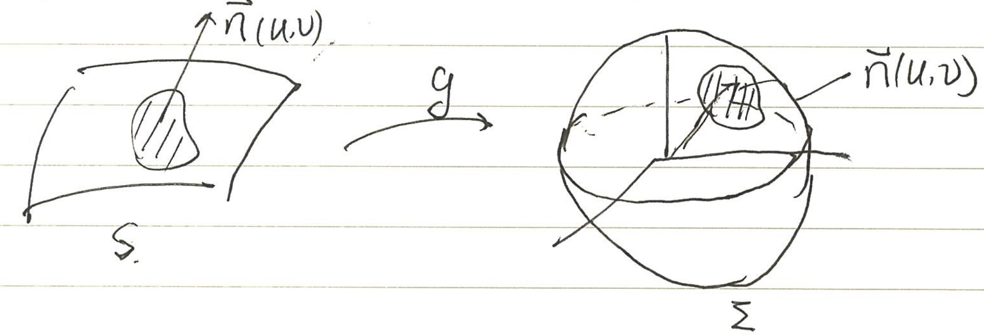
How to determine k_1, k_2, θ_0 for ~~at~~ arbitrary cooridinates ?

10 The Gauss map and Weingarten map

$\vec r = \vec r(u,v)$. ~ regular surface

$g_\circ(\vec r(u,v)) = \vec n(u,v)$, $\quad g: S \to \Sigma$ Gauss map



$g_*: T_pS \to T_{g(p)}\Sigma$

Let $u = u(t)$, $v = v(t)$

$g(\vec r(u(t), v(t))) = \vec n(u(t), v(t))$

$\Rightarrow g_*\left(\frac{d\vec r}{dt}\right) = g_*\left(\cancel{\partial r}\ \vec r_u \frac{du}{dt} + \vec r_v \frac{dv}{dt}\right) = \vec n_u \frac{du}{dt} + \vec n_v \frac{dv}{dt}$

$\Rightarrow g_*(\vec r_u) = \vec n_u, \quad g_*(\vec r_v) = \vec n_v$

$W = -g_*: T_pS \to T_pS$ ~ Weingarten map

Thm 10.1. $\mathrm{II} = W(d\vec r)\cdot d\vec r$

proof: $d\vec r = \vec r_u du + \vec r_v dv$

$W(d\vec r) = -g_*(\vec r_u)du - g_*(\vec r_v)dv$

$= -(\vec n_u du + \vec n_v dv) = -d\vec n$

$\Rightarrow \mathrm{II} = W(d\vec r)\cdot d\vec r$

Thm 10.2: $W(d\vec r)\cdot\delta\vec r = d\vec r\cdot W(\delta\vec r)$

proof: $d\vec r = \vec r_u du + \vec r_v dv$, $\quad \delta\vec r = \vec r_u \delta u + \vec r_v \delta v$

$W(d\vec r) = -(\vec n_u du + \vec n_v dv)$, $\quad W(\delta\vec r) = -(\vec n_u \delta u + \vec n_v \delta v)$

$W(d\vec r)\cdot\delta\vec r = -(\vec n_u du + \vec n_v dv)\cdot(\vec r_u \delta u + \vec r_v \delta v)$

$= -(\vec r_u du + \vec r_v dv)\cdot(\vec n_u \delta u + \vec n_v \delta v) = d\vec r\cdot W(\delta\vec r)$

Let λ and $d\vec{r}$ be the eigenvalue and eigenvector of W

i.e. $W(d\vec{r}) = \lambda d\vec{r}$

Then $k_n = \frac{II}{I} = \frac{W(d\vec{r}) \cdot d\vec{r}}{d\vec{r} \cdot d\vec{r}} = \lambda$

self-conjugate

Proposition: If a linear transformation A satisfies $A(\vec{u}) \cdot \vec{v} = \vec{u} \cdot A(\vec{v})$

Then ① The eigenvalues of A are real.

