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Brief paper

Consensus protocols for discrete-time multi-agent systems with time-varying delays[☆]

Feng Xiao[a,b,*], Long Wang[a]

[a] Intelligent Control Laboratory, Center for Systems and Control, State Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China

[b] School of Information Science and Technology, Beijing Institute of Technology, Beijing 100081, China

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ABSTRACT

This paper addresses consensus problems for discrete-time multi-agent systems with time-varying delays and switching interaction topologies and provides a class of effective consensus protocols that are built on repeatedly using the same state information at two time-steps. We show that those protocols can solve consensus problems under milder conditions than the popular consensus algorithm proposed by Jadbabaie et al., specifically, the presented protocols allow for the case that agents can only use delayed information of themselves, whereas the popular one is invalid. It is proved that if the union of the interaction topologies across the time interval with some given length always has a spanning tree, then in the presence of bounded time-varying delays, those protocols solve consensus problems.

1. Introduction

In this paper, we consider consensus problems for discrete-time multi-agent systems. The typical discrete-time consensus control strategy (protocol) was provided by Jadbabaie, Lin, and Morse (2003), which is a simplified Vicsek model (Vicsek, Czirok, Jacob, Cohen, & Schochet, 1995). Later, Ren and Beard (2005) extended it to the case with switching *directed* interaction topologies and proved that if the union of the interaction topologies has a spanning tree frequently enough as the system evolves, the information consensus can be achieved asympototically. In Xiao and Wang (2006), the effects of transmission time-delays on systems were studied, where the authors showed that if the time-delays among neighboring agents are bounded and time-varying, the same conclusion can be drawn as in Ren and Beard (2005). However, the correctness of the result is on the basis of the assumption that every agent can use its own instantaneous state information. The same assumption is taken by Fang and Antsaklis (2005) in the study of time-delays induced by the asynchrony of agents, and by Angeli and Bliman (2006) in the study of extending the results of Moreau (2005). It is of importance to emphasize that the convergence of systems under such an assumption was also reported by Tsitsiklis (1984) and Tsitsiklis, Bertsekas, and Athans (1986). If for some agents, only delayed information of themselves can be used, then the state consensus can not be guaranteed generally. The vulnerability of discrete-time systems to time-delays is one of the main differences from continuous-time systems. (In continuous-time systems, a small time-delay of all used information does not affect the consensus property of the protocol presented by Olfati-Saber and Murray (2004).) Tanner and Christodoulakis (2005) proposed a consensus control strategy, where each agent uses delayed states of its neighbors and itself. But the interaction topology was supposed to be time-invariant, connected, and undirected, and the information transmission was supposed to be not with time-delays.

The objective of this paper is to provide a class of consensus protocols, which are built on repeatedly using the same state information at two time-steps. We establish that consensus problems are solvable under the same topology conditions as the typical ones, studied in Jadbabaie et al. (2003), Moreau (2005), Ren and Beard (2005), and Xiao and Wang (2006), without the requirement that agents always get their own instantaneous state information.

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[*] Corresponding address: Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China. Tel.: +86 10 62754388; fax: +86 10 62754388.

E-mail addresses: fengxiao@pku.edu.cn (F. Xiao), longwang@pku.edu.cn (L. Wang).

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This paper is organized as follows. In the next section, preliminary notions are provided. The problem is stated in Section 3 and main results are established in Section 4. Finally, concluding remarks are stated in Section 5.

2. Preliminaries

Some preliminary notions in graph theory and matrix theory are provided in this section.

A directed graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and an edge set $\mathcal{E}(\mathcal{G})$, where an edge is an ordered pair of vertices. Let $\mathcal{V}(\mathcal{G})$ be $\{v_1, v_2, \ldots, v_n\}$. If $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$, v_i is defined as the parent vertex. And edge (v_i, v_i) is called the *self-loop* of vertex v_i. The *neighbor* set of vertex v_i is defined by $\mathcal{N}(\mathcal{G}, v_i) = \{v_j : j \neq i, (v_j, v_i) \in \mathcal{E}(\mathcal{G})\}$. A *subgraph* $\mathcal{G}_s$ of a directed graph $\mathcal{G}$ is a directed graph such that the vertex set $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and the edge set $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$, $\mathcal{G}_s$ is called a *spanning subgraph*. If for any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s) \iff (v_i, v_j) \in \mathcal{E}(\mathcal{G})$, $\mathcal{G}_s$ is called an *induced subgraph*. In this case, $\mathcal{G}_s$ is said to be induced by $\mathcal{V}(\mathcal{G}_s)$. A *(directed) path* in a directed graph $\mathcal{G}$ is a sequence $v_{i_1}, \ldots, v_{i_k}$ of vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{V}(\mathcal{G})$ for $j = 1, 2, \ldots, k-1$. A directed graph $\mathcal{G}$ is *strongly connected* if between every pair of distinct vertices v_i, v_j, there exists a path that begins at v_i and ends at v_j. A *directed tree* is a directed graph, where every vertex, except one special vertex without any parent, which is called the *root vertex*, has exactly one parent, and the root vertex can be connected to any other vertices through paths. A *spanning tree* of $\mathcal{G}$ is a directed tree that is a spanning subgraph of $\mathcal{G}$. A directed graph is said to have a spanning tree if a subset of the edges forms a spanning tree. The *union* of a group of directed graphs $\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \ldots, \mathcal{G}_{i_k}$ with a common vertex set $\mathcal{V}$ is a directed graph with the vertex set $\mathcal{V}$ and with the edge set given by the union of $\mathcal{E}(\mathcal{G}_{i_j})$, $j \in \mathcal{I}_k$, where $\mathcal{I}_k = \{1, 2, \ldots, k\}$. A *weighted* directed graph $\mathcal{G}(A)$ is a directed graph $\mathcal{G}$ plus a nonnegative weight matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff a_{ji} > 0$.

