Surfaces: outline

1. Regular surfaces
2. Parameterization of regular surfaces
3. Intrinsic properties (first fundamental form, Gaussian curvature, geodesic curvature, etc.)
4. The Gauss-Bonnet theorem
5. Example: intrinsically curved folds in liquid crystal elastomers

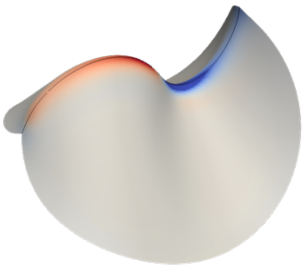


Figure 1: Gaussian curvature generation in liquid crystal elastomer sheets

Mechanics of Surfaces: outline

1. Kirchhoff's plate/membrane theory (rational treatment)
2. Bending energy of a developable surface (Wunderlich functional)
3. Example: curved fold origami

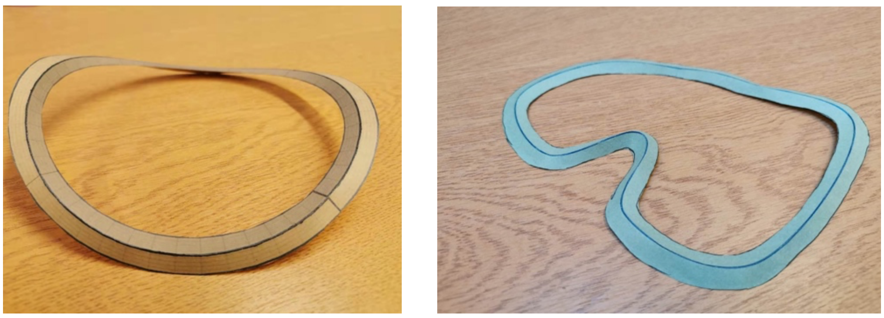


Figure 2: Two curved fold origami. The crease of the left one has non-zero torsion, which the right one has zero torsion.

1. Regular surfaces

Def 1: A subset $S \subset \mathbb{R}^3$ is a regular surface if for $\forall p \in S$, $\exists$ a neighborhood V in $\mathbb{R}^3$ and a map $\vec{x}: U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ s.t.

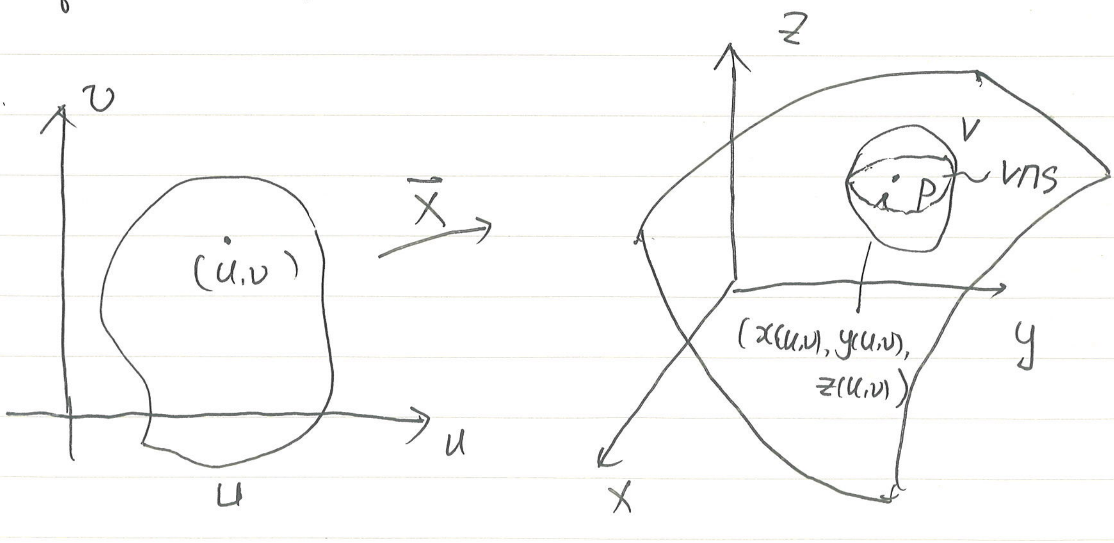
① $\vec{x}$ is differentiable.

$$\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U$$

the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ have continuous partial derivatives

② $\vec{x}$ is a homeomorphism: bijection, continuous, continuous inverse.