$$A(\vec{v}) \cdot \bar{v} = \lambda v \cdot \bar{v} = v \cdot A(\bar{v}) = \bar{\lambda} v \cdot \bar{v}$$

$$\Rightarrow (\bar{\lambda} - \lambda) v \cdot \bar{v} = 0 \quad \Rightarrow \bar{\lambda} = \lambda$$

② if λ_1, λ_2 are two eigenvalues of A, $\lambda_1 \neq \lambda_2$ and the corresponding eigenvalues are v_1, v_2, Then $v_1 \cdot v_2 = 0$

proof: $A(v_1) = \lambda_1 v_1$, $A(v_2) = \lambda_2 v_2$

$$A(v_1) \cdot v_2 = v_1 \cdot A(v_2) \Rightarrow \lambda_1 v_1 \cdot v_2 = \lambda_2 v_1 \cdot v_2$$

$$\Rightarrow (\lambda_1 - \lambda_2) v_1 \cdot v_2 = 0 \quad \Rightarrow v_1 \cdot v_2 = 0$$

Thm 10.3. The two eigenvalues of the Weingarten map are the two principal curvatures, and the corresponding ~~eigenvalues~~ eigenvectors are the two principal tangents.

proof: Let $W(e_1) = \lambda_1 e_1$, $W(e_2) = \lambda_2 e_2$

$\forall e = \cos\theta\, e_1 + \sin\theta\, e_2$

$$W(e) = \cos\theta\, W(e_1) + \sin\theta\, W(e_2) = \lambda_1 \cos\theta\, e_1 + \lambda_2 \sin\theta\, e_2$$

$$\Rightarrow k(\theta) = \frac{W(e) \cdot e}{e \cdot e} = (\lambda_1 \cos\theta\, e_1 + \lambda_2 \sin\theta\, e_2) \cdot (\cos\theta\, e_1 + \sin\theta\, e_2)$$

$$= \lambda_2 + (\lambda_1 - \lambda_2) \cos^2\theta$$

$$\Rightarrow \begin{cases} \theta = 0, & k_n(\theta) = \lambda_2 \\ \theta = \frac{\pi}{2}, & k_n(\theta) = \lambda_1 \end{cases}$$

Computation of principal curvatures and principal directions.

$S: \vec{r} = \vec{r}(u,v)$, $\delta\vec{r} = \vec{r}_u \delta u + \vec{r}_v \delta v$ is a principal direction at (u,v)

$\Rightarrow W(\delta\vec{r}) = \lambda \delta\vec{r}$

$\Rightarrow -(\vec{n}_u \delta u + \vec{n}_v \delta v) = \lambda(\vec{r}_u \delta u + \vec{r}_v \delta v)$

$\cdot \vec{r}_u$, $\cdot \vec{r}_v$ $\Rightarrow \begin{cases} L\delta u + M\delta v = \lambda(E\delta u + F\delta v) \\ M\delta u + N\delta v = \lambda(F\delta u + G\delta v) \end{cases}$

$\therefore (\delta u, \delta v)$ is a nonzero solution to $\begin{cases} (L-\lambda E)\delta u + (M-\lambda F)\delta v = 0 \\ (M-\lambda F)\delta u + (N-\lambda G)\delta v = 0 \end{cases}$

$$\Rightarrow \begin{vmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2(EG-F^2) - \lambda(LG-2MF+NE) + (LN-M^2) = 0$$

$$\Rightarrow K_1 + K_2 = 2H = \frac{LG-2MF+NE}{EG-F^2} \quad \sim \text{平均曲率 } H$$

$$K_1 K_2 = K = \frac{LN-M^2}{EG-F^2} \quad \sim \text{Gauss curvature.}$$

$$\Rightarrow K_1 = H + \sqrt{H^2-K}, \quad K_2 = H - \sqrt{H^2-K}.$$

✡

(#) Fundamental theorem of surfaces — ⑪ Natural coordinates

Regular surfaces $\vec{r} = \vec{r}(u, v)$ — natural coordinates: $\{\vec{r}_u, \vec{r}_v, \vec{n}\}$, $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$\vec{r} = \vec{r}(u^1, u^2)$ $\quad \vec{r}_\alpha = \dfrac{\partial \vec{r}(u^1, u^2)}{\partial u^\alpha}$, $\alpha = 1, 2$.