A stochastic matrix A is called indecomposable and aperiodic (SIA) (or *ergodic*) if there exists a column vector ν such that $\lim_{k\to\infty} A^k = \mathbf{1}\nu^{\mathrm{T}}$, where $\mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}}$ with compatible dimensions. We write $A \geqslant B$ if $A - B$ is nonnegative. Throughout this paper, let $\prod_{i=1}^{k} A_i = A_k A_{k-1} \cdots A_1$, denoting the left product of matrices.

3. Problem statement

The multi-agent system studied in this paper consists of n autonomous agents, labeled 1 through n. All those agents share a common state space $\mathbb{R}$. Let x_i denote the state of agent i. Those agents interact with each other via local information transmission.

Due to the existence of time-delays in the information transmission, we introduce two different interaction topologies. The first one is represented by the directed graph $\mathcal{G}(t)$[1] with the vertex set $\{v_i : i \in \mathcal{I}_n\}$, where the parameter t means that the topology is time-dependent (because of the unreliability of information channels). Vertex v_i represents agent i. $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(t))$ if and only if the state information of agent i (may be with time-delay) successfully reaches the controller of agent j at time t. One notices that the edges of the topology $\mathcal{G}(t)$ correspond to only the information channels, through which the information is successfully transmitted at time t. $\mathcal{N}(\mathcal{G}(t), v_i)$ corresponds to the *neighbors* of agent i at time t. The other interaction topology is also a time-dependent directed graph, denoted by $\mathcal{G}^A(t)$, whose definition is related to the actuators of agents and topology $\mathcal{G}(t)$, and will be given in Section 4. In this paper, the self-loops of $\mathcal{G}(t)$ are not considered and we assume that $\mathcal{G}(t)$ has not self-loops. In addition, we assume that the controller of agent i always has this agent's states (may be stored in memories) to use.

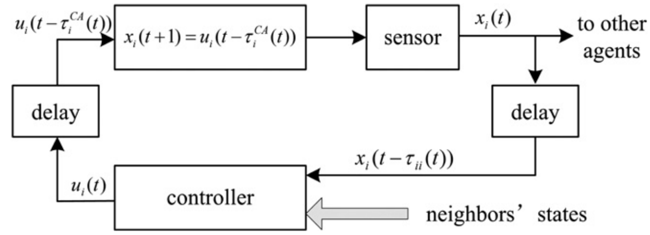

Fig. 1. The dynamics of agent i.

Suppose that agent i takes the following dynamics (see Fig. 1)

$$x_i(t+1) = u_i(t - \tau_i^{CA}(t)), \tag{1}$$

where u_i is a local state feedback, called the *protocol*, to be designed based on the states received from neighbors, and $\tau_i^{CA}(t)$, called the *CA-delay*, is the transmission time-delay from controller to actuator. Notice that CA-delays exist extensively in networked control systems (Yu, Wang, Chu, & Hao, 2004), which are usually larger than one time-step and unknown by controllers.

If for any initial states, there exists an $x^* \in \mathbb{R}$, such that $x_i(t) \to x^*$ as $t \to \infty$ for all $i \in \mathcal{I}_n$, then the system (or the protocol) is said to *solve a consensus problem*. For symbolic simplicity, let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$.

3.1. Motivations

Now, we recall some typical protocols recently studied by researchers.

Xiao and Wang (2006) presented the following model without CA-delays,

$$u_i(t) = \frac{1}{\sum_{v_j \in \{v_i\}\cup\mathcal{N}(\mathcal{G}(t),v_i)} \alpha_{ij}(t)} \times \left(\sum_{v_j \in \{v_i\}\cup\mathcal{N}(\mathcal{G}(t),v_i)} \alpha_{ij}(t) x_j(t - \tau_{ij}(t)) \right), \tag{2}$$

where $\alpha_{ij}(t) > 0$, $i, j \in \mathcal{I}_n$, $t \in \mathbb{N}$, called *weighting factors*, belong to a finite set, and $\tau_{ij}(t) \in \mathbb{N}$ are the transmission time-delays from sensors to controllers or among agents. It was assumed that $\tau_{ii}(t) \equiv 0$ in Xiao and Wang (2006). To distinguish those delays from CA-delays, we call all of them *sensing delays*, no matter what they are. The special case of (2) without sensing delays as well as CA-delays was reported by Ren and Beard (2005). Furthermore, if all weighting factors are 1, the model of Ren and Beard becomes the simplified Vicsek model, studied by Jadbabaie et al. (2003).