③ The regularity condition: for $\forall q \in U$, the differential $dx_q: \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.



Regularity condition: $\vec{x}_u(u_0,v_0) = \left.\frac{\partial \vec{x}}{\partial u}\right|_{(u_0,v_0)}$, $\vec{x}_v(u_0,v_0) = \left.\frac{\partial \vec{x}}{\partial v}\right|_{(u_0,v_0)}$

$\Leftrightarrow \left.\vec{x}_u \times \vec{x}_v\right|_{(u_0,v_0)} \neq 0$

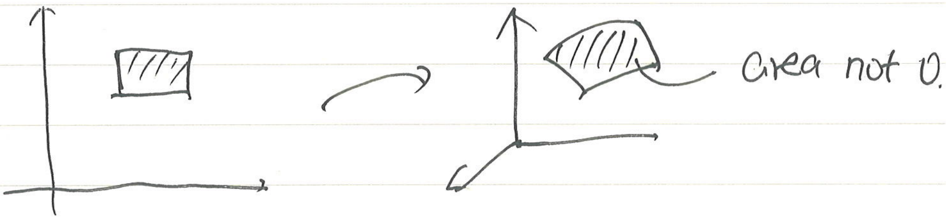
pa: $\vec{x}_u \times \vec{x}_v \Big|_{(u_0,v_0)} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) \Big|_{(u_0,v_0)} \neq 0$

Assume $\frac{\partial(x,y)}{\partial(u,v)} \Big|_{(u_0,v_0)} \neq 0$ Implicit function theorem.
Inverse function thm

$\Rightarrow \exists\ u = u(x,y),\ v = v(x,y)$ s.t.
$x(u(x,y), v(x,y)) \equiv x, \quad y(u(x,y), v(x,y)) \equiv y.$
$z = z(u,v) = z(u(x,y), v(x,y))$

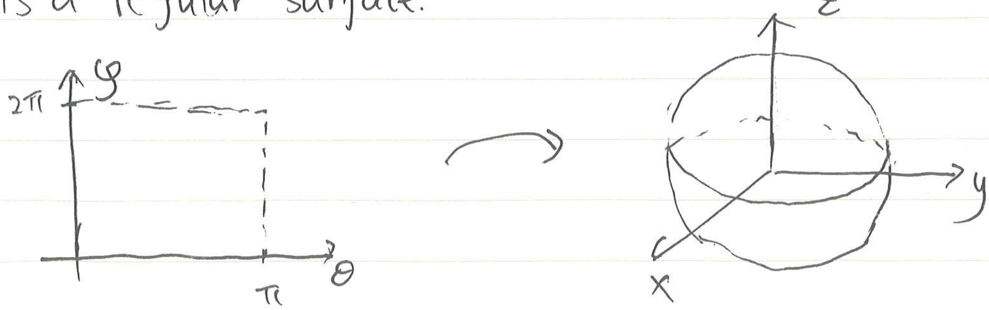
$\Rightarrow (u(x,y), v(x,y)) \Leftrightarrow (x, y, z(x,y))$

Geometrically



Proposition 1: If $f: U \to \mathbb{R}$ is a differentiable function.
$(x, y, f(x,y))$ is a regular surface.

Example 1: Let $V = \{(\theta, \varphi): 0 < \theta < \pi, 0 < \varphi < 2\pi\}$, let $\vec{x}: V \to \mathbb{R}^3$ be
$\vec{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
is a regular surface.



