
\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \leq c_W\mathcal{H}^{n-1}(\partial E\cap\Omega),
$$

as desired. □

Collecting the lower and upper estimates for the approximation of the perimeter functional (Propositions 15.1 and 15.3) we obtain the complete Γ-convergence result.

Comments on Chapter 15

The result in this chapter is essentially due to Modica and Mortola (1977). It has been completed to a convergence result for minimum problems with volume constraints in Modica (1987). The slicing method has been introduced by Ambrosio to treat free-discontinuity problems (see Ambrosio *et al.* (2000)), and has been used to prove many results also within the theory of phase transitions (see the comments to Chapter 6).

16

*AN INTRODUCTION TO THE LOCALIZATION METHOD OF Γ-CONVERGENCE

In this final chapter, which is thought as an introduction to finer and more technical results, we briefly describe the so-called localization method of Γ-convergence. This approach is frequently used in the proof of compactness results for classes of (integral) functionals. Its great advantage is that it reduces the computation of a particular Γ-limit within that class to the characterization of its energy densities. Note that in general such results are not easy to prove since Γ-convergence is not equivalent to any convergence of the integrands. This method stems from the pioneering work of De Giorgi (1975), and in the version which follows is explained in detail in the books by Dal Maso (1993) and by Braides and Defranceschi (1998) (see also Buttazzo (1989) and Fonseca and Leoni (2002) for integral representation results). The description we give has an illustrative purpose, and we will leave out all details.

Step 1*: localization* The starting remark is that in general it can be easily proved (by Proposition 1.42) that from a sequence (F_j) of functionals we can extract a Γ-converging subsequence, but the only property that F_0 is the Γ-limit does not provide any description of its form. In order to gather enough information on F_0, we may 'localize' the dependence of the functionals F_j on the open set of definition. To illustrate this step consider, for example, the case of integral functionals defined on an open subset Ω of $\mathbf{R}^N$ of the usual form

$$F_j(u) = \int_\Omega f_j(x, Du(x))\,dx,$$

defined on a space $X(\Omega)$, for example, on a Sobolev space $W^{1,p}(\Omega;\mathbf{R}^M)$, equipped with the L^p-convergence. For the sake of simplicity suppose that $f_j \geq 0$ for all j. 'Localizing' such F_j means to consider all functionals

$$F_j(u, V) = \int_V f_j(x, Du(x))\,dx$$

as V varies in the family of all open subset of Ω.

At this point, we can apply the compactness result (not only on Ω, but also) on a dense countable family $\mathcal{V}$ of open subsets of Ω. For example we can choose as $\mathcal{V}$ the family of all unions of open polyrectangles with rational vertices. Since $\mathcal{V}$ is countable, by a diagonal argument, upon extracting a subsequence we can suppose that all $F_j(\cdot, V)$ Γ-converge for $V \in \mathcal{V}$. We denote by $F_0(\cdot, V)$ the Γ-limit, whose form may a priori depend on V.

Step 2: inner regularization The next idea is to consider $F_0(u, \cdot)$ as a set function and prove some properties that lead to some (integral) representation. The first property is 'inner regularity' (see below). In general there may be exceptional sets where this property is not valid; hence, in place of F_0, we define the set function $\overline{F}_0(u, \cdot)$, the *inner-regular envelope* of F_0, on all open subsets of Ω by setting

$$\overline{F}_0(u, U) = \sup\{F_0(u, V) : V \in \mathcal{V}, V \subset\subset U\}.$$

In this way, $\overline{F}_0(u, \cdot)$ is automatically *inner regular*: $\overline{F}_0(u, U) = \sup\{\overline{F}_0(u, V) : V \subset\subset U\}$. An alternative approach is directly proving that $F_0(u, \cdot)$ can be extended to an inner-regular set function (which is not always the case).

Step 3: subadditivity A crucial property (see Step 4 below) of $\overline{F}_0$ is subadditivity; that is, that

$$\overline{F}_0(u, U \cup V) \leq \overline{F}_0(u, U) + \overline{F}_0(u, V)$$

(which is enjoyed for example by non-negative integral functionals). This is usually the most technical part to prove that may involve a complex analysis of the behaviour of the functionals F_j. It is usually proved by showing that the sequence F_j satisfies the so-called *fundamental estimate* (with respect to the L^p norm): for all U, Y, Z open subsets of Ω with $Y \subset\subset U$, and for all $\sigma > 0$, there exists $M > 0$ such that for all u, v in the domain of F_j one may find a function w such that $w = u$ in Y, $w = v$ on $Z \setminus U$ (in the case of functionals on Sobolev spaces w is usually of the form $\varphi u + (1 - \varphi)v$ with $\varphi \in C_0^\infty(U; [0, 1])$, $\varphi = 1$ in Y) such that

$$F_j(w, Y \cup Z) \leq (1 + \sigma)(F_j(u, U) + F_j(v, Z)) + M \int_{(U \cap Z) \setminus Y} |u - v|^p \, dx + \sigma\,, \tag{16.1}$$

and $\|u - w\|_{L^p} + \|v - w\|_{L^p} \leq C\|u - v\|_{L^p}$.

If this property is enjoyed then the subadditivity of $\overline{F}_0$ is easily proved by using the definition of Γ-convergence.

