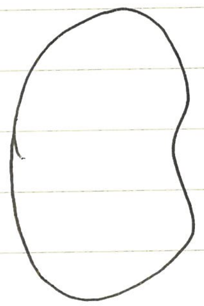
6. Minimal surface 极小曲面. (later)

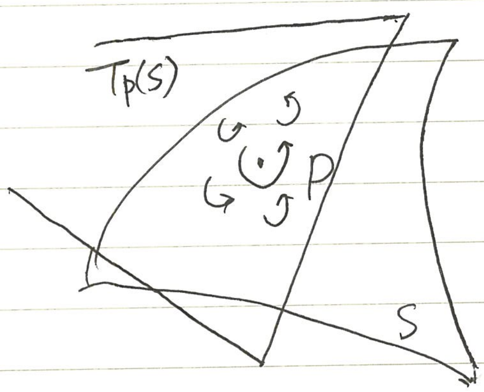


给定曲线C为边界的曲面中面积最小的曲面.

Example: Triply periodic minimal surface
TPMS.
点阵结构.

The second fundamental form

7. Orientation of surfaces. (do Carmo)



$\{\vec{x}_u, \vec{x}_v\}$

new basis $\{\tilde{x}_{\tilde{u}}, \tilde{x}_{\tilde{v}}\}$

$$\tilde{x}_{\tilde{u}} = \vec{x}_u \frac{\partial u}{\partial \tilde{u}} + \vec{x}_v \frac{\partial v}{\partial \tilde{u}}$$

$$\tilde{x}_{\tilde{v}} = \vec{x}_u \frac{\partial u}{\partial \tilde{v}} + \vec{x}_v \frac{\partial v}{\partial \tilde{v}}$$

$$J = \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} > 0$$

Def 7.1: A regular surface is called orientable if it is possible to cover it with a family of coordinate neighborhoods $S_1, S_2, \dots$ s.t. if $p \in S_1 \cap S_2$, the change of cooridinates has positive Jacobian at p.

Example 7.1: Surfaces covered by one coordinate neighborhood.
Sphere.

unit normal: $N = \dfrac{\vec{X}_u \wedge \vec{X}_v}{|\vec{X}_u \wedge \vec{X}_v|}(p)$, $\tilde{X}_{\tilde{u}} \wedge \tilde{X}_{\tilde{v}} = (\vec{X}_u \wedge \vec{X}_v)\dfrac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}$

Proposition: A regular surface $S \subset \mathbb{R}^3$ is orientable if and only if $\exists$ a differentiable field of unit normal vectors $N: S \to \mathbb{R}^3$ on S.

<u>proof</u>: $\Rightarrow$ trivial

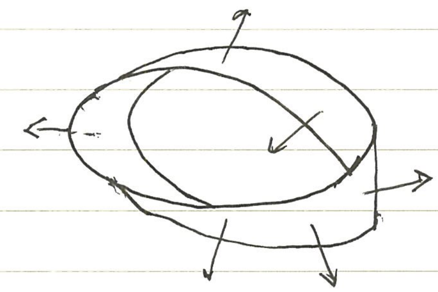
$\Leftarrow$ By interchanging u and v, $N(p) = \dfrac{\vec{X}_u \wedge \vec{X}_v}{|\vec{X}_u \wedge \vec{X}_v|}$

Then $\dfrac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}$ is positive

if not $\dfrac{\vec{X}_u \wedge \vec{X}_v}{|\vec{X}_u \wedge \vec{X}_v|} = N(p) = -\dfrac{\tilde{X}_{\tilde{u}} \wedge \tilde{X}_{\tilde{v}}}{|\tilde{X}_{\tilde{u}} \wedge \tilde{X}_{\tilde{v}}|} = -N(p)$

contradiction!

Example 7.2. Möbius strip is not orientable.



Ref: The shape of a Möbius strip. Nature Materials, 2007

MAR 25.