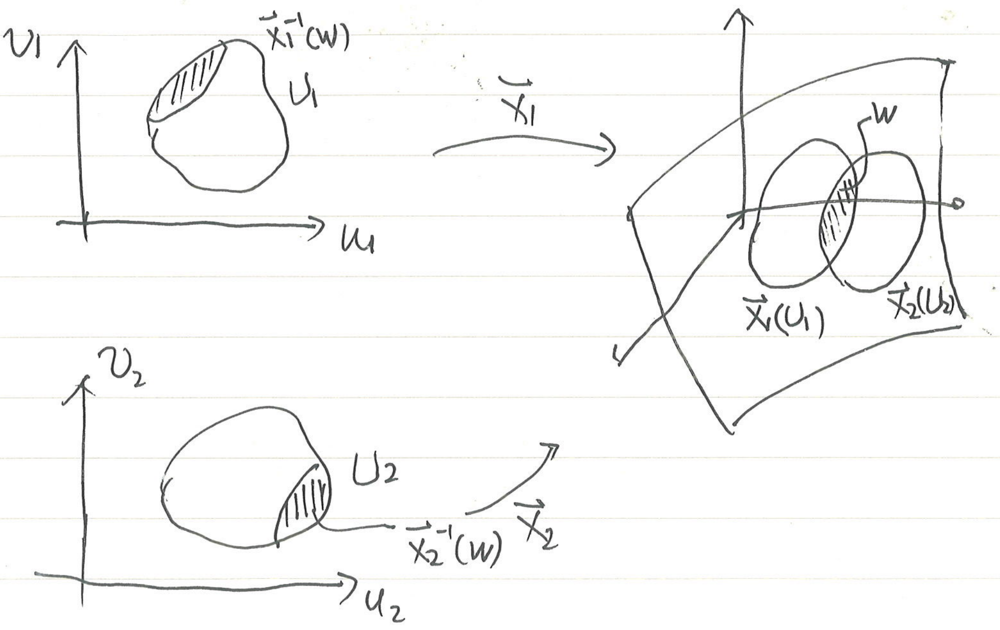
Proof: $\frac{\partial(x,y)}{\partial(\theta,\varphi)} = \sin\theta\cos\theta \quad \frac{\partial(y,z)}{\partial(\theta,\varphi)} = \sin^2\theta\cos\varphi, \quad \frac{\partial(x,z)}{\partial(\theta,\varphi)} = \sin^2\theta\sin\varphi$

$\Rightarrow \cos^2\theta \sin^2\theta + \sin^4\theta \cos^2\varphi + \sin^4\theta \sin^2\varphi = \sin^2\theta \neq 0$

Proposition 2: (change of parameters) S is a regular surface.
$p \in S$. neighborhoods V_1, V_2, $\begin{cases} \vec{x}_1 : U_1 \to V_1 \cap S \\ \vec{x}_2 : U_2 \to V_2 \cap S \end{cases}$

$p \in \vec{x}_1(U_1) \cap \vec{x}_2(U_2) = W$.

Then the change of coordinates $h = \vec{x}_1^{-1} \circ \vec{x}_2 : \vec{x}_2^{-1}(W) \to \vec{x}_1^{-1}(W)$ is a diffeomorphism. 同胚



proof: $\vec{x}_1 = (x_1(u_1,v_1), y_1(u_1,v_1), z_1(u_1,v_1))$ (略)
$\vec{x}_2 = (x_2(u_2,v_2), y_2(u_2,v_2), z_2(u_2,v_2))$