Step 4: measure property The next step is to prove that $\overline{F}_0(u, \cdot)$ is the restriction of a finite Borel measure to the open sets of Ω. To this end it is customary to use the following *De Giorgi Letta Measure Criterion*: if a set function α defined on all open subsets of a set Ω satisfies

(i) $\alpha(U) \leq \alpha(V)$ is $U \subset V$ (α is increasing);
(ii) $\alpha(U) = \sup\{\alpha(V) : V \subset\subset U\}$ (α is inner regular);
(iii) $\alpha(U \cup V) \leq \alpha(U) + \alpha(V)$ (α is subadditive);
(iv) $\alpha(U \cup V) \geq \alpha(U) + \alpha(V)$ if $U \cap V = \emptyset$ (α is superadditive),

then α is the restriction to all open sets of Ω of a regular Borel measure (see De Giorgi and Letta (1977)).

Step 5: integral representation Since $\overline{F}_0(u,\cdot)$ is (the restriction of) a measure we may write it as an integral. For example, in the case of Sobolev spaces if $\overline{F}_0(u,\cdot)$ is absolutely continuous with respect to the Lebesgue measure, then it can be written as

$$\overline{F}_0(u,V) = \int_V f_u(x)\,dx.$$

By combining the properties of $\overline{F}_0$ as a set function with those with respect to u we give a global description. The prototype of such results is the integral representation theorem in Sobolev spaces that we may state as follows:

if $F(u,V)$ is a functional defined for $u \in W^{1,p}(\Omega;\mathbf{R}^M)$ and V open subset of Ω satisfying

(i) (lower semicontinuity) $F(\cdot,V)$ is lower semicontinuous with respect to the L^p convergence;

(ii) (growth estimate) $0 \le F(u,V) \le C\int_V(1+|Du|^p)\,dx$;

(iii) (measure property) $F(u,\cdot)$ is the restriction of a regular Borel measure;

(iv) (locality) F is local: $F(u,V) = F(v,V)$ if $u = v$ a.e. on V,

then there exists a Borel function f such that

$$F(u,V) = \int_V f(x,Du)\,dx.$$

The locality and growth estimates are usually trivially satisfied by $\overline{F}_0$.

Step 6: recovery of the Γ*-limit* The final step is to check that, taking $V = \Omega$, indeed $\overline{F}_0(u,\Omega) = F_0(u,\Omega)$ so that the representation we have found holds for the Γ-limit (and not for its 'inner regularization'). This last step is an inner regularity result on Ω and for some classes of problems is sometime directly proved in Step 2.

The localization method is a powerful tool to show that certain classes of problems are invariant under perturbations, even in the cases when the integrands of the Γ-limit cannot be directly computed from the converging sequence. The same method can be applied to functionals defined on sets of finite perimeter (see Ambrosio and Braides (1990)), or to sequences of Dirichlet problems in arbitrarily-perforated domains (see Dal Maso and Mosco (1987)), or to the description of thin films with varying profile (see Braides *et al.* (2000)), etc. Moreover, the functionals F_j need not be defined all on the same space since this is not a crucial feature of the method. In this way for example we may give a general result describing the gradient theory of phase transitions in inhomogeneous media (see Ansini *et al.* (2002)) or showing the general form of limits of non-local integral functionals (see Cortesani (1998)) or of the continuous limits of discrete systems in the higher-dimensional case. We finally note that this approach has many contact points with issues in theories for random media (see Iosifescu *et al.* (2001) and in statistical mechanics (see Bodineau *et al.* (2000)). But that is another story...

APPENDIX A

SOME QUICK RECALLS

A.1 Convexity

We recall that a function $f : \mathbf{R}^N \to (-\infty, +\infty]$ is *convex* if we have

$$f(tz_1 + (1-t)z_2) \leq t\, f(z_1) + (1-t) f(z_2) \tag{A.1}$$

for all $z_1, z_2 \in \mathbf{R}^N$ and $t \in (0,1)$.

Remark A.1 (a) The convexity of f is equivalent to requiring that *Jensen's inequality* holds:

$$f\Big(\int_X g\, d\mu\Big) \leq \int_X f(g(x)) d\mu \tag{A.2}$$

for all probability spaces (X, μ) and measurable $g : X \to \mathbf{R}^N$.

(b) If $f \in C^1(\mathbf{R}^N)$ then it is convex if and only if

$$f(z) \leq f(w) + \langle f'(z), z - w\rangle \tag{A.3}$$

for all $z, w \in \mathbf{R}^N$.

(c) The supremum of a family of convex functions is convex.

(d) If f is a convex function and f is finite at every point of an open set Ω then f is continuous on Ω and locally Lipschitz continuous on Ω.

(e) If f is convex and there exist $1 \leq p < \infty$ and $c > 0$ such that $0 \leq f(z) \leq c(1 + |z|^p)$ for all $z \in \mathbf{R}^N$, then f satisfies the local Lipschitz condition

$$|f(z) - f(w)| \leq c'(1 + |z|^{p-1} + |w|^{p-1})|z - w| \tag{A.4}$$

for all $z, w \in \mathbf{R}^N$ for some c' depending only on c and p.

(f) If $f_j : \mathbf{R}^N \to \mathbf{R}$ is a sequence of locally equi-bounded convex functions then there exists a subsequence of (f_j) converging uniformly on all compact subsets of $\mathbf{R}^N$.