In Xiao and Wang (2006), it was shown that protocol (2), in the presence of bounded time-varying sensing delays, solves a consensus problem if all CA-delays are zeros and the union of the interaction topology $\mathcal{G}(t)$ across the time interval with a fixed length always has a spanning tree. However, if $\tau_{ii}(t)$, $i \in \mathcal{I}_n$, are allowed to be nonzeros or CA-delays exist, then similar results can not be gotten.

Example 1 (*Counter Example*). Suppose that $n = 4$ and the interaction topology is time-invariant. And suppose that $\mathcal{V}(\mathcal{G}) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$, $\tau_i^{CA} = 1$, $\tau_{ii} = 0$, and $\tau_{ij} = 1$ for $v_j \in \mathcal{N}(\mathcal{G}(t), v_i)$. If the initial states $x(-2) = x(-1) = x(0) = [1, 2, 3, 4]^{\mathrm{T}}$, $\alpha_{ij} = 1$ for $i, j \in \mathcal{I}_4$, and protocol (2) is applied, then $\lim_{k\to\infty} x(2k) = [2.4, 2.6, 2.4, 2.6]^{\mathrm{T}}$. Obviously, this system can not solve any consensus problem.

[1] Note that different meanings between $\mathcal{G}(A)$ and $\mathcal{G}(t)$, which can be distinguished by their parameters.

On many occasions, the existence of time-delays, including the delays from sensors to controllers and the CA-delays, is a reasonable assumption because of the limitations of transmission media. For example, if the multi-agent system consists of n simple networked control systems, which are further connected by a larger communication network. Therefore, we need to find other effective protocols.

3.2. Consensus protocols

For simplicity, we need the following notations. Let $\mathcal{W}$ be the set of weighting factors, which is a compact set of positive real numbers. And let $\gcd(\cdot)$ denote the greatest common divisor. If $\mathcal{S} = \{\zeta_1, \zeta_2, \ldots, \zeta_k\}$ is given, where $\zeta_i \in \mathbb{R}$, $i \in \mathcal{I}_k$, then

$$\mathrm{Co}(\mathcal{S}) = \left\{ \frac{\sum_{i=1}^{k} \omega_i \zeta_i}{\sum_{i=1}^{k} \omega_i} : \omega_i \in \mathcal{W}, i = 1, 2, \ldots, k \right\},$$

which by definition is a compact set.

Suppose that if $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(t))$, then the state of agent j, received by the controller of agent i at time t, is $x_j(t - \tau_{ij}(t))$,[2] where $\tau_{ij}(t)$ is the sensing delay. Define the sensing delays $\tau_{ii}(t)$, $i \in \mathcal{I}_n$, as *internal* sensing delays, and $\tau_{ij}(t)$, $v_j \in \mathcal{N}(\mathcal{G}(t), v_i)$, as *external* sensing delays, and suppose that all agents are identical and the maximum internal sensing delay and the maximum external sensing delay are τ_{in} and $\tau_{\max}$, respectively. We also suppose that the maximum CA-delay exists, denoted by $\tau_{\max}^{CA}$.

Let $\mathcal{D}_i(t)$ be the available state set, which is used to compute $u_i(t)$ by the controller of agent i at time t, and suppose that $\mathcal{D}_i(t)$ satisfies the following assumption

Assumption A. (1) For any $i \in \mathcal{I}_n$, $t \in \mathbb{N}$, $\mathcal{D}_i(t)$ is a subset of $\{x_j(t-k) : j \in \mathcal{I}_n, k = 0, 1, \ldots, \tau_d^o\}$, where τ_d^o is a positive integer;
(2) there exist $\tau_1, \tau_2 \in \mathbb{N}$ such that $\tau_1, \tau_2 \geq \tau_{\max}^{CA}$, $\gcd(\tau_1 + 1, \tau_2 + 1) = 1$, and for any i, t, $x_i(t - \tau_1), x_i(t - \tau_2) \in \mathcal{D}_i(t - \tau_i^{CA}(t))$.

Clearly, $\mathcal{D}_i(t)$ is a finite set and $\tau_1, \tau_2 \leq \tau_d^o + \tau_{\max}^{CA}$.

Then the class of consensus protocols is given by

$$u_i(t) \in \mathrm{Co}(\mathcal{D}_i(t)). \tag{3}$$

Assumption A(1) implies that the information used for feedback at time t is not older than the states of agents at previous τ_d^o time-steps. It can be seen from Assumption A(2) that $x_i(t) \in \mathcal{D}_i(t + \tau_1 - \tau_i^{CA}(t + \tau_1)) \cap \mathcal{D}_i(t + \tau_2 - \tau_i^{CA}(t + \tau_2))$, which means that $x_i(t)$ will be used at at least two time-steps by agent i's actuator if $\tau_1 \neq \tau_2$. In Assumption A, we do not require the nonexistence of CA-delays and do not require that $x_i(t)$ belong to $\mathcal{D}_i(t)$, which implies that protocol (3) allows for nonzero internal sensing delays. This is the main advantage of protocol (3) over the existing ones. Although τ_1 and τ_2 are common for all agents and their property depends on CA-delays, Assumption A(2) is not strong and can be easily satisfied through distributed control strategy, see the next examples of protocol (3).

Next, we present several examples of protocol (3).

The first example is protocol (2), where $\tau_{\max}^{CA} = \tau_1 = \tau_2 = 0$ and $\tau_d^o = \tau_{\max}$.