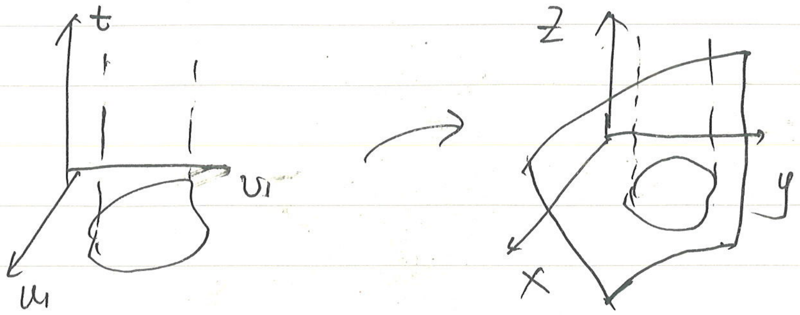
$\frac{\partial(x_1,y_1)}{\partial(u_1,v_1)} \neq 0 \Rightarrow u_1 = f(x,y),\ v_2 = g(x,y)$

regularity $\quad f(x_1(u_1,v_1), y_1(u_1,v_1)) \equiv u_1,\ g(x_1(u_1,v_1), y_1(u_1,v_1)) \equiv v_1$

$\Rightarrow u_1 = f(x_2(u_2,v_2), y_2(u_2,v_2)),\ v_1 = g(x_2(u_2,v_2), y_2(u_2,v_2))$

$$\frac{\partial(u_1,v_1)}{\partial(u_2,v_2)} = \frac{\partial(f,g)}{\partial(x,y)} \cdot \frac{\partial(x_2,y_2)}{\partial(u_2,v_2)} = \frac{\partial(x_2,y_2)}{\partial(u_2,v_2)} \left(\frac{\partial(x_1,y_1)}{\partial(u_1,v_1)}\right)^{-1} \neq 0$$

(do Carmo) $F(u_1,v_1,t) = (x(u_1,v_1),\ y(u_1,v_1),\ z(u_1,v_1)+t)$



F^{-1} exists.

$\Rightarrow h = F^{-1} \circ \vec{x}_2$ is differentiable.

Example 2: Cylindrical surface

$\vec{x} = \vec{x}(u,v) = (a\cos u,\ a\sin u,\ bv)$

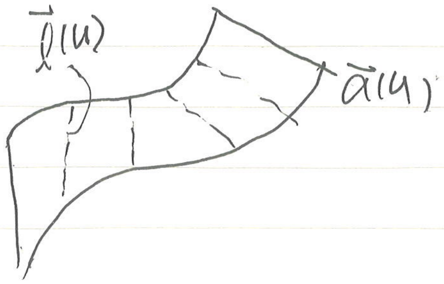
Example 3: Ruled surface.

$\vec{x} = \vec{x}(u,v) = \vec{a}(u) + v\,\vec{l}(u)$

$\vec{x}_u(u,v) = \vec{a}'(u) + u\,\vec{l}'(u)$, $\vec{x}_v(u,v) = \vec{l}(u)$

ruling, 直母线

regularity: $\vec{x}_u \times \vec{x}_v = (\vec{a}'(u) + v\,\vec{l}'(u)) \times \vec{l}(u) \neq 0$

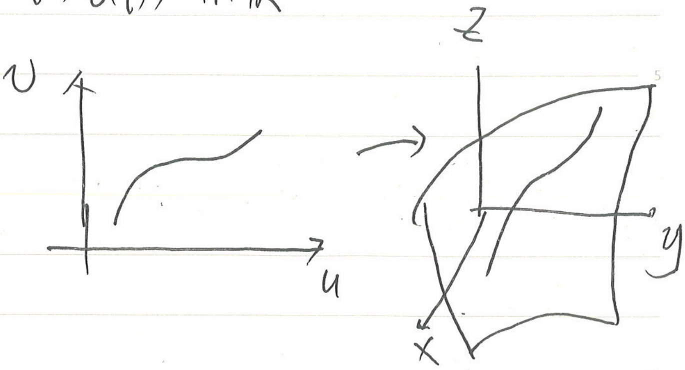


- Cylindrical surface is a ruled surface

2. The tangent plane and normal

Given a ~~g~~ regular surface $\vec{x}=\vec{x}(u,v)$ and a differentiable curve $u=u(t),\ v=v(t)$