The verification of statements (b) and (c) is immediate and is left as an exercise. Jensen's inequality is easily derived from the convexity of f when $f \in C^1(\mathbf{R}^N)$. In that case by (b) we have

$$f\Big(\int_X g\, d\mu\Big) \leq f(g(x)) + \Big\langle f'\Big(\int_X g\, d\mu\Big), \int_X g\, d\mu - g(x)\Big\rangle$$

and it is sufficient to integrate with respect to μ. If f is not C^1 we can proceed by approximation using (c) (the details of the proof, using, for example, Exercise

1.10, are left to the reader). Conversely, convexity trivially follows from Jensen's inequality by choosing $X = \{0, 1\}$, $\mu = t\delta_0 + (1-t)\delta_1$, $g(0) = z_1$ and $g(1) = z_2$. To prove (d), if $\Omega = (a, b)$ is an interval of $\mathbf{R}$ then we may use the well-known monotonicity properties of the difference quotient of a convex function. Thus, if $T > 0$ and $r, s \in [a + 2T, b - 2T]$ with $r < s$. we have

$$\frac{f(s) - f(r)}{s - r} \leq \frac{f(b-T) - f(s)}{b - T - s} \leq 2\sup\{f(x) : x \in [a+T, b-T]\}\,\frac{1}{T} =: C_T,$$

so that $f(s) - f(r) \leq C_T(s - r)$. Symmetrically, we get $f(s) - f(r) \geq -C_T(s-r)$ and hence the local Lipschitz continuity of f. If $N > 1$ then the thesis is proven by arguing as above in each coordinate direction. The proof of (e) can be performed in the same way. Details are left as an exercise. Finally, (f) immediately follows from (d) and Ascoli Arzelà's Theorem.

A.2 Sobolev spaces

In all that follows (a, b) is a bounded open interval of $\mathbf{R}$.

Definition A.2 (weak derivative). *We say that $u \in L^1(a, b)$ is* weakly differentiable *if a function $g \in L^1(a, b)$ exists such the following integration by parts formula holds:*

$$\int_a^b u\varphi'\,dt = -\int_a^b g\varphi\,dt \tag{A.5}$$

for all $\varphi \in C_0^1(a, b)$. If such g exists then it is called the weak derivative of u *and is denoted by u'.*

Remark A.3 The notion of weak derivative is an extension of the notion of classical derivative: if $u \in C^1(a, b)$ and its classical derivative belongs to $L^1(a, b)$ then the classical derivative coincides with its weak derivative. The function $x \mapsto |x|$ is weakly differentiable in any (a, b) but $u \notin C^1(-1, 1)$. and its weak derivative is the function $x \mapsto x/|x|$, which in turn is not weakly differentiable in $(-1, 1)$.

Definition A.4 (Sobolev spaces). *Let $p \in [1, \infty]$; the* Sobolev space $W^{1,p}(a, b)$ *is defined as the space of all weakly differentiable $u \in L^p(a, b)$ such that $u' \in L^p(a, b)$. The* norm *of u in $W^{1,p}(a, b)$ is defined as*

$$\|u\|^p_{W^{1,p}(a,b)} = \|u\|^p_{L^p(a,b)} + \|u'\|^p_{L^p(a,b)}.$$

The space $W^{1,p}_{\text{loc}}(\mathbf{R})$ is defined as the space of $u \in W^{1,p}(I)$ for all bounded open intervals $I \subset \mathbf{R}$.

Remark A.5 The Sobolev space $W^{1,p}(a, b)$ equipped with its norm is a Banach space. This is easily checked upon identifying $W^{1,p}(a, b)$ with the subspace of $L^p(a, b) \times L^p(a, b)$ of all pairs (u, u') with $u \in W^{1,p}(a, b)$. The same identification shows that $W^{1,p}(a, b)$ is separable if $1 \leq p < \infty$.

Theorem A.6 (pointwise value of Sobolev functions). *Let $u \in W^{1,p}(a,b)$; then $\tilde{u} \in C([a,b])$ exists such that $\tilde{u} = u$ a.e. on (a,b) and*

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)\,dt \tag{A.6}$$

for all $x, y \in [a,b]$. We commonly identify u with its continuous representative $\tilde{u}$ *whenever pointwise values are taken into account.*

Remark A.7 (boundary values). If $u \in W^{1,p}(a,b)$ then the boundary values $u(a)$ and $u(b)$ are uniquely defined by the values $\tilde{u}(a)$ and $\tilde{u}(b)$, respectively. We may then extend a function $u \in W^{1,p}(a,b)$ to a function $u \in W^{1,p}_{\rm loc}(\mathbf{R})$ by simply setting $u(t) = u(a)$ for $t \le a$ and $u(t) = b$ for $t \ge b$.

Theorem A.8 (equivalent definitions of Sobolev spaces). *Let $1 < p \le \infty$; then the following statements are equivalent:*

(i) $u \in W^{1,p}(a,b)$;

(ii) *there exists $C \ge 0$ such that* $|\int_a^b u\varphi'\,dt| \le C\|\varphi\|_{L^{p'}(a,b)}$ *if* $\varphi \in C^1_0(a,b)$;

(iii) *there exists $C \ge 0$ such that for all $I \subset\subset (a,b)$ and for all $h \in \mathbf{R}$ such that $|h| \le$* dist $(I, \{a,b\})$ *we have $\|\tau_h u - u\|_{L^p(I)} \le C|h|$, where $\tau_h u(t) = u(t-h)$;*

(iv) *there exists a sequence (u_j) in $C^\infty([a,b])$ such that*

$$\lim_j \|u_j - u\|_{W^{1,p}(a,b)} = 0 \tag{A.7}$$

(v) *there exists a sequence (u_j) in $C^\infty_0(\mathbf{R})$ such that* (A.7) *holds;*

(vi) *there exists a sequence (u_j) in $C^\infty_0(\mathbf{R})$ such that* $\sup_j \|u_j\|_{W^{1,p}(a,b)} < +\infty$ *and* $\lim_j \|u_j - u\|_{L^p(a,b)} = 0$.

