Compatibility, Fundamental theorem, Gauss's Theorema Egregium, geodesic curvature (Liouville equation), Gauss-Bonnet Theorem

13 Compatibility : Gauss-Codazzi equation

- regular surface: $\vec{r} = \vec{r}(u,v)$, or $\vec{r} = \vec{r}(u^1,u^2)$
- natural coordinates: $\{\vec{r}_u, \vec{r}_v, \vec{n}\}$. where $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$
- or $\{\vec{r}_1, \vec{r}_2, \vec{n}\}$. $\vec{r} = \vec{r}(u^1,u^2)$

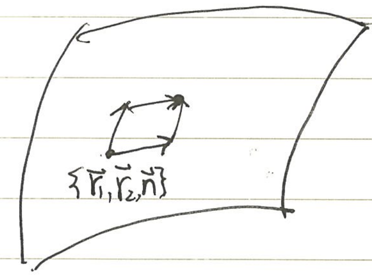
motion of natural coordinates:

$$\begin{cases} \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} \vec{r}_\gamma + b_{\alpha\beta} \vec{n} \\ \dfrac{\partial \vec{n}}{\partial u^\beta} = -b^\gamma_\beta \vec{r}_\gamma \end{cases}$$

$$g_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta, \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$$
$$b_{\alpha\beta} = \vec{r}_{\alpha\beta} \cdot \vec{n} = -\vec{r}_\alpha \cdot \vec{n}_\beta$$
$$b^\gamma_\beta = b_{\beta\xi} g^{\xi\gamma}$$
$$b_{\beta\eta} = b^\gamma_\beta g_{\gamma\eta}$$
$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\xi} \left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\xi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right)$$

Christoffel notation.

- Compatibility



$$\frac{\partial^2 \vec{r}_\alpha}{\partial u^\beta \partial u^\gamma} = \frac{\partial^2 \vec{r}_\alpha}{\partial u^\gamma \partial u^\beta}, \quad \frac{\partial^2 \vec{n}}{\partial u^\beta \partial u^\gamma} = \frac{\partial^2 \vec{n}}{\partial u^\gamma \partial u^\beta} \quad \Rightarrow$$

Gauss equation: $R^\delta_{\alpha\beta\gamma} = b_{\alpha\beta} b^\delta_\gamma - b_{\alpha\gamma} b^\delta_\beta$ or $R_{\alpha\delta\beta\gamma} = b_{\alpha\beta} b_{\delta\gamma} - b_{\alpha\gamma} b_{\delta\beta}$

where $R_{\alpha\delta\beta\gamma} = g_{\delta\eta} R^\eta_{\alpha\beta\gamma}$

$$R^\delta_{\alpha\beta\gamma} = \frac{\partial}{\partial u^\gamma} \Gamma^\delta_{\alpha\beta} - \frac{\partial}{\partial u^\beta} \Gamma^\delta_{\alpha\gamma} + \Gamma^\eta_{\alpha\beta} \Gamma^\delta_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma} \Gamma^\delta_{\eta\beta}$$

Codazzi equation: $\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} = \Gamma^\delta_{\alpha\gamma} b_{\delta\beta} - \Gamma^\delta_{\alpha\beta} b_{\delta\gamma}$

Next ...

Riemann notation: $R_{\alpha\delta\beta\gamma} = \frac{1}{2}\left(\frac{\partial^2 g_{\delta\beta}}{\partial u^\alpha \partial u^\gamma} + \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\delta \partial u^\beta} - \frac{\partial^2 g_{\delta\gamma}}{\partial u^\alpha \partial u^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial u^\delta \partial u^\gamma}\right)$

$+ \Gamma_{\eta\delta\beta}\Gamma^\eta_{\alpha\gamma} - \Gamma_{\eta\delta\gamma}\Gamma^\eta_{\alpha\beta}$

$\Rightarrow R_{\alpha\delta\beta\gamma} = R_{\beta\gamma\alpha\delta} = -R_{\delta\alpha\beta\gamma} = -R_{\alpha\delta\gamma\beta}$