First fundamental form: $I = d\vec{x}(u,v) \cdot d\vec{x}(u,v)$

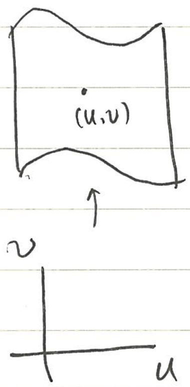
$$= (du, dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

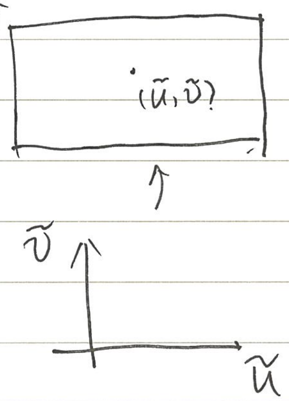
$E = \vec{x}_u \cdot \vec{x}_u \quad F = \vec{x}_u \cdot \vec{x}_v \quad G = \vec{x}_v \cdot \vec{x}_v$

· Isometry. $I_1 = \sigma^* I_2$

① Cylindrical surface: $\vec{x} = \vec{a}(u) + v\vec{l}_0$ $\quad \tilde{u} = \tilde{u}(u) \quad \tilde{v} = \tilde{v}(u)$

$\Rightarrow I_1 = (d\tilde{u})^2 + (d\tilde{v})^2$





$\vec{y} = \tilde{u}e_1 + \tilde{v}e_2$

$\Rightarrow I_2 = (d\tilde{u})^2 + (d\tilde{v})^2$

③ 切线面 $\vec{x}(s,t) = \vec{r}(s) + t\,\vec{r}\,'(s)$

$\vec{x}_s = \vec{r}\,'(s) + t\kappa(s)\vec{n}(s)$, $\vec{x}_t = \vec{t}(s)$

$\Rightarrow E = 1 + t^2\kappa^2$, $F = 1$, $G = 1$

$\Rightarrow I_1 = (1 + t^2\kappa^2)(ds)^2 + 2\,ds\,dt + (dt)^2$

We can find $\vec{y}(\tilde{s},\tilde{t})$ in $\mathbb{R}^2$, $\vec{y}(\tilde{s},\tilde{t}) = \vec{p}(\tilde{s}) + \tilde{t}\,\vec{p}\,'(\tilde{s})$, $\vec{p}(s) \in \mathbb{R}^2$

s.t. $\vec{p}(s)$ and $\vec{r}(s)$ have the same κ

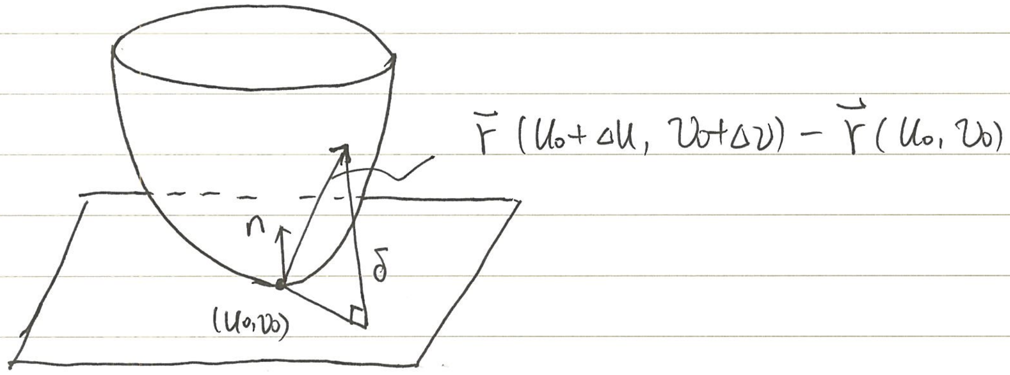
$\Rightarrow \tilde{s} = s$, $\tilde{t} = s$

$I_2 = (1 + \tilde{t}^2\kappa^2)(d\tilde{s})^2 + 2\,d\tilde{s}\,d\tilde{t} + (d\tilde{t})^2$

$= (1 + t^2\kappa^2)(ds)^2 + 2\,ds\,dt + (dt)^2$

The second fundamental form

8 Let $S: \vec{r} = \vec{r}(u,v)$ be a regular surface. Then the unit normal to the tangent plane at (u_0, v_0) is $\vec{n} = \left.\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right|_{(u_0,v_0)}$



- ~~PA~~ Geometrical meaning of the second fundamental form

$$\delta(\Delta u, \Delta v) = \left(\vec{r}(u_0+\Delta u, v_0+\Delta v) - \vec{r}(u_0, v_0)\right)\cdot \vec{n}$$

Taylor expansion: $\vec{r}(u_0+\Delta u, v_0+\Delta v) - \vec{r}(u_0, v_0)$

$$= (\vec{r}_u \Delta u + \vec{r}_v \Delta v) + \frac{1}{2}\left(\vec{r}_{uu}(\Delta u)^2 + 2\vec{r}_{uv}\Delta u \Delta v + \vec{r}_{vv}(\Delta v)^2\right)$$