$d\vec{r} = \vec{r}_\alpha (u^1, u^2)\, du^\alpha$ ~ Einstein notation.

First fundamental form: $g_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta$

Second fundamental form: $b_{\alpha\beta} = \vec{r}_{\alpha\beta} \cdot \vec{n} = -\vec{r}_\alpha \cdot \vec{n}_\beta = -\vec{r}_\beta \cdot \vec{n}_\alpha$

Then the two fundamental forms can be written as

$I = g_{\alpha\beta}\, du^\alpha du^\beta$

$II = b_{\alpha\beta}\, du^\alpha du^\beta$

Recall that $g_{\alpha\beta}$ is positive definite $\Rightarrow$ inverse $g^{\alpha\beta}$ s.t. $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$.

natural coordinates. $\dfrac{\partial \vec{r}}{\partial u^\alpha} = \vec{r}_\alpha$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} = T^\gamma_{\alpha\beta} \vec{r}_\gamma + C_{\alpha\beta} \vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = D^\gamma_\beta \vec{r}_\gamma + D_\beta \vec{n} \end{cases}$$

How to determine $T^\gamma_{\alpha\beta}$, $C_{\alpha\beta}$, D^γ_β, D_β? from $\vec{r}(u^1, u^2)$, $\vec{n}$

- $C_{\alpha\beta} = \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} \cdot \vec{n} = \vec{r}_{\alpha\beta} \cdot \vec{n} = b_{\alpha\beta}$

- $\dfrac{\partial \vec{n}}{\partial u^\beta} \cdot \vec{r}_\xi = D^\gamma_\beta \vec{r}_\gamma \cdot \vec{r}_\xi = D^\gamma_\beta g_{\gamma\xi}$ $\Rightarrow D^\gamma_\beta g_{\gamma\xi} = -b_{\beta\xi}$

$D^\gamma_\beta = -b_{\beta\xi} g^{\xi\gamma}$

Let $b_\beta^\gamma = b_{\beta\xi} g^{\xi\gamma}$. $\sim g^{\xi\gamma}$ $g_{\xi\gamma}$. 指标↑↓

for example: $b_\beta^\gamma g_{\gamma\eta} = b_{\beta\xi} g^{\xi\gamma} g_{\gamma\eta} = b_{\beta\xi} \delta^\xi_\eta = b_{\beta\eta}$

$$\boxed{D_\beta^\gamma = -b_\beta^\gamma}$$

$|\vec{n}| = 1 \Rightarrow \frac{\partial \vec{n}}{\partial u^\beta} \perp \vec{n}$

$$\Rightarrow \begin{cases} \frac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} \vec{r}_\gamma + b_{\alpha\beta} \vec{n} \\ \frac{\partial \vec{n}}{\partial u^\beta} = -b_\beta^\gamma \vec{r}_\gamma \end{cases}$$

How about $\Gamma^\gamma_{\alpha\beta}$?

$$\vec{r}_{\alpha\beta} \cdot \vec{r}_\xi = \Gamma^\gamma_{\alpha\beta} \vec{r}_\gamma \cdot \vec{r}_\xi = \Gamma^\gamma_{\alpha\beta} g_{\gamma\xi}$$

Let $\Gamma^\gamma_{\alpha\beta} g_{\gamma\xi} = \Gamma_{\xi\alpha\beta} \Rightarrow \Gamma^\gamma_{\alpha\beta} = g^{\gamma\xi} \Gamma_{\xi\alpha\beta}$ $g \sim$ 度量矩阵