$\Rightarrow$ Gauss equation: $\boxed{R_{1212} = b_{11}b_{22} - (b_{12})^2}$

Codazzi equation:

$$\begin{cases} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} = -b_{2\delta}\Gamma^\delta_{11} + b_{1\delta}\Gamma^\delta_{12} \\ \frac{\partial b_{21}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} = -b_{2\delta}\Gamma^\delta_{21} + b_{1\delta}\Gamma^\delta_{22} \end{cases} \quad (\star)$$

- 正交曲线参数网 ($\vec{r}_u \cdot \vec{r}_v = 0$) : Gauss: $R_{1212} = -\sqrt{EG}\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right]$

  orthogonal parameterization

- 正交曲率线网 Codazzi: $\frac{\partial L}{\partial v} = H\frac{\partial E}{\partial v}, \quad \frac{\partial N}{\partial u} = H\frac{\partial G}{\partial u}$

  curve along the principal direction. where $H = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)$

(13) Fundamental theorem

$D \subset \mathbb{R}^2$ is connected. Let $\varphi = g_{\alpha\beta}\, du^\alpha du^\beta$, $\psi = b_{\alpha\beta}\, du^\alpha du^\beta$ be two differential forms. $g_{\alpha\beta}$ $b_{\alpha\beta}$ differentiable, $g_{\alpha\beta} = g_{\beta\alpha}$. $b_{\alpha\beta} = b_{\beta\alpha}$, $g_{\alpha\beta}$ positive definite.

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\gamma\beta}}{\partial u^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right), \quad \Gamma_{\alpha\beta}^\gamma = g^{\gamma\delta}\Gamma_{\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma}^\delta = \frac{\partial \Gamma_{\alpha\beta}^\delta}{\partial u^\gamma} - \frac{\partial \Gamma_{\alpha\gamma}^\delta}{\partial u^\beta} + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\gamma}^\delta - \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\beta}^\delta, \quad R_{\alpha\delta\beta\gamma} = g_{\delta\eta} R_{\alpha\beta\gamma}^\eta$$

Thm. If φ, ψ satisfy the Gauss-Codazzi equations (☆) then for $\forall\, (u_0^1, u_0^2) \in D$ $\exists$ its neighborhood $U \subset D$ and a regular surface $S: \vec{r} = \vec{r}(u^1, u^2)$, $(u^1, u^2) \in U$ s.t. the first and second fundamental forms of S are $\varphi|_U$ and $\psi|_U$. Furthermore, any two surfaces satisfying the above conditions differ by a rigid motion.

proof: Uniqueness proved. ① $\sigma(\vec{r}_1^{(2)}(u_0), \vec{r}_2^{(2)}(u_0), \vec{n}^{(2)}(u_0))$
$= (\vec{r}_1^{(1)}(u_0), \vec{r}_2^{(1)}(u_0), \vec{n}^{(1)}(u_0))$

$u_0 = (u_0^1, u_0^2)$

② $f_{\alpha\beta}(u) = (\vec{r}_\alpha^{(1)} - \vec{r}_\alpha^{(2)}) \cdot (\vec{r}_\beta^{(1)} - \vec{r}_\beta^{(2)})$

$f_\alpha(u) = (\vec{r}_\alpha^{(1)} - \vec{r}_\alpha^{(2)}) \cdot (\vec{n}^{(1)} - n^{(2)})$

$f(u) = (\vec{n}^{(1)} - \vec{n}^{(2)})^2$

③ $f_{\alpha\beta}(u_0) = f_\alpha(u_0) = f(u_0) = 0.$

$\Rightarrow f_{\alpha\beta}(u) \equiv 0$, $f_\alpha(u) \equiv 0$ $f(u) \equiv 0$

by $\frac{\partial f_{\alpha\beta}}{\partial u^\gamma} = \cdots$ $\frac{\partial f_\alpha}{\partial u^\gamma} = \cdots$ $\frac{\partial f}{\partial u^\gamma} = \cdots$

Existence:

① Construct the 1st order PDE system

$$\frac{\partial \vec{r}}{\partial u^\beta} = \vec{r}_\beta \qquad \frac{\partial \vec{n}}{\partial u^\beta} = -b_\beta^\gamma \vec{r}_\gamma$$

$$\frac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma \vec{r}_\gamma + b_{\alpha\beta}\vec{n}$$

$\vec{r}, \vec{r}_1, \vec{r}_2, \vec{n}$ unknown, u^1, u^2 variables.