+ higher order terms

$$\Rightarrow \delta(\Delta u, \Delta v) = \frac{1}{2}\left[L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right] + \text{higher order terms}$$

where
$$L = \vec{r}_{uu}\cdot\vec{n}$$
$$M = \vec{r}_{uv}\cdot\vec{n}$$
$$N = \vec{r}_{vv}\cdot\vec{n}$$

$$\vec{r}_u\cdot\vec{n} = \vec{r}_v\cdot\vec{n} = 0 \Rightarrow \begin{cases} \vec{r}_{uu}\cdot\vec{n} + \vec{r}_u\cdot\vec{n}_u = 0 \\ \vec{r}_{uv}\cdot\vec{n} + \vec{r}_u\cdot\vec{n}_v = 0 \\ \vec{r}_{vu}\cdot\vec{n} + \vec{r}_v\cdot\vec{n}_u = 0 \\ \vec{r}_{vv}\cdot\vec{n} + \vec{r}_v\cdot\vec{n}_v = 0 \end{cases}$$

$$\therefore L = -\vec{r}_u\cdot\vec{n}_u, \quad M = -\vec{r}_u\cdot\vec{n}_v = -\vec{r}_v\cdot\vec{n}_u, \quad N = -\vec{r}_v\cdot\vec{n}_v$$

The second fundamental form: $\mathrm{II} = d^2\vec{r}\cdot\vec{n} = -d\vec{r}\cdot d\vec{n}$

$$= L(du)^2 + 2M\,du\,dv + N(dv)^2$$

$\mathrm{II} \approx 2\vec{\delta}(du, dv)$

Thm 8.1. The second fundametal form is independent of the choices of parameterization.

proof: $u = u(\tilde{u}, \tilde{v})$, $v = v(\tilde{u}, \tilde{v})$ and $\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} > 0$

$$\vec{r}_{\tilde{u}} = \vec{r}_u \frac{\partial u}{\partial \tilde{u}} + \vec{r}_v \frac{\partial v}{\partial \tilde{u}}, \quad \vec{r}_{\tilde{v}} = \vec{r}_u \frac{\partial u}{\partial \tilde{v}} + \vec{r}_v \frac{\partial v}{\partial \tilde{v}}$$

$$\Rightarrow \vec{r}_{\tilde{u}} \times \vec{r}_{\tilde{v}} = \frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} \vec{r}_u \times \vec{r}_v, \quad \tilde{n} = \vec{n}$$

$$\vec{n}_{\tilde{u}} = \vec{n}_u \frac{\partial u}{\partial \tilde{u}} + \vec{n}_v \frac{\partial v}{\partial \tilde{u}}, \quad \vec{n}_{\tilde{v}} = \vec{n}_u \frac{\partial u}{\partial \tilde{v}} + \vec{n}_v \frac{\partial v}{\partial \tilde{v}}$$

$$\Rightarrow \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = -\begin{pmatrix} \vec{r}_{\tilde{u}} \\ \vec{r}_{\tilde{v}} \end{pmatrix} (\vec{n}_{\tilde{u}}, \vec{n}_{\tilde{v}}) = -J \begin{pmatrix} \vec{r}_u \\ \vec{r}_v \end{pmatrix} (\vec{n}_u, \vec{n}_v) J^T$$

$$= J \begin{pmatrix} L & M \\ M & N \end{pmatrix} J^T$$

$(du, dv) = (d\tilde{u}, d\tilde{v}) J$

$$\Rightarrow \mathrm{II} = (du, dv) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = (d\tilde{u}, d\tilde{v}) \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix}$$

Example 8.1 Cylindrical surface.

$$\vec{r} = (a\cos\tfrac{u}{a}, a\sin\tfrac{u}{a}, v)$$

$\vec{r}_u = (-\sin\frac{u}{a}, \cos\frac{u}{a}, 0)$, $\vec{r}_v = (0, 0, 1)$

$\vec{n} = \vec{r}_u \times \vec{r}_v = (\cos\frac{u}{a}, \sin\frac{u}{a}, 0)$

$\vec{r}_{uu} = (-\frac{1}{a}\cos\frac{u}{a}, -\frac{1}{a}\sin\frac{u}{a}, 0)$, $\vec{r}_{uv} = \vec{r}_{vv} = \vec{0}$

