Rods with intrinsic curvature

① $E = \int_\ell d\xi \left(\frac{1}{2} EI_1 (v_1 - \cancel{v_{10}})^2 + \frac{1}{2} EI_2 (v_2 - \cancel{v_{20}})^2 + \frac{1}{2} \mu J (v_3 - \cancel{v_{30}})^2 \right).$

ξ: arc length

$\Rightarrow \begin{cases} p(\xi) + n'(\xi) = 0 \\ m'(\xi) + r'(\xi) \times n(\xi) + q(\xi) = 0 \end{cases}$ where

$m = (EI_1(v_1 - \cancel{v_{10}}),\ EI_2(v_2 - \cancel{v_{20}}),\ \mu J(v_3 - \cancel{v_{30}}))$

in $\{d_1, d_2, d_3\}$ basis

② $\{E_1, E_2, E_3\}$. $\{D_1, D_2, D_3\}$, $\{d_1, d_2, d_3\}$

~~$d = PP_0 E$~~ $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = PP_0 \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$

P_0 is known from the reference configuration.

③ Solving the ODE system in ①, we obtain $v_k - v_{k0}$ (ξ)

④ Euler angles $P = Q(\alpha_3, \hat{e}_3)\, Q(\alpha_2, \hat{e}_2)\, Q(\alpha_1, \hat{e}_1)$

⑤ $\alpha_1 = \alpha_{10}$, $\alpha_2 = \alpha_{20}$ $\alpha_3 = \alpha_{30}$ initial condition

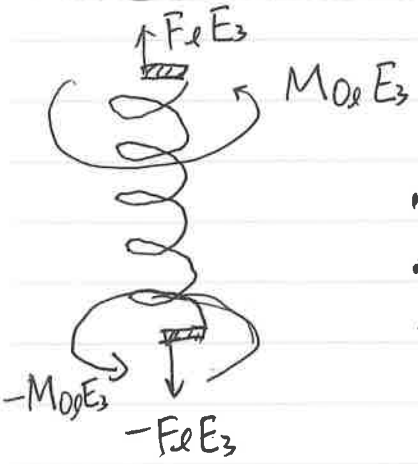
$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial \xi} \\ \frac{\partial \alpha_2}{\partial \xi} \\ \frac{\partial \alpha_3}{\partial \xi} \end{pmatrix} = \begin{pmatrix} -\operatorname{cosec}\alpha_2 \cos\alpha_3 & \operatorname{cosec}\alpha_2 \sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ \cot\alpha_2 \cos\alpha_3 & -\cot\alpha_2 \sin\alpha_3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\Rightarrow \alpha_1(\xi),\ \alpha_2(\xi),\ \alpha_3(\xi)$

⑥ $d\alpha' = (P(v + v_0)) \times d\alpha$

⑦ $r'(\xi) = d_3(\xi) \Rightarrow r = \int_0^\xi d_3(\xi) d\xi + d_3(0)$

Example: spiral springs



$$\vec{M}_{0\ell} = M_{0\ell} E_3 = M_\ell + r(\ell) \times F_\ell$$

- e_n is in the plane of E_1, E_2
- In principle, $\{D_1, D_2, D_3\}$ can be chosen arbitrarily. However, it is convenient to choose $\{D_1, D_2, D_3\} = \{e_n, e_b, e_t\}$, where $\{e_t, e_n, e_b\}$ is the Frenet frame.

- $P_0 = Q(\varphi_2, \cos\varphi_1 E_1 + \sin\varphi_1 E_2)\, Q(\varphi_1, E_3)$

  $e_n = \cos\varphi_1 E_1 + \sin\varphi_1 E_2$

Following the previous discussion:

② Compute P_0: — Mathematica.

Compute v_0: $v_0 = ax(P_0' P_0^T) = v_{10} D_1 + v_{20} D_2 + v_{30} D_3$

$= \varphi_1' \sin\varphi_2\, D_2 + \varphi_1' \cos\varphi_2\, D_3$

- If the spring: $R(s) = R\, e_{\theta_0} + R\,\theta_0(s) \tan(\zeta)\, E_3$

  $e_{\theta_0} = \cos(\theta_0(s)) E_1 + \sin(\theta_0(s)) E_2$

  $\Rightarrow \theta_0(s) = \varphi_1(s) - \pi, \quad \zeta = \frac{\pi}{2} - \varphi_2(s) \quad \Rightarrow \varphi_2'(s) = 0$!