Another example of protocol (3) is to let $\mathcal{D}_i(t)$ be the set of the states of agent i's neighbors received at time t and the states of agent i obtained by agent i at time $t, t-1, \ldots, t - \tau_{\mathrm{in}} - \tau_{\max}^{CA} - 1$. If for any t, $x_i(t)$ can always be obtained by agent i sometime (it may be with time-delay), then $x_i(t - \tau_{\mathrm{in}}), x_i(t - \tau_{\mathrm{in}} - 1), \ldots, x_i(t - \tau_{\mathrm{in}} - \tau_{\max}^{CA} - 1)$ belong to $\mathcal{D}_i(t)$, and thus $\{x_i(t - \tau_{\mathrm{in}} - \tau_{\max}^{CA}), x_i(t - \tau_{\mathrm{in}} - \tau_{\max}^{CA} - 1)\} \subset \mathcal{D}_i(t) \cap \mathcal{D}_i(t-1) \cap \cdots \cap \mathcal{D}_i(t - \tau_{\max}^{CA})$. Therefore, $\{x_i(t - \tau_{\mathrm{in}} - \tau_{\max}^{CA}), x_i(t - \tau_{\mathrm{in}} - \tau_{\max}^{CA} - 1)\} \subset \mathcal{D}_i(t - \tau_i^{CA}(t))$ and $\mathcal{D}_i(t)$ satisfies Assumption A with $\tau_1 = \tau_{\mathrm{in}} + \tau_{\max}^{CA}$, $\tau_2 = \tau_1 + 1$ and $\tau_d^o = \max\{2\tau_{\mathrm{in}} + \tau_{\max}^{CA} + 1, \tau_{\max}\}$.

Protocol (4) is also a special case of protocol (3) with $\tau_1 = \tau_{\mathrm{in}} + \tau_{\max}^{CA}$, $\tau_2 = \tau_1 + 1$ and $\tau_d^o = \max\{\tau_{\mathrm{in}} + \tau_{\max}^{CA} + 1, \tau_{\max}\}$.

$$\begin{aligned} u_i(t) = {} & \frac{1}{\sum_{v_j \in \mathcal{N}(\mathcal{G}(t), v_i)} \alpha_{ij}(t) + (\tau_{\max}^{CA} + 2)\alpha_{ii}(t)} \\ & \times \left(\sum_{v_j \in \mathcal{N}(\mathcal{G}(t), v_i)} \alpha_{ij}(t) x_j(t - \tau_{ij}(t)) \right. \\ & \left. + \ \alpha_{ii}(t) \sum_{t^{\Delta}=0}^{\tau_{\max}^{CA}+1} x_i(t - \tau_{\mathrm{in}} - t^{\Delta}) \right), \end{aligned} \tag{4}$$

where $\alpha_{ij}(t) > 0$, $i, j \in \mathcal{I}_n$, are the weighting factors.

It may seem that protocol (4) is of some restrictions in its applications. However, there are at least two cases, in which this protocol is applicable.

Case (1): Internal sensing delays are time-invariant and all equal; and $x_i(t - \tau_{ii}(t))$, obtained at time t by agent i, precisely by the controller of agent i, is stored in memories so that it can be used again at next several time-steps;

Case (2): Internal information is time-stamped, i.e., internal sensing delays are also known; $x_i(t)$ can always be obtained by the controller of agent i sometime, though it is usually with time-delay; and each agent is equipped with memories to store information.

In practice, the assumptions of the above two cases can be satisfied easily, since internal information channels are more reliable than external ones and stamping the internal information by times is not a difficult thing.

4. Main results

In this section, we present the main results.

4.1. Convergence results

We first give the definition of the topology $\mathcal{G}^A(t)$ before presenting the convergence result regarding protocol (3). Its vertex set is also $\{v_i : i \in \mathcal{I}_n\}$, while its edge set is $\{(v_i, v_j) : i \neq j, x_i(t') \in \mathcal{D}_j(t - \tau_j^{CA}(t)) \text{ for some } t'\}$, that is, $(v_i, v_j) \in \mathcal{E}(\mathcal{G}^A(t))$ if and only if the state information of agent i is used by the actuator of agent j to set the value of $x_j(t+1)$. Till now, we have defined two interaction topologies, namely, $\mathcal{G}(t)$ and $\mathcal{G}^A(t)$, with different practical meanings. In the following part, we will refer to them explicitly to avoid misunderstanding.

Theorem 2. *If there exists a positive integer T with the property that for any t, the union of interaction topologies $\mathcal{G}^A(t), \mathcal{G}^A(t+1), \ldots, \mathcal{G}^A(t+T-1)$ has a spanning tree, then protocol (3), satisfying Assumption A, solves a consensus problem.*

To judge the solvability of the consensus problem by the property of $\mathcal{G}(t)$, we need the following assumption.

Assumption B. For any t, if $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(t))$, then $x_j(t - \tau_{ij}(t)) \in \mathcal{D}_i(t')$ for some $t' \geq t$.

[2] We can make the similar discussion when more different states of agent j are received simultaneously by agent i, and we exclude this case in the derivation of the main results.

Assumption B means that if agent i receives its neighbors' states, then those information will be used for feedback at once or at some future times.