$\Rightarrow E = 1$, $F = 0$, $G = 1$, $L = -\frac{1}{a}$, $M = 0$, $N = 0$

$\mathrm{I} = (du)^2 + (dv)^2$, $\mathrm{II} = -\frac{1}{a}(du)^2$

Example 8.2: Plane $\vec{r} = (u, v, 0)$

$\vec{n} = (0, 0, 1)$

$\Rightarrow \mathrm{I} = (du)^2 + (dv)^2 \qquad \mathrm{II} = -d\vec{r} \cdot d\vec{n} = 0.$

Example 8.3: ~~Sh.~~ Sphere $(\vec{r}(u,v) - \vec{r}_0)^2 = R^2$

$\Rightarrow d\vec{r} \cdot (\vec{r}(u,v) - r_0) = 0 \quad \Rightarrow \vec{n} = \frac{1}{R}(\vec{r}(u,v) - \vec{r}_0)$

$\mathrm{II} = -dr \cdot dn = -\frac{1}{R} d\vec{r} \cdot d\vec{r} = -\frac{1}{R}\mathrm{I}.$

9 Normal curvature and Principal Curvature.

curves on a surface $\Rightarrow$ out of plane property.

$S: \vec{r} = \vec{r}(u,v)$. curve C: $u = u(s)$ $v = v(s)$

$\vec{r} = \vec{r}(u(s), v(s)) \qquad \vec{\alpha}(s) = \frac{d\vec{r}}{ds} = \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds}$ — tangent

curvature: $\frac{d\vec{\alpha}}{ds} = k\vec{\beta}$

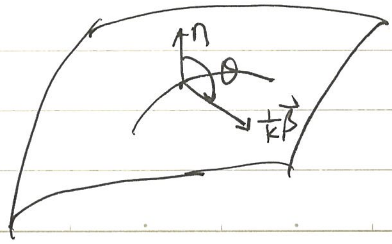
$(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ - Frenet.

$$= \vec{r}_{uu}\left(\frac{du}{ds}\right)^2 + 2\vec{r}_{uv}\frac{du}{ds}\frac{dv}{ds} + \vec{r}_{vv}\left(\frac{dv}{ds}\right)^2 + r_u \frac{d^2u}{ds^2} + r_v \frac{d^2v}{ds^2}$$

$$k_n = \frac{d\vec{\alpha}}{ds} \cdot \vec{n} = L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

$\downarrow$

normal curvature 法曲率 $\quad k_n = k\cos\theta$



$\vec{r}(s) = \vec{r}(u(s), v(s))$ · all curves with the same tangent have the same k_n.

$$|\vec{r}\,'(s)|^2 = 1 = E\left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2$$

$$\Rightarrow ds^2 = E(du)^2 + 2F\,du\,dv + G(dv)^2$$

$$\Rightarrow k_n = \frac{L(du)^2 + 2M\,du\,dv + N(dv)^2}{ds^2} = \frac{\mathrm{II}}{\mathrm{I}}$$

Def 9.1 $k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L(du)^2 + 2M\,du\,dv + N(dv)^2}{E(du)^2 + 2F\,du\,dv + G(dv)^2}$ is called the normal curvature at (u,v), along (du, dv)

(du, dv) $\vec{n}(u,v)$ $\Rightarrow$ 法截面 normal section

Example 9.1. Compute the normal curvatures of plane. cylindrical surface and spherical surface.

① $\mathrm{I} = (du)^2 + (dv)^2$, $\mathrm{II} = -\frac{1}{a}(du)^2$. $\vec{r} = (a\cos\frac{u}{a}, a\sin\frac{u}{a}, v)$

$$k_n = -\frac{(du)^2}{a[(du)^2 + (dv)^2]} = -\frac{1}{a}\cos^2\theta$$

② $k_n = -\frac{1}{R}$

Thm: 9.1. k_n reaches maximum and minimum along two perpendicular tangents.

proof:

$S: \vec{r} = \vec{r}(u,v)$ Recalling the existence of $\boxed{\text{orthonormal basis}}$ (§4, 3.4)

$$\mathrm{I} = E(du)^2 + G(dv)^2, \quad \mathrm{II} = L(du)^2 + 2M\,du\,dv + N(dv)^2$$

Let θ be the angle between (du, dv) and u-line.

$$\cos\theta = \frac{\sqrt{E}\,du}{\sqrt{E(du)^2 + G(dv)^2}} \qquad \sin\theta = \frac{\sqrt{G}\,dv}{\sqrt{E(du)^2 + G(dv)^2}}$$

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L(du)^2 + 2M\,du\,dv + N(dv)^2}{E(du)^2 + G(dv)^2}$$

$$= \cdots$$

$$= \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + \underbrace{\sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}}_{A} \cos 2(\theta - \theta_0)$$

where $\cos 2\theta_0 = \frac{1}{2A}\left(\frac{L}{E} - \frac{N}{G}\right)$, $\sin 2\theta_0 = \frac{M}{A\sqrt{EG}}$.