③ Balance law.

$$\begin{cases} n' = 0 \\ m' + r' \times n = 0 \end{cases} \qquad m = EI_1 v_1 d_1 + EI_2 v_2 d_2 + \mu J v_3 d_3$$

$n' = 0 \Rightarrow n = \text{const.} = F_0 = F_{01} d_1 + F_{02} d_2 + F_{03} d_3$

$d_k' = \left(\sum_{i=1}^{3} (v_i + v_{i0}) d_i\right) \times d_k \quad \sim ⑥$

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$$m' = EI_1 v_1' d_1 + EI_2 v_2' d_2 + \mu J v_3' d_3$$
$$+ EI_1 v_1 [-(v_2+v_{20}) d_3 + (v_3+v_{30}) d_2]$$
$$+ EI_2 v_2 [(v_1+v_{10}) d_3 - (v_3+v_{30}) d_1]$$
$$+ \mu J v_3 [-(v_1+v_{10}) d_2 + (v_2+v_{20}) d_1]$$

$$\Rightarrow \begin{cases} EI_1 v_1' - EI_2 v_2 (v_3+v_{30}) + \mu J v_3 (v_2+v_{20}) = +F_{02} \\ EI_2 v_2' - \mu J (v_1+v_{01}) v_3 + EI_1 (v_3+v_{03}) v_1 = -F_{01} \\ \mu J v_3' - EI_1 (v_2+v_{02}) v_1 + EI_2 (v_1+v_{01}) v_2 = 0 \end{cases}$$

③ For our springs problem: $I_1 = I_2 = I$

BCs: $n(\ell^-) = F_\ell E_3$, $m(\ell^-) = M_{0\ell} E_3 - r(\ell^-) \times F_\ell E_3$

strain: $v_{01} = 0$, $v_{02} = \varphi_1' \sin\varphi_2$, $v_{03} = \varphi_1' \cos\varphi_2$

$$\Rightarrow \begin{cases} EI v_1' - EI v_2 (v_3+v_{03}) + \mu J (v_2+v_{02}) v_3 = d_2 \cdot F_\ell E_3 \\ EI v_2' - \mu J v_1 v_3 + EI (v_3+v_{03}) v_1 = -d_1 \cdot F_\ell E_3 \\ \mu J v_3' - EI v_{02} v_1 = 0 \end{cases} \quad \star$$

We seek the boundary conditions F_ℓ and $M_{0\ell}$ that will deform $R(s)$ to

$$r(s) = r(\cos\theta(s) E_1 + \sin\theta(s) E_2) + r\theta(s) \tan\sigma E_3$$

$$\boxed{\text{Assumption: } \tilde{e}_t = d_3,\ \tilde{e}_n = d_1,\ \tilde{e}_b = d_2}$$

$$PP_0 = Q\left(\tfrac{\pi}{2} - \sigma, \cos(\theta+\pi) E_1 + \sin(\theta+\pi) E_2\right) Q(\theta+\pi, E_3)$$

$\Rightarrow P = PP_0 \cdot P_0^{-1}$ (see Mathematica)

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$\Rightarrow$ $v_1 = 0$, $v_2 = \kappa - \kappa_0$ $v_3 = \tau - \tau_0$ in $\{D_1, D_2, D_3\}$

where $\kappa = \dfrac{\cos^2\sigma}{r}$, $\tau = \dfrac{\cos\sigma \sin\sigma}{r}$, $\kappa_0 = \dfrac{\cos^2\xi}{R}$, $\tau_0 = \dfrac{\cos\xi \sin\xi}{R}$

$$\star \Rightarrow \begin{cases} d_1 \cdot F_\ell E_3 = 0, \ v_1 = 0 \qquad (\text{substitute } v_1' = v_2' = v_3' = 0) \\ EI v_2 (v_3 + v_{03}) - \mu J (v_2 + v_{02}) v_3 + d_2 \cdot F_\ell E_3 = 0 \end{cases}$$

$$\Rightarrow \cos\sigma F_\ell = \mu J \kappa (\tau - \tau_0) - EI \tau (\kappa - \kappa_0) \qquad \sim (1)$$

$$\vec{m} = EI(\kappa - \kappa_0) d_2 + \mu J (\tau - \tau_0) d_3$$

$$\Rightarrow M_{0\ell} = EI(\kappa - \kappa_0)\cos\sigma \pm \mu J (\tau - \tau_0) \sin\sigma \qquad \sim (2)$$

(1) and (2) are the BCs that deform $R(s)$ into $r(s)$ $\square$.