Corollary 3. *Suppose that $\mathcal{D}_i(t)$ satisfies Assumptions A and B and all CA-delays are zeros. If there exists a positive integer T with the property that for any t, the union of interaction topologies $\mathcal{G}(t), \mathcal{G}(t+1), \ldots, \mathcal{G}(t+T-1)$ has a spanning tree, then protocol (3) solves a consensus problem.*

Proof. It follows from Assumption B that for any t, $\mathcal{E}(\mathcal{G}(t)) \in \mathcal{E}(\mathcal{G}^A(t)) \cup \mathcal{E}(\mathcal{G}^A(t+1)) \cup \cdots \cup \mathcal{E}(\mathcal{G}^A(t+\tau_d^o))$. Hence, the union of $\mathcal{G}^A(t), \mathcal{G}^A(t+1), \ldots, \mathcal{G}^A(t+T+\tau_d^o-1)$ has a spanning tree and by Theorem 2, this system solves a consensus problem. ■

Remark. Since protocol (3) is robust against CA-delays, a potential application is in the network consisting of n networked control systems. In addition, Theorem 2 and Corollary 3 are also correct if we relax the requirement of the compactness of $\mathcal{W}$ by the weaker condition that there exist positive uniform lower and upper bounds on the weighting factors. The reason is given as follows. Under the weaker condition, $\mathcal{W}$ is a subset of some compact set of positive real number set, such as the closure of $\mathcal{W}$. Since we have proved that if the weighting factors are taken from the closure of $\mathcal{W}$, the above results hold, we have that if the weighting factors are taken from $\mathcal{W}$, which is a subset of the closure of $\mathcal{W}$, then the above results also hold.

In order to prove Theorem 2, we transform the system under protocol (3) into its equivalent augmented system.

Let $\tau_d = \tau_d^o + \tau_{\max}^{CA}$. Note that all possible $\mathcal{D}_i(t-\tau_i^{CA}(t))$ are subsets of $\{x_j(t-k), j \in \mathcal{I}_n, k = 0, 1, \ldots, \tau_d\}$. Therefore, there exist nonnegative row vectors $d^{(i,k)}(t) = [d_1^{(i,k)}(t), d_2^{(i,k)}(t), \ldots, d_n^{(i,k)}(t)]$ such that $u_i(t-\tau_i^{CA}(t)) = \sum_{k=0}^{\tau_d} d^{(i,k)}(t)x(t-k)$ and $\sum_{k,j} d_j^{(i,k)}(t) = 1$ for any $i \in \mathcal{I}_n$, where $d^{(i,k)}(t), k = 0, 1, \ldots, \tau_d$, are *uniquely* determined by the coefficients before the elements of $\mathcal{D}_i(t-\tau_i^{CA}(t))$ in the expression of $u_i(t-\tau_i^{CA}(t))$.

Let $y(t) = [x(t)^{\mathrm{T}}, x(t-1)^{\mathrm{T}}, \ldots, x(t-\tau_d)^{\mathrm{T}}]^{\mathrm{T}}$, $D_k(t) = [d^{(1,k)}(t)^{\mathrm{T}}, d^{(2,k)}(t)^{\mathrm{T}}, \ldots, d^{(n,k)}(t)^{\mathrm{T}}]^{\mathrm{T}}$, $k = 0, 1, \ldots, \tau_d$, and let

$$\Xi(t) = \begin{bmatrix} D_0(t) & D_1(t) & \cdots & D_{\tau_d-1}(t) & D_{\tau_d}(t) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \tag{5}$$

where I is the $n \times n$ identity matrix.

Then we have

$$y(t+1) = \Xi(t)y(t). \tag{6}$$

By Assumption A and the definition of $\mathrm{Co}(\mathcal{D}_i(t))$, it is not difficult to get that $\Xi(t)$ has the following properties.

Lemma 4. (1) *$\Xi(t)$ is stochastic;*

(2) *$D_{\tau_1}, D_{\tau_2} \geq 2\mu I$, where $\mu = \frac{\min_{\omega\in\mathcal{W}}\omega}{2(\tau_d^o+1)n\max_{\omega\in\mathcal{W}}\omega}$, which is larger than zero by the compactness of $\mathcal{W}$; moreover, all nonzero entries of $\Xi(t)$ are not less than 2μ;*

(3) *$\mathcal{E}(\mathcal{G}^A(t)) \subset \mathcal{E}(\mathcal{G}(\sum_{k=0}^{\tau_d} D_k(t)))$.*

(4) *all possible $\Xi(t)$ constitute a compact set[3] $\mathcal{X}$.*

[3] The set of all $r \times s$ matrices can be viewed as the metric space $\mathbb{R}^{rs}$ and compact set is equivalent to bounded closed set.

System (6) is the augmented system of the studied system. It also can be seen as a multi-agent system, consisting of $(\tau_d+1)n$ agents, where the communication topology is $\mathcal{G}(\Xi(t))$. And it is not difficult to obtain that protocol (3) solves a consensus problem if and only if system (6) solves a consensus problem. Due to the special structures of $\Xi(t)$ (it may happen that $\mathrm{diag}(\Xi(t)) = 0$), all the existing results, such as those in Angeli and Bliman (2006), Jadbabaie et al. (2003), Moreau (2005), Ren and Beard (2005), and Xiao and Wang (2006), can not be applied. Nevertheless, we can employ the same arguments as in Moreau (2005) to prove that system (6) as well as system (1) are stable, whose definition is given in Moreau (2005).