$\theta = \theta_0$, $k_1 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) + \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}$,

$\theta = \theta_0 + \frac{\pi}{2}$, $k_2 = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) - \sqrt{\left(\frac{1}{2}\left(\frac{L}{E} - \frac{N}{G}\right)\right)^2 + \left(\frac{M}{\sqrt{EG}}\right)^2}$

$A = 0$, $k_n = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)$

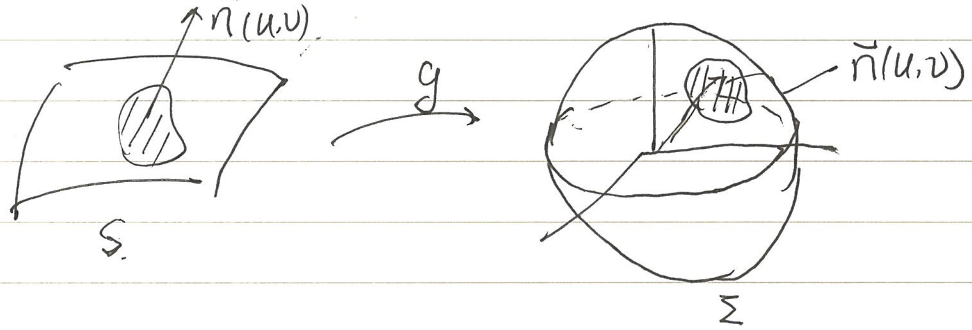
$$\boxed{k_n(\theta) = k_1 \cos^2(\theta - \theta_0) + k_2 \sin^2(\theta - \theta_0)}$$ Euler equation

How to determine k_1, k_2, θ_0 for ~~at~~ arbitrary coordinates ?

10 The Gauss map and Weingarten map

$\vec{r} = \vec{r}(u,v)$. ~ regular surface

$g \cdot (\vec{r}(u,v)) = \vec{n}(u,v)$. $\quad g : S \to \Sigma \quad$ Gauss map



$g_* : T_pS \to T_{g(p)}\Sigma$

Let $u = u(t)$, $v = v(t)$

$g(\vec{r}(u(t), v(t))) = \vec{n}(u(t), v(t))$

$\Rightarrow g_*\left(\frac{d\vec{r}}{dt}\right) = g_*\left(\vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt}\right) = \vec{n}_u \frac{du}{dt} + \vec{n}_v \frac{dv}{dt}$

$\Rightarrow g_*(\vec{r}_u) = \vec{n}_u, \quad g_*(\vec{r}_v) = \vec{n}_v$

$W = -g_* : T_pS \to T_pS \quad$ ~ Weingarten map

Thm 10.1. $\mathrm{II} = W(d\vec{r}) \cdot d\vec{r}$

proof: $d\vec{r} = \vec{r}_u du + \vec{r}_v dv$

$W(d\vec{r}) = -g_*(\vec{r}_u)du - g_*(\vec{r}_v)dv$

$= -(\vec{n}_u du + \vec{n}_v dv) = -d\vec{n}$

$\Rightarrow \mathrm{II} = W(d\vec{r}) \cdot d\vec{r}$

Thm 10.2. $W(d\vec{r}) \cdot \delta\vec{r} = d\vec{r} \cdot W(\delta\vec{r})$

proof: $d\vec{r} = \vec{r}_u du + \vec{r}_v dv$, $\delta\vec{r} = \vec{r}_u \delta u + \vec{r}_v \delta v$

$W(d\vec{r}) = -(\vec{n}_u du + \vec{n}_v dv)$, $W(\delta\vec{r}) = -(\vec{n}_u \delta u + \vec{n}_v \delta v)$

$W(d\vec{r}) \cdot \delta\vec{r} = -(\vec{n}_u du + \vec{n}_v dv) \cdot (\vec{r}_u \delta u + \vec{r}_v \delta v)$