② Existence of solution to first-order PDE systems. (陈 附录1)

$$\frac{\partial}{\partial u^\gamma}\left(\frac{\partial \vec{r}}{\partial u^\beta}\right)=\frac{\partial}{\partial u^\beta}\left(\frac{\partial \vec{r}}{\partial u^\gamma}\right), \quad \frac{\partial}{\partial u^\gamma}\left(\frac{\partial \vec{r}_\alpha}{\partial u^\beta}\right)=\frac{\partial}{\partial u^\beta}\left(\frac{\partial \vec{r}_\alpha}{\partial u^\gamma}\right)$$

$$\frac{\partial}{\partial u^\gamma}\left(\frac{\partial \vec{n}}{\partial u^\beta}\right)=\frac{\partial}{\partial u^\beta}\left(\frac{\partial \vec{n}}{\partial u^\gamma}\right)$$

$\Rightarrow$ for $\forall$ intitial values $\vec{r}^0, \vec{r}_1^0, \vec{r}_2^0, \vec{n}^0$ $\exists\ \vec{r}, \vec{r}_1, \vec{r}_2, \vec{n}$ in a neighborhood

③ Choose $\{\vec{r}^0, \vec{r}_1^0, \vec{r}_2^0, \vec{n}^0\}$ s.t.

$$\vec{r}_\alpha^0\cdot\vec{r}_\beta^0 = g_{\alpha\beta}(u_0^1, u_0^2)$$
$$\vec{r}_\alpha^0\cdot\vec{n}^0=0$$
$$\vec{n}^0\cdot\vec{n}^0=1$$
$$(\vec{r}_1^0, \vec{r}_2^0, \vec{n}^0)>0$$

Consider

$$f_{\alpha\beta}=\vec{r}_\alpha\cdot\vec{r}_\beta-g_{\alpha\beta}$$
$$f_\alpha=\vec{r}_\alpha\cdot\vec{n}$$
$$f=\vec{n}\cdot\vec{n}-1$$

use $\begin{cases}\dfrac{\partial f_{\alpha\beta}}{\partial u^\gamma}=\cdots \\ \dfrac{\partial f_\alpha}{\partial u^\gamma}=\cdots \\ \dfrac{\partial f}{\partial u^\gamma}=\cdots\end{cases}$ and $f_{\alpha\beta}(u^0)=f_\alpha(u^0)=f(u^0)=0$

$\Rightarrow f_{\alpha\beta}\equiv 0, \quad f_\alpha\equiv 0, \quad f\equiv 0$

$\Rightarrow (\vec{r}_1, \vec{r}_2, \vec{n})^2=\det(g_{\alpha\beta})>0$

$\because$ continuity $\Rightarrow (\vec{r}_1, \vec{r}_2, \vec{n})>0$

④ $\vec{r}=\vec{r}(u^1,u^2)$ is a regular surface with φ, ψ being the first and second fundamental forms.

⑭ Gauss's Theorema Egregium. (Remarkable Thm)

Gauss equation: $b_{11}b_{22}-(b_{12})^2 = R_{1212}$

$$K = \frac{b_{11}b_{22}-(b_{12})^2}{g_{11}g_{22}-(g_{12})^2} = \frac{R_{1212}}{g_{11}g_{22}-(g_{12})^2}$$

only depends on the first fundamental form

Theorema Egregium: The Gauss curvature K of a surface is invariant by local isometries.