4.2. Technical proof

First, we introduce an important concept, namely, scrambling matrix, and extend the main result of Wolfowitz (1963) about SIA matrices.

Let A, B be $r \times r$ nonnegative matrices and let $\delta(A) = \max_j \max_{i_1,i_2} |a_{i_1j} - a_{i_2j}|$. Thus, $\delta(A)$ measures how different the rows of A are. If the rows of A are identical, $\delta(A) = 0$ and conversely. We say that A, B are of the same type, $A \sim B$, if they have zero elements and positive elements in the same place. Let $\boldsymbol{n}(r)$ be the number of different types of all $r \times r$ SIA matrices. Define $\lambda(A) = 1 - \min_{i_1,i_2} \sum_j \min(a_{i_1,j}, a_{i_2,j})$, where A is stochastic. If $\lambda(A) < 1$, A is called a *scrambling matrix*.

The next lemma generalizes the main result of Wolfowitz (1963).

Lemma 5. *Let $\mathcal{A}$ be a compact set consisting of $r \times r$ SIA matrices with the property that for any nonnegative integer k and any $A_1, A_2, \ldots, A_k \in \mathcal{A}$ (repetitions permitted), $\prod_{i=1}^k A_i$ is SIA. Then if $k > \boldsymbol{n}(r)$, $\prod_{i=1}^k A_i$ is scrambling. And given any infinite sequence $A_1, A_2, A_3, \ldots$ (repetitions permitted) of matrices from $\mathcal{A}$, there exists a column vector ν such that $\lim_{l\to\infty} \prod_{i=1}^l A_i = \mathbf{1}\nu^{\mathrm{T}}$.*

Proof. Since $k > \boldsymbol{n}(r)$, there exist $k_1 < k_2$, such that $\prod_{i=k_1}^k A_i \sim \prod_{i=k_2}^k A_i$. It follows from Lemma 3 of Wolfowitz (1963) and $\prod_{i=k_1}^{k_2-1} A_i$ being SIA that $\prod_{i=k_2}^k A_i$ is a scrambling matrix. Thus, $\prod_{i=1}^k A_i$ is also a scrambling matrix by Lemma 1 of Wolfowitz (1963).

Let k be fixed. Then $\lambda_m = \max_{A_i\in\mathcal{A}} \lambda(\prod_{i=1}^k A_i)$ exists and $\lambda_m < 1$ by the continuity of function $\lambda(\cdot)$ and the product of matrices and by the compactness of $\mathcal{A}$.

Consider the infinite sequence of matrices $A_1, A_2, A_3, \ldots$. For any positive integer l, there exists $j \in \mathbb{N}$, such that $jk \leq l < (j+1)k$. By Lemma 2 of Wolfowitz (1963) $\delta(\prod_{i=1}^l A_i) \leq \prod_{p=0}^{j-1} \lambda(\prod_{i=1+pk}^{(p+1)k} A_i) \leq \lambda_m^j$. Therefore, $\lim_{l\to\infty} \delta(\prod_{i=1}^l A_i) = \lim_{j\to\infty} \lambda_m^j = 0$, which leads to that there exists a column vector ν such that $\lim_{l\to\infty} \prod_{i=1}^l A_i = \mathbf{1}\nu^{\mathrm{T}}$. ■

The following lemma provides us a way to judge SIA matrices by graph theory.

Lemma 6 (*Xiao and Wang (2006), Lemma 1*). *Let A be a stochastic matrix. If $\mathcal{G}(A)$ has a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(A)$, then A is SIA.*

To prove Theorem 2, the properties of $\Xi(t)$ need to be further investigated. To characterize the matrix $\Xi(t)$, we introduce two constant matrices Δ and Λ and give their properties. And then we study the products of the state matrices of system (6).

Let $\{e_1, e_2, \ldots, e_{\tau_d+1}\}$ denote the standard basis of $\mathbb{R}^{\tau_d+1}$, in which e_i has a 1 as its ith component and 0 elsewhere, let $\otimes$ denote the Kronecker product, and suppose that $\tau_1 \le \tau_2$. Then we define $(\tau_d+1)n \times (\tau_d+1)n$ matrices

$$\Delta = \left[e_{\tau_1+1}+e_{\tau_2+1}, e_1, e_2, \ldots, e_{\tau_d}\right]^{\mathrm{T}} \otimes I$$

and

$$\Lambda = \left[\sum_{k=1}^{\tau_2+1} e_k, \sum_{k=1}^{\tau_2+1} e_k, \ldots, \sum_{k=1}^{\tau_2+1} e_k\right]^{\mathrm{T}} \otimes I.$$

Lemma 7. (1) *There exists $\kappa^o \in \mathbb{N}$ such that for any $k \ge \kappa^o$, $\Delta^k \ge \Lambda$;*

(2) *let $M=[M_{ij}]$ and $N=[N_{ij}]$ be nonnegative $(\tau_d+1)\times(\tau_d+1)$ block matrices, where $M_{ij}, N_{ij} \in \mathbb{R}^{n\times n}$. Then for any $p, q \in \mathbb{N}$, if $N \ge 0$ and $M = \Delta^p N \Delta^q$, then $\sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} M_{ij} \ge \sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} N_{ij}$;*

(3) *if $\mathcal{G}(\sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} N_{ij})$ has a spanning tree, then $\mathcal{G}(\Lambda+N)$ has a spanning tree and the corresponding root vertex has a self-loop.*

Duo to the space limitation, we only provide the sketch of proof.