Then $\vec{r}_{\alpha\beta} \cdot \vec{r}_\xi = \Gamma_{\xi\alpha\beta}$

$$\boxed{\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}, \quad \Gamma_{\xi\alpha\beta} = \Gamma_{\xi\beta\alpha}}$$

$$g_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta \Rightarrow \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} = \vec{r}_{\alpha\gamma} \cdot \vec{r}_\beta + \vec{r}_\alpha \cdot \vec{r}_{\beta\gamma}$$

$$= \Gamma_{\beta\alpha\gamma} + \Gamma_{\alpha\beta\gamma}$$

$$\Rightarrow \begin{cases} \frac{\partial g_{\alpha\gamma}}{\partial u^\beta} = \Gamma_{\gamma\alpha\beta} + \Gamma_{\alpha\gamma\beta}, \\ \frac{\partial g_{\gamma\beta}}{\partial u^\alpha} = \Gamma_{\beta\gamma\alpha} + \Gamma_{\gamma\beta\alpha} \end{cases}$$

$$2\Gamma_{\gamma\alpha\beta} = \frac{\partial g_{\alpha\gamma}}{\partial u^\beta} + \frac{\partial g_{\gamma\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}$$

$$\Rightarrow \Gamma_{\gamma\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial u^\beta} + \frac{\partial g_{\gamma\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right)$$

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\xi}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\xi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}\right)$$

$\uparrow$

Christoffel notation

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\vec{r}_\gamma + b_{\alpha\beta}\vec{n} \\[2ex] \dfrac{\partial \vec{n}}{\partial u^\beta} = -b^\gamma_\beta \vec{r}_\gamma \end{cases}$$

Recalling Frenet equatio

Example: $z = f(x,y)$ Christoffel notion

$\vec{r} = (x, y, f(x,y))$

Sol: $\vec{r}_x = (1, 0, f_x)$, $\vec{r}_y = (0, 1, f_y)$, $\vec{r}_{xx} = (0, 0, f_{xx})$
$\vec{r}_{xy} = (0, 0, f_{xy})$, $\vec{r}_{yy} = (0, 0, f_{yy})$

$\Rightarrow g_{11} = 1 + f_x^2$, $g_{12} = f_x f_y$, $g_{22} = 1 + f_y^2$, $g = 1 + f_x^2 + f_y^2$

$$g^{11} = \frac{1+f_y^2}{1+f_x^2+f_y^2}, \quad g^{12} = -\frac{f_x f_y}{1+f_x^2+f_y^2}, \quad g^{22} = \frac{1+f_x^2}{1+f_x^2+f_y^2}$$

$\Rightarrow \Gamma_{111}$, $\Gamma_{112} = \Gamma_{121}$, Γ_{122}, Γ_{211}, Γ_{222}, $\Gamma_{212} = \Gamma_{221}$

$\Rightarrow \Gamma^1_{11}$, $\Gamma^1_{12} = \Gamma^1_{21}$, Γ^1_{22}, Γ^2_{11}, $\Gamma^2_{12} = \Gamma^2_{21}$, Γ^2_{22}

Thm: Let D_1, D_2 be regular surfaces from $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$. If for $\forall (u^1, u^2) \in D$, S_1 and S_2 have the same first and second fundamental forms. Then S_1 and S_2 differ by a rigid motion

sketch of proof:

$l_1 = \{ \vec{r}_1^{(1)}(u_0), \vec{r}_2^{(1)}(u_0), \vec{n}^{(1)}(u_0) \}$ $\qquad l_2 = \{ \vec{r}_1^{(2)}(u_0), \vec{r}_2^{(2)}(u_0), \vec{n}^{(2)}(u_0) \}$

① $\exists \sigma$: $\quad \sigma(l_1) \to l_2$

② $f_{\alpha\beta}(u) = (\vec{r}_\alpha^{(1)} - \vec{r}_\alpha^{(2)}) \cdot (\vec{r}_\beta^{(1)} - \vec{r}_\beta^{(2)})$
$f_\alpha(u) = (\vec{r}_\alpha^{(1)} - \vec{r}_\alpha^{(2)}) \cdot (\vec{n}^{(1)} - \vec{n}^{(2)})$
$f(u) = (n^{(1)} - n^{(2)})^2$

