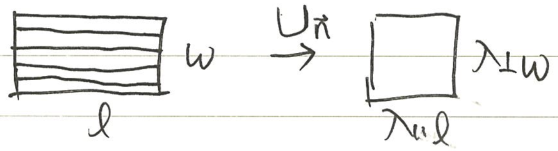
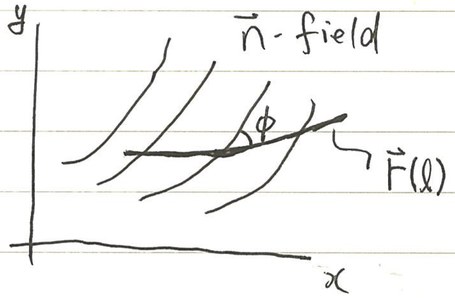
## ⑰ Intrinsic property of activated LCE sheets

LCE actuation



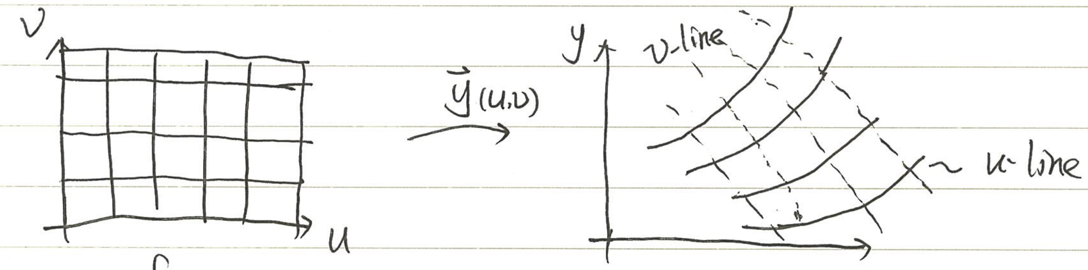

nematic isotropic state

Question:




what is the geodesic curvature of $\vec{F}(l)$ after actuation?

idea:




reference

u-line is along $\vec{n}$ $\quad$ $\vec{U}_n$

$$\frac{\partial \vec{y}}{\partial u} = \vec{y}_u = \alpha \vec{n}, \quad \frac{\partial \vec{y}}{\partial v} = \vec{y}_v = \beta \vec{n}_\perp$$

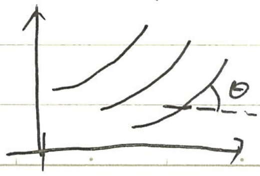
?

$$dl^2 = (du, dv) \begin{pmatrix} \alpha^2 & \\ & \beta^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \sim \vec{n} \text{ field}$$

activated domain

$$dl_A^2 = (du, dv) \begin{pmatrix} \lambda_\parallel^2 \alpha^2 & \\ & \lambda_\perp^2 \beta^2 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \sim \text{activated domain}$$

In the $\vec{n}$- fied




$$\vec{n} = (\cos\theta, \sin\theta)$$

$$b = \nabla \times \vec{n} = \vec{n} \cdot \nabla\theta \; (= |\vec{n} \cdot \nabla \vec{n}|), \quad \text{bend}$$

$$s = \nabla \cdot \vec{n} = \vec{n}_\perp \cdot \nabla\theta, \quad \text{splay}$$

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$$\vec{n}\cdot\nabla f = \frac{1}{\alpha}\frac{\partial f}{\partial u}\Big|_v \qquad (\vec{n}_\perp\cdot\nabla f) = \frac{1}{\beta}\frac{\partial f}{\partial v}\Big|_u$$

in $\vec{n}$ field.

$$\vec{y}_u = \alpha\vec{n} \qquad \vec{y}_v = \beta\vec{n}_\perp$$

$$\vec{y}_{uu} = \alpha_u\vec{n} + \alpha\vec{n}_u = \alpha_u\vec{n} + \alpha\cdot(\alpha b)\vec{n}_\perp = \Gamma_{11}^1\vec{y}_u + \Gamma_{11}^2\vec{y}_v$$
$$\vec{y}_{uv} = \alpha_v\vec{n} + \alpha\vec{n}_v = \alpha_v\vec{n} + \alpha\cdot(\beta s)\vec{n}_\perp = \Gamma_{12}^1\vec{y}_u + \Gamma_{12}^2\vec{y}_v$$