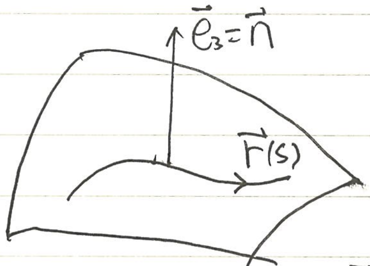
Orthogonal parameterization: $K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right]$

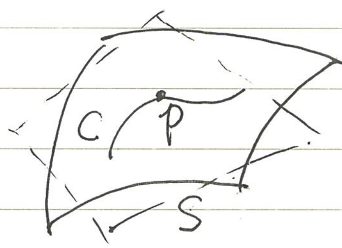
Thm 14.1: 无脐点 (umbilical point) surface is developal $\Leftrightarrow K=0$

Example: Spherical surface and cylindrical surface

⑮ Geodesic curvature 测地曲率 and Liouville equation.

Geodesic curvature: curvature of a curve projected on the tangent plane





Frenet frame $\{\vec{\alpha}(s), \vec{\beta}(s), \vec{\gamma}(s)\}$

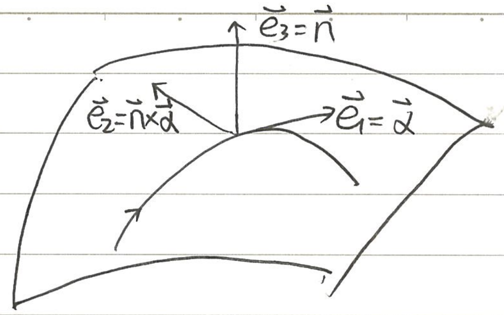
$\vec{\alpha}(s) = \vec{r}\,'(s)$

$\vec{\alpha}\,'(s) = \kappa(s)\vec{\beta}(s)$

$\vec{\gamma}(s) = \vec{\alpha}(s)\times\vec{\beta}(s)$

Frenet equation:

$$\begin{pmatrix}\vec{\alpha}\,'(s)\\ \vec{\beta}\,'(s)\\ \vec{\gamma}\,'(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{\alpha}(s)\\ \vec{\beta}(s)\\ \vec{\gamma}(s)\end{pmatrix}$$



Darboux frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$\vec{e}_1 = \vec{\alpha}(s)$

$\vec{e}_3 = \vec{n}(s)$

$\vec{e}_2 = \vec{e}_3 \times \vec{e}_1 = \vec{n}(s) \times \vec{\alpha}(s)$

Darboux formula:
$$\begin{pmatrix} \vec{e}_1'(s) \\ \vec{e}_2'(s) \\ \vec{e}_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

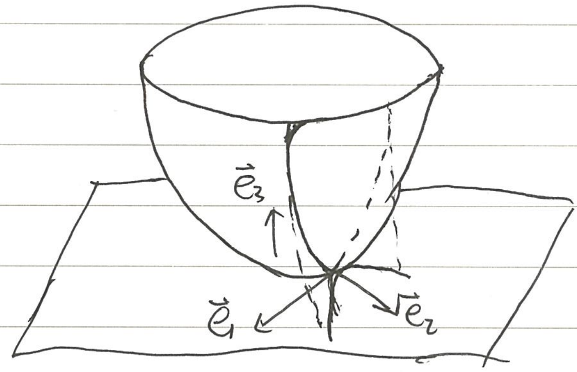
$k_n = \vec{r}''(s) \cdot \vec{n}$ ~ normal curvature.

$k_g = \vec{r}''(s) \cdot (\vec{n}(s) \times \vec{r}'(s)) = (\vec{n}(s), \vec{r}'(s), \vec{r}''(s))$ ~ geodesic curvature

$\tau_g = \vec{e}_2'(s) \cdot \vec{n}(s) = \frac{d}{ds}(\vec{n} \times \vec{r}'(s)) \cdot \vec{n} = (\vec{n}(s), \vec{n}'(s), \vec{r}'(s))$ ~ geodesic torsion

$k_g = \vec{n} \cdot (\vec{\alpha}(s) \times (k\vec{\beta}(s))) = k\vec{n} \cdot \vec{\gamma}(s)$