Sketch of proof. (i) Let $P=e_1^{\mathrm{T}}\otimes I$. It can be observed that for any $k_1, k_2 \in \mathbb{N}$ and any nonnegative $(\tau_d+1)n\times(\tau_d+1)n$ matrix A,

$$P\Delta^{k_1(\tau_1+1)+k_2(\tau_2+1)}A \ge PA. \tag{7}$$

Since $\gcd(\tau_1+1, \tau_2+1)=1$, there exist integers r_1, r_2 such that $r_1(\tau_1+1)+r_2(\tau_2+1)=1$. Without loss of generality, assume that $r_1 \le 0$. Then $r_2 \ge 0$. For any $k \ge (-r_1)(\tau_1+1)^2$, there exist $k_1, k_2 \in \mathbb{N}$, such that $k = k_1(\tau_1+1)+k_2$, where $0 \le k_2 < \tau_1+1$. Then

$$\begin{aligned} k &= k_1(\tau_1+1)+k_2(r_1(\tau_1+1)+r_2(\tau_2+1)) \\ &= (k_1+k_2r_1)(\tau_1+1)+k_2r_2(\tau_2+1), \end{aligned} \tag{8}$$

where $k_1+k_2r_1 > -r_1(\tau_1+1)+r_1(\tau_1+1)=0$. By (7) and (8), we have for any nonnegative matrix A and any $k \ge (-r_1)(\tau_1+1)^2$,

$$P\Delta^k A \ge PA.$$

If $k \ge (-r_1)(\tau_1+1)^2+\tau_2+1$, then for any $0\le k' \le \tau_2$,

$$P\Delta^k = P\Delta^{k-k'-1}\Delta^{k'+1} \ge P\Delta^{k'+1} \ge e_{\tau_2+1-k'}^{\mathrm{T}} \otimes I.$$

Therefore,

$$P\Delta^k \ge (e_1+e_2+\cdots+e_{\tau_2+1})^{\mathrm{T}} \otimes I.$$

Let $\kappa^o = (-r_1)(\tau_1+1)^2+\tau_2+\tau_d+1$. Then if $k \ge \kappa^o$, $\Delta^k \ge \Lambda$.

(ii) By induction, to prove statement (2) holds, we only need to prove it holds when $p=0, q=1$, and when $p=1, q=0$. Those two cases hold obviously.

(iii) Statement (3) can be proved by some straightforward arguments. Details are omitted. ■

Next lemma characterizes the products of the matrices in $\mathcal{X}$. Before presenting it, we introduce some notations. Let $\kappa \triangleq \max\{T, \kappa^o\}$, where κ^o is given in Lemma 7. Let Π denote the set of products of κ matrices from $\mathcal{X}$ with the property that for any $\pi = \prod_{k=1}^{\kappa} \Xi^{(k)} \in \Pi$, $\mathcal{G}(\sum_{k=1}^{\kappa}\sum_{i=0}^{\tau_d} D_i^{(k)})$ has a spanning tree, where $\Xi^{(k)} \in \mathcal{X}$, $D_i^{(k)} \in \mathbb{R}^{n\times n}$, and the first n rows of $\Xi^{(k)}$ is $[D_0^{(k)}, D_2^{(k)}, \ldots, D_{\tau_d}^{(k)}]$.

Lemma 8. *Π is compact and for any nonnegative integer k and any $\pi_1, \pi_2, \ldots, \pi_k \in \Pi$ (repetitions permitted), $\prod_{i=1}^{k}\pi_i$ is SIA.*

Proof. (i) The compactness of Π follows from that $\mathcal{X}$ is compact and that the nonzero entries of matrices in $\mathcal{X}$ are bounded below by 2μ.

(ii) Let $\pi = \prod_{k=1}^{\kappa}\Xi^{(k)} \in \Pi$, where $\Xi^{(k)} \in \mathcal{X}$. By Lemma 4, $\Xi^{(k)} \ge 2\mu\Delta$ and thus $\Xi^{(k)} = \frac{1}{2}\Xi^{(k)}+\frac{1}{2}\Xi^{(k)} \ge \mu(\Xi^{(k)}+\Delta)$. Then $\pi = \prod_{k=1}^{\kappa}\Xi^{(k)} \ge \prod_{k=1}^{\kappa}\mu(\Xi^{(k)}+\Delta)$. Therefore, by Lemma 7(1),

$$\pi \ge \mu^\kappa\Delta^\kappa \ge \mu^\kappa\Lambda. \tag{9}$$

Furthermore, for any k', $1\le k' \le \kappa$,

$$\pi \ge \mu^\kappa \Delta^{\kappa-k'}\Xi^{(k')}\Delta^{k'-1}. \tag{10}$$

Let $\pi = [\pi_{ij}]$ and the first n rows of $\Xi^{(k)}$ be $[D_0^{(k)}, D_1^{(k)}, \ldots, D_{\tau_d}^{(k)}]$, where $\pi_{ij}, D_0^{(k)}, \ldots, D_{\tau_d}^{(k)} \in \mathbb{R}^{n\times n}$. By Lemma 7(2) and inequality (10), $\sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} \pi_{ij} \ge \mu^\kappa \sum_{i=0}^{\tau_d} D_i^{(k')}$. Therefore,

$$\sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} \pi_{ij} \ge \frac{\mu^\kappa}{\kappa}\sum_{k=1}^{\kappa}\sum_{i=0}^{\tau_d} D_i^{(k)}. \tag{11}$$