Remark A.9 (a) The best constant C in (ii) and (iii) above is $\|u'\|_{L^p(a,b)}$.

(b) If $p = 1$ then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). Note that the function $x \mapsto x/|x|$ satisfies (ii)–(vi) with $p = 1$ but does not belong to $W^{1,1}(-1,1)$.

(c) By (iii) we easily see that $W^{1,\infty}(a,b)$ coincides with the space Lip(a,b) of all Lipschitz functions on (a,b), and $\|u'\|_{L^\infty(a,b)}$ is the best Lipschitz constant for u.

Theorem A.10 (embedding results). *There exists $C = C(a,b)$ such that*

$$\|u\|_{L^\infty(a,b)} \le C\|u\|_{W^{1,p}(a,b)}. \tag{A.8}$$

Moreover we have the compact embeddings

$$W^{1,p}(a,b) \subset C^0([a,b]) \tag{A.9}$$

for $1 < p \le \infty$, and

$$W^{1,1}(a,b) \subset L^q(a,b) \tag{A.10}$$

for all $q \ge 1$.

Definition A.11 *The space* $W_0^{1,p}(a,b)$ *is defined as the closure of* $C_0^\infty(a,b)$ *in the* $W^{1,p}$*-norm, or, equivalently, as the set of those* $u \in W^{1,p}(a,b)$ *with boundary values* $u(a) = u(b) = 0$.

Theorem A.12 (Poincaré's inequality). *There exists a constant* $C = C(a,b)$ *such that*

$$\|u\|_{W^{1,p}(a,b)} \leq C\|u'\|_{L^p(a,b)} \tag{A.11}$$

for all $u \in W^{1,p}(a,b)$ *such that* $\tilde{u}(x) = 0$ *for some* $x \in [a,b]$. *In particular this holds for* $u \in W_0^{1,p}(a,b)$.

Note that if we apply a similitude of ratio ρ to the domain (a,b), the constant C is multiplied by ρ. This remark holds in any space dimension.

Definition A.13 *Let* $u : (a,b) \to \mathbf{R}$ *be a measurable function. The* total variation *of* u *on* (a,b) *is defined as*

$$\mathrm{Var}(u,(a,b)) = \inf\Big\{\sup\Big\{\sum_{i=1}^{N} |v(t_{i+1}) - v(t_i)| \\ : \ a < t_0 < \ldots < t_N < b, \ N \in \mathbf{N}\Big\} : v = u \ \text{a.e. on } (a,b)\Big\} \tag{A.12}$$

If $\mathrm{Var}(u,(a,b)) < +\infty$ *then we say that* u *is a* function of bounded variation. *We simply write* $\mathrm{Var}\, u$ *if* (a,b) *is fixed.*

Remark A.14 If $u \in W^{1,1}(a,b)$ then $\mathrm{Var}(u,(a,b)) = \int_a^b |u'|\,dt$; in particular, u is a function of bounded variation. Note that also $v(x) = x/|x|$ is a function of bounded variation with $\mathrm{Var}(v,(-1,1)) = 2$.

A.3 *Sets of finite perimeter

A classical problem in the Calculus of Variations is that of the computation of the set of least perimeter and given area. The attack of such a problem by the direct methods need a definition of surface area which is lower semicontinuous under a convergence of sets which ensures also a compactness property. It is clear that a bound on the area of a sequence of sets does not ensure any continuity, even though all sets are smooth. We give a quick introduction to this subject, sufficient to the exemplificatory use that we make in Chapter 15. We refer to the book of Ambrosio *et al.* (2000) for a complete treatment and to Morgan (1988) for a quick introduction.

The simplest way to have a definition of *perimeter* which is lower semicontinuous by the L^1-convergence of the sets is by *relaxation*: if $E \subset \mathbf{R}^N$ is of class C^1 define the perimeter $\mathcal{P}(E,\Omega)$ of the set E inside the open set Ω in a classical way, and then for an arbitrary set, define

$$\mathcal{P}(E,\Omega) = \inf\{\liminf_j \mathcal{P}(E_j,\Omega) : \chi_{E_j} \to \chi_E \text{ in } L^1(\Omega)\}.$$

Another choice leading to the same definition is to start with E_j of polyhedral type, for example. This definition coincides with the distributional definition of perimeter

$$\mathcal{P}(E,\Omega) = \sup\Big\{\int_E \operatorname{div} g\,dx :\ g \in (C_0^1(\Omega))^N,\ |g| \le 1\Big\}.$$

If $\mathcal{P}(E,\Omega) < +\infty$ then we say that E is a *set of finite perimeter* in Ω. For such sets it is possible to define a notion of measure-theoretical boundary, where a normal is defined, so that we may heuristically picture those sets as having a smooth boundary. In order to make these concepts more precise we recall the definition of the *k-dimensional Hausdorff measure* (in this context we will limit ourselves to $k \in \mathbf{N}$). If E is a Borel set in $\mathbf{R}^N$ then we define

$$\mathcal{H}^k(E) = \sup_{\delta>0} \frac{\omega_k}{2^k} \inf\Big\{\sum_{i\in\mathbf{N}} (\operatorname{diam} E_i)^k :\ \operatorname{diam} E_i \le \delta,\ E \subseteq \bigcup_{i\in\mathbf{N}} E_i\Big\},$$

where ω_k is the Lebesgue measure of the unit ball in $\mathbf{R}^k$ and $\operatorname{diam} B$ is the diameter of B. Note that the k-dimensional Hausdorff measure coincides with the elementarily-defined k-dimensional surface measure on k-dimensional subspaces. In particular, the N-dimensional Hausdorff measure coincides with the Lebesgue on $\mathbf{R}^N$.