$\Rightarrow \boxed{k\beta(s) = k_g \vec{e}_2 + k_n \vec{n}}$



Thm: k_g is intrinsic ~ k_g is invariant under isometries.

proof: $S: \vec{r} = \vec{r}(u^1, u^2)$, curve $C: u^1 = u^1(s), u^2 = u^2(s)$

$\vec{r}(s) = \vec{r}(u^1(s), u^2(s))$

$\vec{e}_1(s) = \vec{\alpha}(s) = \vec{r}_\alpha \frac{du^\alpha(s)}{ds}$

$$\frac{d\vec{e}_1(s)}{ds} = \vec{r}_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} + \vec{r}_\alpha \frac{d^2 u^\alpha(s)}{ds^2}$$

$$= \left(\frac{d^2 u^\gamma(s)}{ds^2} + \Gamma^\gamma_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} \right) \vec{r}_\gamma + b_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} \vec{n}$$

$$k_g = \frac{d\vec{e}_1(s)}{ds} \cdot \vec{e}_2(s)$$

$\vdots$

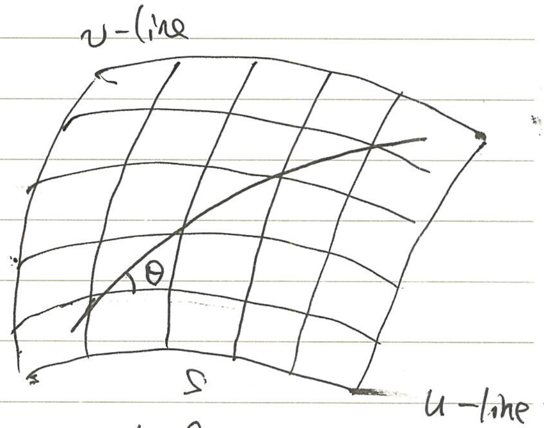
$$k_g = |\vec{r}_1 \times \vec{r}_2| \left[\frac{du^1}{ds}\left(\frac{d^2u^2}{ds^2} + \Gamma^2_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\right) - \frac{du^2}{ds}\left(\frac{d^2u^1}{ds^2} + \Gamma^1_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\right)\right]$$

$$= \sqrt{g_{11}g_{22}-(g_{12})^2} \begin{vmatrix} \frac{du^1}{ds} & \frac{d^2u^1}{ds^2} + \Gamma^1_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} \\ \frac{du^2}{ds} & \frac{d^2u^2}{ds^2} + \Gamma^2_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} \end{vmatrix}$$

$\sim$ intrinsic !

Liouville formula

(u,v) orthogonal coordinates



S: $I = E(du)^2 + G(dv)^2$

C: $u = u(s)$, $v = v(s)$

k_g of C: $$k_g = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

HW3 $$= \frac{d\theta}{ds} + \frac{1}{2\sqrt{EG}}\left[G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right]$$

- τ_g geodesic torsion is NOT intrinsic. $\tau_g = (\vec{n}(s), \vec{n}'(s), \vec{r}'(s))$

16 The Gauss-Bonnet Theorem.

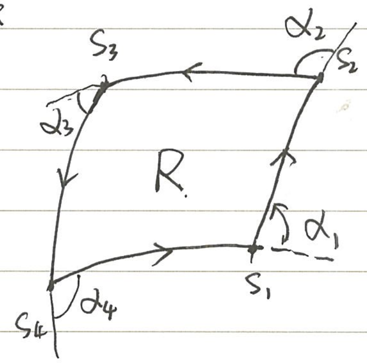
(Local) Gauss-Bonnet Theorem:

Let $S: \vec{r}(u,v)$ be an orthogonal parameterized surface

$R \subset \vec{r}(U)$ ~~be~~ is a simple region of S,

Then

$$\oint_{\partial R} k_g \, ds + \iint_R K \, d\sigma = 2\pi - \sum_{i=1}^{n} \alpha_i$$



α_i is the external angle.

proof:

$$\oint_{\partial R} k_g \, ds = \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} k_g(s) \, ds$$