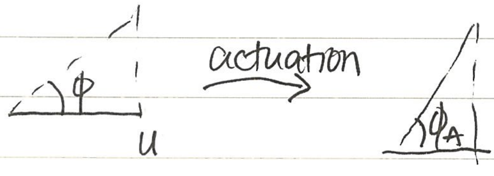
$$\Gamma_{11}^2 = \frac{1}{2}g^{2\xi}\left(\frac{\partial g_{1\xi}}{\partial u} + \frac{\partial g_{\xi 1}}{\partial u} - \frac{\partial g_{11}}{\partial u_\xi}\right) \quad \text{where} \quad g = \begin{pmatrix}\alpha^2 & \\ & \beta^2\end{pmatrix} \quad g^{-1} = \begin{pmatrix}\frac{1}{\alpha^2} & \\ & \frac{1}{\beta^2}\end{pmatrix}$$

$$\Rightarrow \boxed{s = \frac{\beta_u}{\alpha\beta}, \quad b = -\frac{\alpha_v}{\alpha\beta}}$$

Liouville's formula: $k_g = \phi_A' - \frac{1}{2\sqrt{EG}}(u'E_v - v'G_u)$ ✡

$\phi_A$ : angle between $\vec{F}(l)$ and u-line in the activated state

$$✡ \Rightarrow \int k_{gA}\,dl_A = \int d\phi_A + \int\left(\frac{\lambda_\perp}{\lambda_\parallel}\frac{\beta_u}{\alpha\beta}\beta\,dv - \frac{\lambda_\parallel}{\lambda_\perp}\frac{\alpha_v}{\alpha\beta}\alpha\,du\right)$$




$$\tan\phi_A = \frac{\lambda_\perp}{\lambda_\parallel}\tan\phi$$

$$\Rightarrow \int d\phi_A = \int d\left(\tan^{-1}\left[\frac{\lambda_\perp}{\lambda_\parallel}\tan\phi\right]\right) = \int \frac{\lambda_\parallel\lambda_\perp}{\lambda_\parallel^2\cos^2\phi + \lambda_\perp^2\sin^2\phi}\,d\phi$$

$\vec{n}$-field: $d\vec{l} = \alpha\,du\,\vec{n} + \beta\,dv\,\vec{n}_\perp$

$$\Rightarrow \int k_{gA}\,dl_A = \int d\phi_A + \int\left(\frac{\lambda_\perp}{\lambda_\parallel}\frac{\beta_u}{\alpha\beta}\vec{n}_\perp - \frac{\lambda_\parallel}{\lambda_\perp}\frac{\alpha_v}{\alpha\beta}\vec{n}\right)\cdot d\vec{l}$$

Date

$$= \int \frac{\lambda_{\parallel}\lambda_{\perp}}{\lambda_{\parallel}^2\cos^2\phi + \lambda_{\perp}^2\sin^2\phi}\, d\phi + \int \left(\frac{\lambda_{\perp}}{\lambda_{\parallel}} s\vec{n}_{\perp} + \frac{\lambda_{\parallel}}{\lambda_{\perp}} b\vec{n}\right)\cdot d\vec{l}$$

$$dl_A = \sqrt{\lambda_{\parallel}^2\cos^2\phi + \lambda_{\perp}^2\sin^2\phi}\; dl$$

$$\Rightarrow\ kg_A = \frac{\lambda_{\parallel}\lambda_{\perp}}{(\lambda_{\parallel}^2\cos^2\phi + \lambda_{\perp}^2\sin^2\phi)^{3/2}}\frac{d\phi}{dl} + \frac{(\lambda_{\perp}/\lambda_{\parallel})\, s\sin\phi + (\lambda_{\parallel}/\lambda_{\perp})\, b\cos\phi}{\sqrt{\lambda_{\parallel}^2\cos^2\phi + \lambda_{\perp}^2\sin^2\phi}} \quad \square$$