By the definition of Π, $\mathcal{G}(\sum_{k=1}^{\kappa}\sum_{i=0}^{\tau_d} D_i^{(k)})$ has a spanning tree. Thus, inequality (11) implies that $\mathcal{G}(\sum_{i\in\mathcal{I}_{\tau_2+1}, j\in\mathcal{I}_{\tau_d+1}} \pi_{ij})$ has a spanning tree. By inequality (9), $\pi = \frac{1}{2}(\pi+\pi) \ge \frac{\mu^\kappa}{2}\Lambda + \frac{1}{2}\pi \ge \frac{\mu^\kappa}{2}(\Lambda+\pi)$. Therefore, by Lemma 7 (3), $\mathcal{G}(\pi)$ has a spanning tree with the property that the root vertex has a self-loop. Obviously, π is stochastic and therefore, by Lemma 6, π is SIA.

With the same arguments, we can show that for any $\pi_1, \pi_2, \ldots, \pi_k \in \Pi$, $\prod_{i=1}^{k}\pi_i$ is SIA. ■

Proof of theorem 2. Let $\pi_1 = \prod_{k=0}^{\kappa-1}\Xi(k)$, $\pi_2 = \prod_{k=\kappa}^{2\kappa-1}\Xi(k)$, $\pi_3 = \prod_{k=2\kappa}^{3\kappa-1}\Xi(k), \ldots$. By Lemma 4(3) and the assumption of this theorem, $\pi_1, \pi_2, \pi_3, \ldots \in \Pi$. By Lemmas 5 and 8, there exists a column vector ν such that $\lim_{l\to\infty}\prod_{i=1}^{l}\pi_i = \mathbf{1}\nu^{\mathrm{T}}$.

For any t, there exists $l \in \mathbb{N}$, such that $l\kappa \le t < (l+1)\kappa$. If $l \ge 1$, then $\prod_{k=0}^{t}\Xi(k) - \mathbf{1}\nu^{\mathrm{T}} = (\prod_{k=l\kappa}^{t}\Xi(k))(\prod_{i=1}^{l}\pi_i - \mathbf{1}\nu^{\mathrm{T}})$. Clearly, all possible $\prod_{k=l\kappa}^{t}\Xi(k)$ are bounded. And hence $\lim_{t\to\infty}\prod_{k=1}^{t}\Xi(k) - \mathbf{1}\nu^{\mathrm{T}} = 0$, which implies that system (6) solves a consensus problem. ■

5. Conclusion

We have provided a class of consensus protocols for discrete-time systems with switching topology and illustrated their robustness against time-delays. Those kinds of protocols are based on repeatedly using the same data at two time-steps. The same idea also sheds light on designing other robust protocols for time-delayed systems. Moreover, the useful result for the product of infinite SIA matrices given by Wolfowitz (1963), which is a key lemma in drawing the main conclusions in Jadbabaie et al. (2003) and Ren and Beard (2005), has been extended, and by that, we prove that the weighting factors are allowed to be any positive real numbers as long as they are bounded below and above by positive real numbers.

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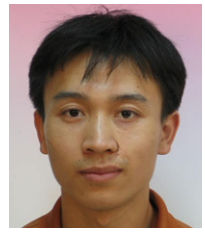

Feng Xiao received the B.S. and M.S. degrees in Mathematics from Inner Mongolia University in 2001 and 2004, respectively. He is currently working toward the Ph.D. degree in complex systems at Peking University. His current research interests are in the fields of coordination of multi-agent systems, consensus problems, formation control and swarm dynamics. Email:fengxiao@pku.edu.cn.

Long Wang was born in Xian, China on Feb. 13, 1964. He received his Bachelor, Master, and Doctor's degrees in Dynamics and Control from Tsinghua University and Peking University in 1986, 1989, and 1992, respectively. He has held research positions at the University of Toronto, the University of Alberta, Canada, and the German Aerospace Center, Munich, Germany. He is currently Cheung Kong Chair Professor of Dynamics and Control, Director of Center for Systems and Control of Peking University, and Director of Center for Intelligent Aerospace Systems, Academy for Advanced Technology, Peking University. He is also Vice-Chairman of Chinese Intelligent Aerospace Systems Committee, and Vice-Director of National Key Laboratory of Complex Systems and Turbulence. He is a panel member of the Division of Information Science, National Natural Science Foundation of China, and a member of IFAC (International Federation of Automatic Control) Technical Committee on Networked Systems. He is in the editorial boards of Progress in Natural Science, Journal of Intelligent Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Control and Decision, Information and Control, Journal of Applied Mathematics and Computation, International Journal of Computer Systems, etc. His research interests are in the fields of networked systems, hybrid systems, swarm dynamics, cognitive science, collective intelligence, and bio-mimetic robotics. Email: longwang@pku.edu.cn.