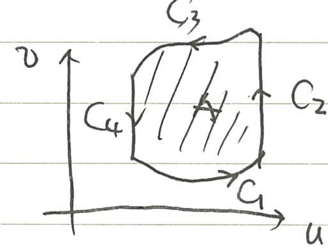
$$k_g(s) = \frac{1}{2\sqrt{EG}} \left[G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] + \frac{d\theta_i}{ds}$$, θ_i is the angle from $\vec{r}_u$ to $\partial R(s)$

$$\sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} k_g(s) \, ds = \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} \left(\frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du}{ds} \right) ds + \sum_{i=0}^{k} \int_{S_i}^{S_{i+1}} \frac{d\theta_i}{ds} \, ds$$

Gauss-Green Theorem: If $P(u,v)$, $Q(u,v)$ are differentiable functions in a simple domain $A \subset \mathbb{R}^2$, the boundary of which is $u = u(s)$, $v = v(s)$

Then
$$\sum_{i=1}^{n} \int_{S_i}^{S_{i+1}} \left(P \frac{du}{ds} + Q \frac{dv}{ds} \right) ds = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du \, dv$$

$$\oint P \, du = \iint_A \left(-\frac{\partial P}{\partial v} \right) dA, \quad \oint Q \, dv = \iint \frac{\partial Q}{\partial u} \, dA$$



or $\oint \vec{F} \cdot d\vec{r} = \iint \nabla \times \vec{F} \cdot \hat{n} \, dA$

$$\Rightarrow \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} k_g(s) \, ds = \iint \left[\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right] du \, dv + \sum_{i=0}^{n} \int_{S_i}^{S_{i+1}} \frac{d\theta}{ds} \, ds$$

$$\iint_{F^{-1}(R)} \left[\left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right] du\,dv = -\iint_{F^{-1}(R)} K\sqrt{EG}\,du\,dv = -\iint_R K\,d\sigma$$

$\uparrow$

for orthogonal parameterization, $K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right]$

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{d\theta}{ds}\,ds = 2\pi - \sum_{i=0}^{n} \alpha_i$$

$$\Rightarrow \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=0}^{n} \alpha_i = 2\pi \qquad \text{Q.E.D.}$$

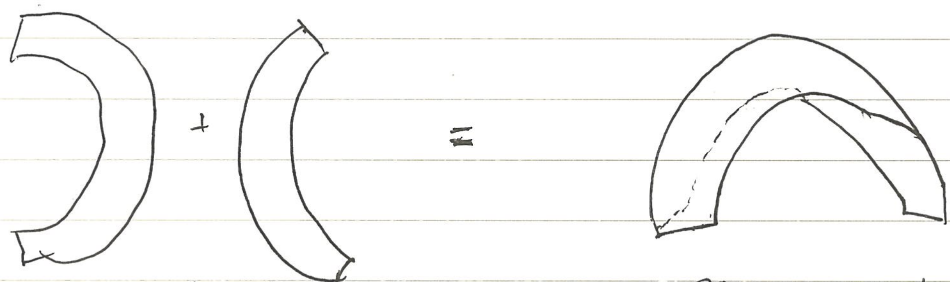
or $\oint_{\partial R} k_g\,ds + \iint_R K\,d\sigma = 2\pi - \sum_{i=1}^{n} \alpha_i$

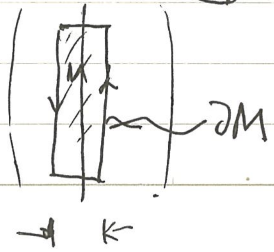
(Global) Gauss-Bonnet Theorem.

Let $R \subset S$ be a regular region of an oriented surface and let $C_1 \cdots C_n$ be the closed, simple, piecewise regular curves which form ∂R. Suppose each C_i is positively oriented and let $\theta_1 \cdots \theta_p$ be the external angles of $C_1 \cdots C_n$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi\chi(R)$$

Example: intrinsically curved folds





$\Omega = \iint_M K\,ds$, $\frac{d\Omega}{ds} = k_{g_1} - k_{g_2}$