$= -(\vec{r}_u du + \vec{r}_v dv) \cdot (\vec{n}_u \delta u + \vec{n}_v \delta v) = d\vec{r} \cdot W(\delta\vec{r})$

Let λ and $d\vec{r}$ be the eigenvalue and eigenvector of W

i.e. $W(d\vec{r}) = \lambda d\vec{r}$

Then $k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{W(d\vec{r}) \cdot d\vec{r}}{d\vec{r} \cdot d\vec{r}} = \lambda$

self-conjugate

Proposition: If a linear transformation A satisfies $A(\vec{u}) \cdot \vec{v} = \vec{u} \cdot A(\vec{v})$

Then ① The eigenvalues of A are real.

$$A(\vec{v}) \cdot \bar{v} = \lambda v \cdot \bar{v} = v \cdot A(\bar{v}) = \bar{\lambda} v \cdot \bar{v}$$

$$\Rightarrow (\bar{\lambda} - \lambda) v \cdot \bar{v} = 0 \quad \Rightarrow \bar{\lambda} = \lambda$$

② if λ_1, λ_2 are two eigenvalues of A, $\lambda_1 \neq \lambda_2$ and the corresponding eigenvalues are v_1, v_2, Then $v_1 \cdot v_2 = 0$

proof: $A(v_1) = \lambda_1 v_1$, $A(v_2) = \lambda_2 v_2$

$$A(v_1) \cdot v_2 = v_1 \cdot A(v_2) \Rightarrow \lambda_1 v_1 \cdot v_2 = \lambda_2 v_1 \cdot v_2$$

$$\Rightarrow (\lambda_1 - \lambda_2) v_1 \cdot v_2 = 0 \quad \Rightarrow v_1 \cdot v_2 = 0$$

Thm 10.3. The two eigenvalues of the Weingarten map are the two principal curvatures, and the corresponding ~~eigenvalues~~ eigenvectors are the two principal tangents

proof: Let $W(e_1) = \lambda_1 e_1$, $W(e_2) = \lambda_2 e_2$

$\forall\, e = \cos\theta\, e_1 + \sin\theta\, e_2$

$W(e) = \cos\theta\, W(e_1) + \sin\theta\, W(e_2) = \lambda_1 \cos\theta\, e_1 + \lambda_2 \sin\theta\, e_2$

$$\Rightarrow k(\theta) = \frac{W(e) \cdot e}{e \cdot e} = (\lambda_1 \cos\theta\, e_1 + \lambda_2 \sin\theta\, e_2) \cdot (\cos\theta\, e_1 + \sin\theta\, e_2)$$

$$= \lambda_2 + (\lambda_1 - \lambda_2)\cos^2\theta$$

$$\Rightarrow \begin{cases} \theta = 0, & k_n(\theta) = \lambda_2 \\ \theta = \frac{\pi}{2}, & k_n(\theta) = \lambda_1 \end{cases}$$

Computation of principal curvatures and principal directions.

$S: \vec{r}=\vec{r}(u,v)$, $\delta\vec{r}=\vec{r}_u\delta u+\vec{r}_v\delta v$ is a principal direction at (u,v)

$\Rightarrow W(\delta\vec{r})=\lambda\delta\vec{r}$

$\Rightarrow -(\vec{n}_u\delta u+\vec{n}_v\delta v)=\lambda(\vec{r}_u\delta u+\vec{r}_v\delta v)$

$\cdot\vec{r}_u$, $\cdot\vec{r}_v$ $\Rightarrow \begin{cases} L\delta u+M\delta v=\lambda(E\delta u+F\delta v) \\ M\delta u+N\delta v=\lambda(F\delta u+G\delta v)\end{cases}$

$\therefore (\delta u,\delta v)$ is a nonzero solution to $\begin{cases}(L-\lambda E)\delta u+(M-\lambda F)\delta v=0 \\ (M-\lambda F)\delta u+(N-\lambda G)\delta v=0\end{cases}$

$$\Rightarrow \begin{vmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G\end{vmatrix}=0$$

$$\Rightarrow \lambda^2(EG-F^2)-\lambda(LG-2MF+NE)+(LN-M^2)=0$$

$$\Rightarrow k_1+k_2=2H=\frac{LG-2MF+NE}{EG-F^2}$$ ~ 平均曲率 H

$$k_1k_2=K=\frac{LN-M^2}{EG-F^2}$$ ~ Gauss curvature.

$$\Rightarrow k_1=H+\sqrt{H^2-K},\quad k_2=H-\sqrt{H^2-K}.$$ ✡