We say that $x \in E$ is a *point of density* $t \in [0,1]$ if there exists the limit

$$\lim_{\rho\to 0+} \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n} = t\,.$$

The set of all points of density t will be denoted by E_t. If E is a set of finite perimeter in Ω then the De Giorgi's *essential boundary* of E, denoted by $\partial^* E$, is defined as the set of points $x \in \Omega$ with density $1/2$.

Theorem A.15 (De Giorgi's rectifiability theorem). *Let $E \subset \mathbf{R}^N$ be a set of finite perimeter in Ω. Then $\partial^* E$ is* rectifiable; *i.e., there exists a countable family (Γ_i) of graphs of C^1 functions of $(N-1)$ variables such that $\mathcal{H}^{N-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$. Moreover the perimeter of E in $\Omega' \subseteq \Omega$ is given by*

$$\mathcal{P}(E,\Omega') = \mathcal{H}^{N-1}(\partial^* E \cap \Omega').$$

By the previous theorem and the Implicit Function Theorem a *internal normal* $\nu_E(x)$ to E is defined at $\mathcal{H}^{N-1}$-almost all points of $\partial^* E$ as the normal of the corresponding Γ_i. The following *generalized Gauss–Green formula* holds

$$\int_E \operatorname{div} g\,dx = -\int_{\partial^* E} \langle \nu, g\rangle\, d\mathcal{H}^{N-1} \tag{A.13}$$

holds for all $g \in (C_0^1(\Omega))^N$, which states that the distributional derivative of χ_E is a vector measure given by

$$D\chi_E(B) = \int_B \nu \, d\mathcal{H}^{N-1};$$

In particular, we have $\mathcal{P}(E,\Omega) = |D\chi_E|(\Omega)$, the total variation of the measure $D\chi_E$ on Ω.

The following theorem essentially states that sets of finite perimeter are characterized as those sets (almost all) whose one-dimensional sections are finite unions of intervals (i.e. one-dimensional sets of finite perimeter). We use the notation for one-dimensional sections introduced in Chapter 15 and that for piecewise-constant functions introduced in Chapter 5.

Theorem A.16 (a) *Let E be a set of finite perimeter in a smooth open set $\Omega \subset \mathbf{R}^N$ and let $u = \chi_E$. Then for all $\xi \in S^{n-1}$ and for $\mathcal{H}^{N-1}$-a.a. $y \in \Pi_\xi$ the function $u_{\xi,y}$ belongs to $PC(\Omega_{\xi,y})$. Moreover, for such y we have*

$$S(u_{\xi,y}) = \{t \in \mathbf{R}: \ y + t\xi \in S(u)\}, \tag{A.14}$$

and for all Borel functions g

$$\int_{\Pi_\xi} \sum_{t \in S(u_{\xi,y})} g(t) \, d\mathcal{H}^{N-1}(y) = \int_{S(u)} g(x) |\langle \nu_E, \xi \rangle| d\mathcal{H}^{N-1}; \tag{A.15}$$

(b) *Conversely, if $E \subset \Omega$ and for all $\xi \in \{e_1, \ldots, e_N\}$ and for $\mathcal{H}^{N-1}$-a.a. $y \in \Pi_\xi$ the function $u_{\xi,y}$ is piecewise constant in each interval of $\Omega_{\xi,y}$ and*

$$\int_{\Pi_\xi} \#(S(u_{\xi,y})) \, d\mathcal{H}^{n-1}(y) < +\infty, \tag{A.16}$$

then E is a set of finite perimeter in Ω.

The following theorem states that (if Ω is regular) sets of finite perimeter can be approximated by smooth sets in $\mathbf{R}^N$.

Proposition A.17 *Let Ω be a Lipschitz set. If E is a set of finite perimeter in Ω then there exists a sequence (E_j) of sets of finite perimeter in Ω, such that*

$$\lim_j |E \Delta E_j| = 0, \qquad \lim_j \mathcal{P}(E_j, \Omega) = \mathcal{P}(E, \Omega), \tag{A.17}$$

and for every open set Ω' with $\Omega \subset\subset \Omega'$ there exist sets E'_j of class C^∞ in Ω' and such that $E'_j \cap \Omega = E_j$.

We finally recall the *coarea formula* on the open set $A \subset \mathbf{R}^N$

$$\int_A f(x) |Dd| \, dx = \int_{-\infty}^{+\infty} \int_{\{d=t\}} f(y) d\mathcal{H}^{N-1}(y) \, dt \tag{A.18}$$

valid if d is a Lipschitz function and f a Borel function. Note that if $d(x) = x_j$ this is a particular case of Fubini's theorem.

APPENDIX B

CHARACTERIZATION OF Γ-CONVERGENCE FOR 1D INTEGRAL PROBLEMS

In this section we prove the Characterization Theorem 2.35. The proof will follow from some compactness and convex analysis arguments, which will occupy the rest of the section. We first remark some compactness properties of the class $\mathcal{F}$.

Proposition B.1 *Let (F_j) be a sequence in $\mathcal{F}$. Then there exists a subsequence (not relabelled) of (F_j) and a convex functional $F : W^{1,p} \to [0,+\infty)$ such that $F = \Gamma(L^p(a,b))\text{-}\lim_j F_j$. Moreover, for all $u \in W^{1,p}(a,b)$ there exists a sequence u_j such that $u_j - u \in W_0^{1,p}(a,b)$ and converges to 0 weakly in $W^{1,p}(a,b)$, and $F(u) = \lim_j F_j(u_j)$.*

Proof The existence of a Γ-converging subsequence follows directly from Proposition 1.42 and the convexity of F follows from Exercise 1.6. The proof of the last statement is contained in Proposition 2.37. □

Remark B.2 Note that all functionals $F \in \mathcal{F}$ are continuous on $W^{1,p}(a,b)$ with respect to the strong convergence in $W^{1,p}(a,b)$.