$\Rightarrow \vec{x}(u(t),v(t))$ in $\mathbb{R}^3$.



$$\left.\frac{d\vec{x}(u(t),v(t))}{dt}\right|_{t=0}$$

$$=\vec{x}_u\left.\frac{du(t)}{dt}\right|_{t=0}+\vec{x}_v\left.\frac{dv(t)}{dt}\right|_{t=0}$$

$$\begin{cases} u(t)=u_0+at \\ v(t)=v_0+bt \end{cases} \Rightarrow \left.\frac{d\vec{x}(u(t),v(t))}{dt}\right|_{t=0}=a\vec{x}_u+b\vec{x}_v$$

$\vec{x}_u\times\vec{x}_v\neq 0.\ \Rightarrow$ Tangent space 切空间 T_pS.

Tangent plane : $\vec{X}(\lambda,\mu)=\vec{x}(u,v)+\lambda\vec{x}_u(u,v)+\mu\vec{x}_v(u,v)$

Normal 法向量: $\vec{n}(u,v)=\dfrac{\vec{x}_u(u,v)\times\vec{x}_v(u,v)}{|\vec{x}_u(u,v)\times\vec{x}_v(u,v)|}$

$\{\vec{x}_u(u,v),\vec{x}_v(u,v),\vec{n}(u,v)\}$ 自然标架 natural coordinates

3. Parameterization of regular surfaces

① Regular parameterization $(u,v)\in U$, $\vec{x}: U \to \mathbb{R}^3$
$\vec{x}(u,v)$

② Implicit parameterization

$f(x,y,z)$ $\qquad \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \neq 0$

$f(x,y,z) = c$ $\sim$ regular surface

implicit function thm

Assume $\left.\frac{\partial f}{\partial z}\right|_p \neq 0$ $\Rightarrow z = g(x,y) \Rightarrow f(x,y,g(x,y)) = c$

$\Rightarrow \vec{x}(x,y) = (x, y, g(x,y))$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the normal

Example 4: $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ $\qquad$ $\lambda \in (-\infty, c)$ 椭球
$\lambda \in (c, b)$ 单叶双曲
$\lambda \in (b, a)$ 双叶双曲.

4. The first fundamental form

regular surface $S: \vec{x} = \vec{x}(u,v)$

$d\vec{x}(u,v) = \vec{x}_u(u,v)\,du + \vec{x}_v(u,v)\,dv$

Let $E(u,v) = \vec{x}_u(u,v) \cdot \vec{x}_u(u,v)$
$F(u,v) = \vec{x}_u(u,v) \cdot \vec{x}_v(u,v) = \vec{x}_v(u,v) \cdot \vec{x}_u(u,v)$
$G(u,v) = \vec{x}_v(u,v) \cdot \vec{x}_v(u,v)$

- $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ positive definite

- $I = d\vec{x}(u,v) \cdot d\vec{x}(u,v)$

$$= \cdots$$

$$= E(u,v)(du)^2 + 2F(u,v)\,du\,dv + G(u,v)(dv)^2$$

$$= (du, dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$\sim$ First fundamental form

Thm 2: The first fundamental form is independent of the choices of ~~para~~ regular parameterization.

<u>proof</u> : 略

$$u = u(\tilde{u}, \tilde{v}), \quad v = v(\tilde{u}, \tilde{v})$$

$$\vec{x} = \vec{x}(\tilde{u}, \tilde{v}) = \vec{x}(u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v}))$$

$$d\vec{x} = \vec{x}_{\tilde{u}}(\tilde{u}, \tilde{v})\,d\tilde{u} + \vec{x}_{\tilde{v}}(\tilde{u}, \tilde{v})\,d\tilde{v}$$

$$\Rightarrow (d\tilde{u}, d\tilde{v})\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}\begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = (du, dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$