③ prove: $f_{\alpha\beta}(u) \equiv 0$, $f_\alpha(u) \equiv 0$, $f(u) \equiv 0$

$f_{\alpha\beta}(u_0) = 0$, $f_\alpha(u_0) = 0$, $f(u_0) = 0$
$$\begin{cases} \dfrac{\partial f_{\alpha\beta}}{\partial u^\gamma} = \cdots \\ \dfrac{\partial f_\alpha}{\partial u^\gamma} = \cdots \\ \dfrac{\partial f}{\partial u^\gamma} = \cdots \end{cases}$$

First order

⑫ Compatibility : Gauss-Codazzi equation

? Given $\varphi = g_{\alpha\beta}\, du^\alpha du^\beta$ $\psi = b_{\alpha\beta}\, du^\alpha du^\beta$ is there a regular surface in $\mathbb{R}^3$: $f : D \to \mathbb{R}^3$ s.t. φ, ψ are the first and second fundamental forms ?

$\Rightarrow$ compatibility equation.

$\{\vec{r}_1, \vec{r}_2, \vec{n}\}$ natural coordinates

$$\begin{cases} \dfrac{\partial \vec{r}}{\partial u^\alpha} = \vec{r}_\alpha \\ \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\, \vec{r}_\gamma + b_{\alpha\beta}\, \vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = - b^\gamma_\beta\, \vec{r}_\gamma \end{cases}$$

where. $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\xi}\left(\dfrac{\partial g_{\alpha\xi}}{\partial u^\beta} + \dfrac{\partial g_{\xi\beta}}{\partial u^\alpha} - \dfrac{\partial g_{\alpha\beta}}{\partial u^\xi}\right)$,

$b^\gamma_\beta = g^{\gamma\xi}\, b_{\xi\beta}$

Regular surfaces: $\dfrac{\partial^2 \vec{r}_\alpha}{\partial u^\beta \partial u^\gamma} = \dfrac{\partial^2 \vec{r}_\alpha}{\partial u^\gamma \partial u^\beta}$, $\dfrac{\partial^2 \vec{n}}{\partial u^\beta \partial u^\gamma} = \dfrac{\partial^2 \vec{n}}{\partial u^\gamma \partial u^\beta}$

$$\begin{cases} \dfrac{\partial}{\partial u^\gamma}\left(\Gamma^\delta_{\alpha\beta}\, \vec{r}_\delta + b_{\alpha\beta}\, \vec{n}\right) = \dfrac{\partial}{\partial u^\beta}\left(\Gamma^\delta_{\alpha\gamma}\, \vec{r}_\delta + b_{\alpha\gamma}\, \vec{n}\right) \\ \dfrac{\partial}{\partial u^\gamma}\left(b^\delta_\beta\, \vec{r}_\delta\right) = \dfrac{\partial}{\partial u^\beta}\left(b^\delta_\gamma\, \vec{r}_\delta\right) \end{cases}$$

$$\Rightarrow \left(\frac{\partial}{\partial u^\gamma}\Gamma^\delta_{\alpha\beta} - \frac{\partial}{\partial u^\beta}\Gamma^\delta_{\alpha\gamma} + \Gamma^\eta_{\alpha\beta}\Gamma^\delta_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\delta_{\eta\beta} - b_{\alpha\beta} b^\delta_\gamma + b_{\alpha\gamma} b^\delta_\beta\right)\vec{r}_\delta$$
$$+ \left(\Gamma^\delta_{\alpha\beta} b_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} b_{\delta\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \cancel{\partial b_{\alpha\gamma}}\frac{\partial b_{\alpha\gamma}}{\partial u^\beta}\right)\vec{n} = \vec{0}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial u^\gamma}\Gamma^\delta_{\alpha\beta} - \dfrac{\partial}{\partial u^\beta}\Gamma^\delta_{\alpha\gamma} + \Gamma^\eta_{\alpha\beta}\Gamma^\delta_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\delta_{\eta\beta} = b_{\alpha\beta} b^\delta_\gamma - b_{\alpha\gamma} b^\delta_\beta \\ \dfrac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \dfrac{\partial b_{\alpha\gamma}}{\partial u^\beta} = \Gamma^\delta_{\alpha\gamma} b_{\delta\beta} - \Gamma^\delta_{\alpha\beta} b_{\delta\gamma} \end{cases}$$