In order to apply some arguments of convex analysis, we will have to deal with conjugate functions with respect to a duality, of which Definition 2.34 is a particular case.

Definition B.3 *Let V be a topological vector space, and let V^* denote its dual. If $F : V \to \mathbf{R}$, its* conjugate function *is $F^* : V^* \to [-\infty,+\infty]$ given by*

$$F^*(v^*) = \sup\{\langle v^*, v\rangle - F(v) :\ v \in V\}. \tag{B.1}$$

With this definition, $f^*(t,\cdot)$ as in Definition 2.34 is the conjugate of $f(t,\cdot)$ on $\mathbf{R}$ (with respect to the duality given by the product), and t acts as a parameter.

Remark B.4 If F is convex and lower semicontinuous then $F = (F^*)^*$; that is,

$$F(v) = \sup\{\langle v^*, v\rangle - F^*(v^*) : v^* \in V^*\} \tag{B.2}$$

for all $v \in V$.

Having the previous definition in mind, we take

$$V = \{v \in W^{1,p}(a,b) :\ v(b) = 0\}, \tag{B.3}$$

equipped with the $L^p(a,b)$ norm. Clearly, since all functionals in $\mathcal{F}$ satisfy the property of invariance by addition of a constant $F(u+c) = F(u)$, it is sufficient to characterize the Γ-convergence on this space. Note that by the embedding of V in $L^\infty(a,b)$, V^* contains $L^1(a,b)$.

Proposition B.5 *If $F \in \mathcal{F}$ with integrand f is considered as a function on V, then $F^* : V^* \to \mathbf{R}$ is represented as*

$$F^*(\varphi) = \int_a^b f^*\Big(t, -\int_a^t \varphi(s)ds\Big)\, dt \tag{B.4}$$

for all $\varphi \in L^1(a,b)$.

Proof Let at first $f(t,\cdot) \in C^1(\mathbf{R})$ for all $t \in (a,b)$. Note that the supremum in (2.27) is actually a maximum, obtained at z satisfying $z^* = \frac{\partial f}{\partial z}(t,z)$; that is,

$$z^* - \frac{\partial f}{\partial z}(t,z) = 0 \qquad \text{if and only if} \qquad f^*(t,z^*) = z^*z - f(t,z). \tag{B.5}$$

Let $\varphi \in L^1(a,b)$ and let $\Phi \in W^{1,1}(a,b)$ be defined by

$$\Phi(t) = -\int_a^t \varphi(s)ds. \tag{B.6}$$

Note that $\Phi' = -\varphi$ and $\Phi(a) = 0$. By (B.1) we have

$$F^*(\varphi) = \sup_{v\in V}\Big\{\int_a^b \Big(\varphi v - f(t,v')\Big)\, dt\Big\} = \sup_{v\in V}\Big\{\int_a^b \Big(\Phi v' - f(t,v')\Big)\, dt\Big\}.$$

This supremum is actually a maximum, and, by computing the Euler equations for the maximum point u, we get

$$\int_a^b \Big(\Phi - \frac{\partial f}{\partial z}(t,u')\Big) v'\, dt = 0$$

for all $v \in V$, which implies that $\Phi - \frac{\partial f}{\partial z}(t,u') = c$, $\frac{\partial f}{\partial z}(a,u'(a)) = 0$, and then $\Phi = \frac{\partial f}{\partial z}(t,u')$ a.e. on (a,b). Note that by (B.5) we have, taking, for each fixed t $z^* = \Phi(t)$ and $z = u'(t)$,

$$\Phi - \frac{\partial f}{\partial z}(t,u') = 0 \qquad \text{if and only if} \qquad f^*(t,\Phi) = \Phi u' - f(t,u'),$$

so that $F^*(\varphi) = \displaystyle\int_a^b (\Phi u' - f(t,u'))\, dt = \int_a^b f^*(t,\Phi(t))\, dt$, as desired.

In the general case, we can reason by approximation. We can find a sequence of convex functions (f_k), each one smooth in the second variable, converging to f uniformly on compact sets and such that $f_k \geq f$ for all k. We can take, for example, $f_k(\cdot,z) = \rho_k * f(\cdot,z)$, where (ρ_k) is a sequence of mollifiers, the condition $f_k \geq f$ following from Jensen's inequality. Note that the conditions on f_k imply that $\lim_k f_k^*(t,z^*) = f^*(t,z^*)$ for all $t \in (a,b)$ and $z^* \in \mathbf{R}$. From what proved above, we have

$$F_k^*(\varphi) = \int_a^b f_k^*\Big(t, -\int_a^t \varphi(s)ds\Big)\,dt$$

for all $\varphi \in L^1(a,b)$. By the Dominated Convergence Theorem and the inequality $F^* \geq F_k^*$ we then get $F^*(\varphi) \geq \lim_k F_k^*(\varphi) = \int_a^b f^*\Big(t, -\int_a^t \varphi(s)ds\Big)\,dt$.