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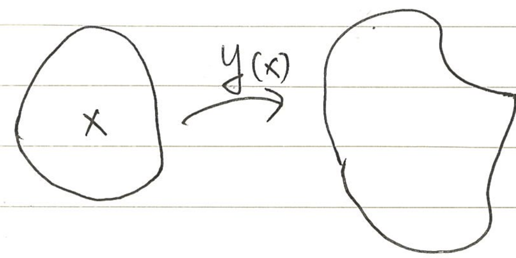
Kirchhoff's nonlinear plate theory

① Liouville's rigidity theorem: if a smooth mapping $v: \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, satisfies $\nabla v \in SO(n)$, then it is affine, $v(x) = Rx + c$.

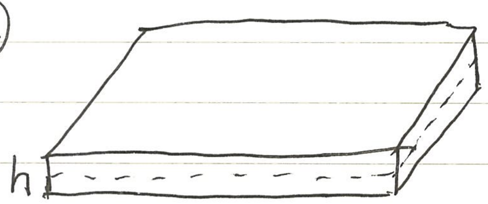
physics ?



if $y(x) \in SO(3)$ for $\forall x$, then $y(x) = Rx + c$

Ref: A Theorem on Gemetric Rigidity and the Derivation of Nonlinear Plate theory from Three-Dimensional Elasticity, by Friesecke, James and Müller.

②



$\Omega_h = (-1,1)^2 \times (-\frac{h}{2}, \frac{h}{2})$

$\frac{1}{h^3} \int_{\Omega_h} W(\nabla v^{(h)}(x)) dx \to ?$

- Isotropic energy $W(F) = W(QFR)$, $Q, R \in SO(3)$

$$\frac{\partial^2 W}{\partial F^2}(I)(A, A) = 2\mu |e|^2 + \lambda (\mathrm{tr}\, e)^2, \quad e = \frac{A + A^T}{2}$$

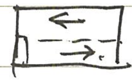
$\sim$ Hooke's law.

$$\star \quad I^0(v) = \begin{cases} \frac{1}{24} \int_S \left(2\mu |\mathrm{II}|^2 + \frac{\lambda\mu}{\mu + \lambda/2} (\mathrm{tr}\,\mathrm{II})^2\right), & \text{isometres } v: S \to \mathbb{R}^3 \\ +\infty & \text{Otherwise} \end{cases}$$

or $\frac{h^3}{24} \int_S 2\mu |\mathrm{II}|^2 + \frac{\lambda\mu}{\mu + \lambda/2} (\mathrm{tr}\,\mathrm{II})^2 \quad \propto h^3$

$\frac{1}{h}\int_{\Omega_h} W(\nabla v(x))\,dx \longrightarrow$ Cosserat model, shear etc.

$\frac{1}{h^3}\int_{\Omega_h} W(\nabla v(x))\,dx \rightarrow$ Kirchhoff's model, no shear...



- $\mathrm{II} = (\nabla v)^T \nabla b$. $b$ is the surface normal.

☆: $v$ isometric, but not limited to $v_1 = v_2 = 0$ (classic Kirchhoff)

- $\Gamma$-convergence: minimizers converge to minimizers.
  If $F_n$ $\Gamma$-converge to $F$, and $x_n$ is a minimizer for $F_n$, then $x_n \to x$ is a minimizer of $F$.

③ 3D energy $\int_\Omega W(\nabla v(z))\,dz$, $v: \Omega \to \mathbb{R}^3$

(1). $W \in C^0(M^{3\times3})$, $W \in C^2$ in a neighborhood of $SO(3)$

$\uparrow$

$SO(3)$ 附近二阶可导

(2) frame indifference:

$W(F) = W(RF)$, $F \in M^{3\times3}$, $R \in SO(3)$

(3) $W(F) \overset{?}{\geq} C\,\mathrm{dist}^2(F, SO(3))$

$W(F) = 0$ if $F \in SO(3)$ $\sim$ $\mathrm{dist}(A, SO(n)) = |\sqrt{A^TA} - I|$

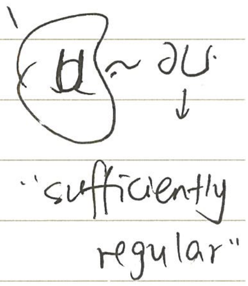
④. $x_1 = z_1$, $x_2 = z_2$, $x_3 = \frac{1}{h} z_3$

$y: \Omega_1 \to \mathbb{R}^3$ $y(x) = v(z(x))$

$$\int_{\Omega_h} W(\nabla v(z))\,dz = h\int_{\Omega_1} W\left(\nabla' y, \frac{1}{h} y_{,3}\right)dx =: E^{(h)}(y)$$

Date

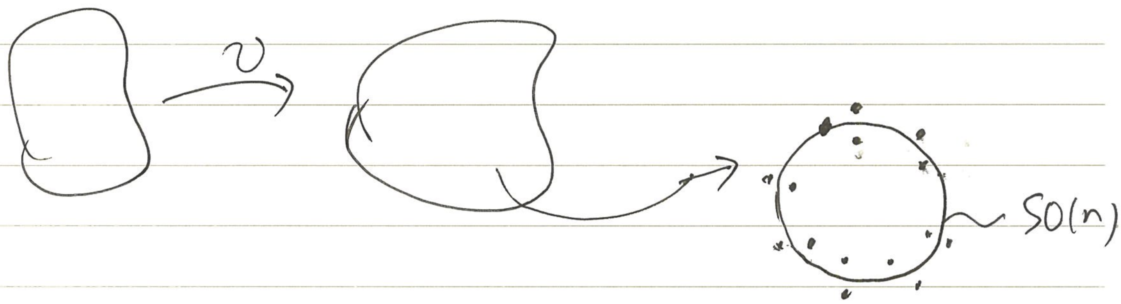
⑤ Rigidity theorem: $U$ - bounded Lipschitz domain in $\mathbb{R}^n$.



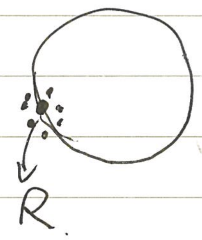

$n \geq 2$. $\exists$ constant $C(U)$ s.t.
For $\forall\, v \in W^{1,2}(U, \mathbb{R}^n)$, there is an
a associated rotation $R \in SO(n)$ s.t.

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \|\mathrm{dist}(\nabla v, SO(n))\|_{L^2(U)}$$

Physics:




rigidity $\Rightarrow$




⑥ Finite bending energy $\Rightarrow$ convergent sequence

$y^{(h)} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ has finite bending energy

$$\limsup_{h \to 0} \frac{1}{h^2} \int_\Omega \mathrm{dist}^2\left(\left(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}\right), SO(3)\right) dx < +\infty$$

Then $\nabla_h y^{(h)} = \left(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}\right)$ is precompact

$\exists$ $\nabla_h y^{(h)} \to \underbrace{(\nabla' y, b)}_{\text{与厚度无关 independent of } x_3} \in L^2(\Omega)$ with $(\nabla' y, b) \in SO(3)$ a.e.

⑦ $h \to 0, \quad \frac{1}{h^3} E^{(h)} \to ?$

(a). lower bound. if $y^{(h)} \subset W^{1,2}(\Omega; \mathbb{R}^3) \to y$ in $W^{1,2}$.

Then $\liminf_{h\to 0} \frac{1}{h^3} E^{(h)}(y^{(h)}) \geq I^0(y)$

(b) Attainment of lower bound. $\exists\, y^{(h)} \to y$ in $W^{1,2}$

s.t. $\lim_{h\to 0} \frac{1}{h^3} E^{(h)}(y^{(h)}) = I^0(y)$

where $I^0(y) := \begin{cases} \frac{1}{24}\int_S Q_2(\mathrm{II})\, dx', & \text{if } y(x) \text{ is independent of } x_3 \text{ and } y \in A \\ +\infty & \text{otherwise.} \end{cases}$

Here $A = \{ y \in W^{2,2} : |y_{,1}| = |y_{,2}| = 1,\ y_{,1} \cdot y_{,2} = 0 \}$ — isometries

$\mathrm{II} = (\nabla' y)^T \nabla' b$ second fundamental form

$Q_2(G) := \min_{c \in \mathbb{R}^3} Q_3(\hat{G} + c \otimes e_3) \longrightarrow$ a shear.