Riemann notation: $R^{\delta}_{\alpha\beta\gamma} = \frac{\partial}{\partial u^\gamma}\Gamma^{\delta}_{\alpha\beta} - \frac{\partial}{\partial u^\beta}\Gamma^{\delta}_{\alpha\gamma} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\delta}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\cdot\Gamma^{\delta}_{\eta\beta}$

$R_{\alpha\delta\beta\gamma} = g_{\delta\eta} R^{\eta}_{\alpha\beta\gamma}$, $R^{\delta}_{\alpha\beta\gamma} = g^{\delta\eta} R_{\alpha\eta\beta\gamma}$.

$\Rightarrow R^{\delta}_{\alpha\beta\gamma} = b_{\alpha\beta} b^{\delta}_{\gamma} - b_{\alpha\gamma} b^{\delta}_{\beta}$

or $R_{\alpha\delta\beta\gamma} = b_{\alpha\beta} b_{\delta\gamma} - b_{\alpha\gamma} b_{\delta\beta}$ — Gauss equation

$\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} = \Gamma^{\delta}_{\alpha\gamma} b_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} b_{\delta\gamma}$ — Codazzi equation.

I equation in Gauss:

$$R_{\alpha\delta\beta\gamma} = g_{\delta\eta}\left(\frac{\partial}{\partial u^\gamma}\Gamma^{\eta}_{\alpha\beta} - \frac{\partial}{\partial u^\beta}\Gamma^{\eta}_{\alpha\gamma} + \Gamma^{\xi}_{\alpha\beta}\Gamma^{\eta}_{\xi\gamma} - \Gamma^{\xi}_{\alpha\gamma}\Gamma^{\eta}_{\xi\beta}\right)$$

$$= \frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^\beta} - \frac{\partial g_{\delta\eta}}{\partial u^\gamma}\Gamma^{\eta}_{\alpha\beta} + \frac{\partial g_{\delta\eta}}{\partial u^\beta}\Gamma^{\eta}_{\alpha\gamma} + \Gamma^{\xi}_{\alpha\beta}\Gamma_{\delta\xi\gamma}$$
$$- \Gamma^{\xi}_{\alpha\gamma}\Gamma_{\delta\xi\beta}$$

$$= \frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^\beta} + \Gamma_{\eta\delta\beta}\Gamma^{\eta}_{\alpha\gamma} - \Gamma_{\eta\delta\gamma}\Gamma^{\eta}_{\alpha\beta}$$

$$R_{\alpha\delta\beta\gamma} = \frac{1}{2}\left(\frac{\partial^2 g_{\delta\beta}}{\partial u^\alpha \partial u^\gamma} + \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\delta \partial u^\beta} - \frac{\partial^2 g_{\delta\gamma}}{\partial u^\alpha \partial u^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial u^\delta \partial u^\gamma}\right)$$
$$+ \Gamma_{\eta\delta\beta}\Gamma^{\eta}_{\alpha\gamma} - \Gamma_{\eta\delta\gamma}\Gamma^{\eta}_{\alpha\beta}$$

$\Rightarrow R_{\alpha\delta\beta\gamma} = R_{\beta\gamma\alpha\delta} = -R_{\delta\alpha\beta\gamma} = -R_{\alpha\delta\gamma\beta}$

$\Rightarrow R_{1212} = b_{11} b_{22} - (b_{12})^2$ ~ Gauss

$\Rightarrow$ Gauss Theorema ~~Egre~~ Egregium

2 equations in Codazzi.

$$\begin{cases} \dfrac{\partial b_{11}}{\partial u^2} - \dfrac{\partial b_{12}}{\partial u^1} = -b_{2\delta}\Gamma_{11}^{\delta} + b_{1\delta}\Gamma_{12}^{\delta} \\ \dfrac{\partial b_{21}}{\partial u^2} - \dfrac{\partial b_{22}}{\partial u^1} = -b_{2\delta}\Gamma_{21}^{\delta} + b_{1\delta}\Gamma_{22}^{\delta} \end{cases}$$