On the other hand, by (2.27) we have $f^*(t,z^*) \geq z^*z - f(t,z)$ for all t, z and z^*. With fixed $v \in V$, by taking $z^* = \Phi(t)$ and $z = v'(t)$ we have

$$\int_a^b f^*(t,\Phi)\,dt \geq \int_a^b \Big(\Phi v' - f(t,v')\Big)\,dt = \int_a^b \Big(\varphi v - f(t,v')\Big)\,dt.$$

By the arbitrariness of $v \in V$ we have the inequality $\int_a^b f^*(t,\Phi)\,dt \geq F^*(\varphi)$, which concludes the proof. □

The next two lemmas will give the final ingredients for the proof of Theorem 2.35.

Lemma B.6 *Let* $1 < q < \infty$, *let* $k_1, k_2, k_3 > 0$, *and for all* $j \in \mathbf{N}$ *let* $g_j \in \mathcal{F}(q,k_1,k_2,k_3)$. *If* $g_j(\cdot,z)$ *weakly*-converges to* $g(\cdot,z))$ *for all* $z \in \mathbf{R}$ *then* $g_j(\cdot,v(\cdot))$ *weakly*-converges to* $g(\cdot,v(\cdot))$ *for all* $v \in C^0([a,b])$.

Proof Note that $g(t,\cdot)$ is convex and $g \in \mathcal{F}(q,k_1,k_2,k_3)$. Let $v \in C^0([a,b])$, $\varphi \in L^1(a,b)$ and $N \in \mathbf{N}$. Then we have

$$\begin{aligned}\Big|\int_a^b (g_j(t,v) - g(t,v))\varphi\,dt\Big| \leq{}& \sum_{i=1}^N \Big|\int_{(t_{i-1},t_i)} (g_j(t,v(t)) - g_j(t,v(t_i)))\varphi\,dt\Big| \\ &+ \sum_{i=1}^N \Big|\int_{(t_{i-1},t_i)} (g_j(t,v(t_i)) - g(t,v(t_i)))\varphi\,dt\Big| \\ &+ \sum_{i=1}^N \Big|\int_{(t_{i-1},t_i)} (g(t,v(t_i)) - g(t,v(t)))\varphi\,dt\Big|, \qquad \text{(B.7)}\end{aligned}$$

where $t_i = a + i(b-a)/N$. By hypothesis, in particular, we have

$$\lim_j \int_{(t_{i-1},t_i)} (g_j(t,v(t_i)) - g(t,v(t_i)))\varphi\,dt = 0$$

for all $i = 1,\dots,N$. By the uniform local Lipschitz continuity of g_j (see Remark A.1(e)), we have

$$\Big|\int_{(t_{i-1},t_i)} (g_j(t,v(t)) - g(t,v(t_i)))\varphi\,dt\Big|$$

$$\leq c \int_{(t_{i-1},t_i)} (1 + |v(t)|^{p-1} + |v(t_i)|^{p-1})|v(t) - v(t_i)||\varphi|\,dt$$
$$\leq c(1 + \|v\|^{p-1}_{L^\infty(t_{i-1},t_i)})\|v - v(t_i)\|_{L^\infty(t_{i-1},t_i)}\|\varphi\|_{L^1(t_{i-1},t_i)}.$$

The same inequality holds with g in place of g_j. Hence, plugging back these estimates in (B.7) and noting that $\lim_j \|v - v(t_i)\|_{L^\infty(t_{i-1},t_i)} = 0$, we get that $\lim_j \int_a^b (g_j(t,v) - g(t,v))\varphi\,dt = 0$, as desired. □

Lemma B.7 *Let* $1 < q < \infty$, *let* $k_1, k_2, k_3 > 0$, *and for all* $j \in \mathbf{N}$ *let* $g_j \in \mathcal{F}(q, k_1, k_2, k_3)$. *Then there exists a subsequence (not relabelled) of* (g_j) *and* $g : (a,b) \times \mathbf{R} \to [0,+\infty)$ *such that* $g_j(\cdot, z)$ *weakly*-converges to* $g(\cdot, z)$ *for all* $z \in \mathbf{R}$.

Proof With fixed $z \in \mathbf{R}$ the sequence $(g_j(\cdot,z))$ is bounded in $L^\infty(a,b)$; hence it admits a weakly*-converging subsequence. Hence, we may suppose, upon relabelling (g_j), that $(g_j(\cdot,z))$ weakly* converges for all $z \in \mathbf{Q}$. Define $g : (a,b)\times\mathbf{Q} \to [0,+\infty)$ in such a way that $g(\cdot,z)$ is the weak*-limit of $(g_j(\cdot,z))$. $g(t,\cdot)$ is convex and hence locally Lipschitz continuous (see Remark A.1(e)) on $\mathbf{Q}$. If we still denote by $g(t,\cdot)$ its continuous extension, then it is easily checked that $g_j(\cdot,z)$ weakly*-converges to $g(\cdot,z)$ for all $z \in \mathbf{R}$. □

Proof (Theorem 2.35)

Step 1 Suppose that (ii) holds. We can apply Lemma B.6 and obtain that $f_j^*(\cdot,\Phi)$ weakly* converges to $f^*(\cdot,\Phi)$ for all $\varphi \in L^1(a,b)$ (Φ as in (B.6)). Hence, from Proposition B.5 we have that $F^*(\varphi) = \lim_j F_j^*(\varphi)$ for all $\varphi \in L^1(a,b)$.