$\uparrow$ quadratic form

$$Q_3(F) = \frac{\partial^2 W}{\partial F^2}(I)(F,F) = \sum \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(I) F_{ij} F_{kl}.$$

$\Rightarrow$ for $W$ isotropic, $Q_3(F) = 2\mu |e|^2 + \lambda (\mathrm{tr}\, e)^2, \quad e = \frac{F + F^T}{2}$

$$Q_2(G) = 2\mu \left| \frac{G + G^T}{2} \right|^2 + \frac{\lambda \mu}{\mu + \lambda/2} (\mathrm{tr}\, G)^2$$

$\Rightarrow I^0(y) = \begin{cases} \frac{1}{24}\int_S \left(2\mu |\mathrm{II}|^2 + \frac{\lambda\mu}{\mu + \lambda/2} (\mathrm{tr}(\mathrm{II}))^2\right) dx' & y \text{ is independen[t] of } x_3,\ y \in A \\ +\infty & \text{otherwise} \end{cases}$

Date

- different from the classic Kirchhoff's plate theory, in which $y_1 = y_2 = 0$.

proof (a) "Lattice of squares" $S_h'$

$$G^{(h)}(x', x_3) = \frac{R^{(h)}(x')^T \nabla_h y^{(h)}(x', x_3) - I}{h}$$

$$\Rightarrow \|G^{(h)}\|_{L^2(S_h' \times (-\frac{1}{2}, \frac{1}{2}))} \le C \quad \sim \text{rigidity.}$$

$$G^{(h)} \to G \text{ in } L^2(\Omega)$$

$$W(I + hA) \geq \frac{1}{2} Q_3(hA) - \omega(|hA|)$$

$\uparrow$ $\sup_{|A| \le t} |\eta(A)|$

where $W(I+A) = \frac{1}{2} Q_3(A) + \eta(A)$.

$$\Rightarrow \liminf_{h \to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)}) dx \geq \frac{1}{2} \int_\Omega Q_3(G) dx.$$

$$\geq \frac{1}{2} \int_{S \times (-\frac{1}{2}, \frac{1}{2})} Q_2(G') \, dx$$

$$G'(x', x_3) = G'(x', 0) + x_3 \, \mathrm{II}(x'),$$

$$\mathrm{II}(x') = (\nabla' y)^T \nabla' b.$$

(b) $$\frac{1}{24} \int_S Q_2(R^T(\nabla' b, d^j)) \le I^0(y) + \frac{1}{j}$$

Date

⑧ Strong convergence of the Rescaled Nonlinear Strain for Low-energy Sequences.

Thm 7.1: Assume $\nabla_h y^{(h)} = \cancel{(\nabla' y^{(h)}, \frac{1}{h} y)}$

$$= (\nabla' y^{(h)}, \tfrac{1}{h} y^{(h)}_{,3}) \to (\nabla' y, b) \text{ in } L^2,$$

and

$$\lim_{h \to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx = I^0(y) < \infty$$

Then $y \in A$ and

$$\frac{[\nabla_h y^{(h)T} \nabla_h y^{(h)}]^{\frac{1}{2}} - I}{h} \to \cancel{x_3 \left(\widehat{\mathrm{II}(x')} + C_{min}(x') \otimes e_3\right.}$$

$$x_3 \left( \widehat{\mathrm{II}}(x') + \frac{C_{min}(x') \otimes e_3 + e_3 \otimes C_{min}(x')}{2} \right)$$

in $L^2(\Omega)$

Where $C_{min}$ is the pointwise minimizer of the problem $\min_c Q_3(\hat{\mathrm{II}} + c \otimes e_3)$