Step 2 We now remark that the pointwise convergence of F_j^* to F^* on $L^1(a,b)$ is equivalent to the Γ-convergence of F_j to F. In fact, if $F = \Gamma\text{-}\lim_j F_j$ then for all $\varphi \in L^1(a,b)$ we have by Theorem 1.21

$$\lim_j F_j^*(\varphi) = \lim_j \sup_{u\in V}\Big\{\int_a^b u\varphi\,dt - F_j(u)\Big\} = -\lim_j \inf_{u\in V}\Big\{F_j(u) - \int_a^b u\varphi\,dt\Big\}$$
$$= -\inf_{u\in V}\Big\{F(u) - \int_a^b u\varphi\,dt\Big\} = \sup_{u\in V}\Big\{\int_a^b u\varphi\,dt - F(u)\Big\} = F^*(u).$$

Vice versa, if F_j^* converges pointwise to F^* on $L^1(a,b)$ and $G = \Gamma\text{-}\lim_j F_j$ (which we may suppose upon extracting a subsequence by Proposition 1.42), then, by what seen above, F_j^* converges pointwise to G^* on $L^1(a,b)$, so that $F^* = G^*$ and $G = F$ by Remarks B.4 and B.2. Note that we can replace (a,b) by I in all the reasonings above.

Step 3 From Steps 1 and 2 above we deduce that (ii) implies (i).

Step 4 Let (i) hold. By Lemma B.7 we may suppose also that there exists $g : (a,b)\times\mathbf{R} \to [0,+\infty)$ such that for all $z^* \in \mathbf{R}$, $g^*(\cdot,z^*)$ is the weak*-limit of the sequence $(f_j^*(\cdot,z^*))$. Then by Step 3 we have that, denoted by G the functional in $\mathcal{F}$ with integrand g, $G(\cdot,I) = \Gamma(L^p(I)\text{-}\lim_j F_j(\cdot,I)$ for all I open subintervals of (a,b). Hence $\int_I f(t,u'(t))\,dt = \int_I g(t,u'(t))\,dt$ for all $u \in W^{1,p}(a,b)$ and I open subintervals of (a,b), so that $f = g$ and the proof is concluded. □

LIST OF SYMBOLS

Sets, numbers, measures

$a \vee b$ ($a \wedge b$) the maximum (minimum) between a and b

$A \subset\subset B$ means that the closure of A is contained in the interior of B

$B_\rho(x)$ the open ball of centre x and radius ρ

c (if not otherwise stated) a strictly positive constant independent from the parameters of the problem, whose value may vary from line to line

$|E|$ the Lebesgue measure of the set E

$e_1, \ldots, e_N$ canonical base of $\mathbf{R}^N$ (N-dimensional Euclidean space)

$\mathcal{H}^k$ the k-dimensional Hausdorff measure

$\lim_j$ ($\liminf_k$, etc.) limit as the discrete parameter j tends to $+\infty$

$\mu \llcorner E$ the restriction of the measure μ to E

Ω (if not otherwise stated) a bounded open subset of an Euclidean space

p' dual exponent of p

$\overline{\mathbf{R}} = [-\infty, +\infty]$ extended real line

$[t]$ integer part of $t \in \mathbf{R}$

$\langle \xi, \eta \rangle$ scalar product of ξ and $\eta \in \mathbf{R}^N$

$|\xi|$ norm of ξ

Function spaces

(in the text often $\Omega = (a, b)$)

$C^k(\Omega; \mathbf{R}^N)$ the space of k-times differentiable $\mathbf{R}^N$-valued functions (k omitted if 0; $\mathbf{R}^N$ omitted if $N = 1$)

$L^p(\Omega; \mathbf{R}^N)$ the space of $\mathbf{R}^N$-valued p-summable functions on Ω ($\mathbf{R}^N$ omitted if $N = 1$)

$PC(a, b)$ the space of piecewise-constant functions on (a, b)

$P\text{-}W^{1,p}(a, b)$ the space of piecewise-Sobolev functions on (a, b)

$\|u\|_{L^p(\Omega;\mathbf{R}^N)}$ or simply $\|u\|_p$ the L^p norm of u

$\|u\|_{W^{1,p}(\Omega;\mathbf{R}^N)}$ or simply $\|u\|_{1,p}$ the $W^{1,p}$ norm of u

$W^{1,p}(\Omega)$ the space of Sobolev functions with p-summable derivatives on Ω

$W_0^{1,p}(\Omega; \mathbf{R}^N)$ the closure of $C_0^\infty(\Omega; \mathbf{R}^N)$ in $W^{1,p}(\Omega; \mathbf{R}^N)$

$X_{\text{loc}}(\Omega; \mathbf{R}^N)$ $\{u : \Omega \to \mathbf{R}^N : u \in X(U; \mathbf{R}^N)$ for all open $U \subset\subset \Omega\}$ (X a generic notation for a function space)

Functions

χ_E the characteristic function of the set E ($\chi_E(x) = 1$ if $x \in E$ $\chi_E(x) = 0$ if $x \notin E$)

Du (u' in dimension one) the weak derivative of u

f^* the conjugate (Legendre transform) of f

f^{**} the convex and lower semicontinuous envelope of f

f^∞, ϑ^0 recession functions of f and ϑ

$f \triangle \vartheta$ inf-convolution of f and ϑ

$\nu_u(x)$ the normal to $S(u)$ at x

ρ a mollifier; ρ_γ the scaled mollifier given by $\rho_\gamma(x) = \frac{1}{\gamma^n}\rho(\frac{x}{\gamma})$

$S(u)$ the set of essential discontinuity points of u (jump set)

$T_\lambda f$ the Yosida transform of f

$u^\pm(x)$ the approximate limits of u at x

$u(t\pm)$ the right-/left-hand-side limits of u at t

$u_j \to u$ u_j converges strongly to u

$u_j \rightharpoonup u$, $u_j \overset{*}{\rightharpoonup} u$ u_j converges weakly to u, u_j converges weakly* to u

$\mathrm{Var}\, u$ the variation of the function u

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