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Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays

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Abstract

In this paper, we discuss average consensus problem in undirected networks of dynamic agents with fixed and switching topologies as well as multiple time-varying communication delays. By employing a linear matrix inequality method, we prove that all the nodes in the network achieve average consensus asymptotically for appropriate communication delays if the network topology is connected. Particularly, several feasible linear matrix inequalities are established to determine the maximal allowable upper bound of time-varying communication delays. Numerical examples are given to demonstrate the effectiveness and the sharpness of the theoretical results.

Keywords: Multiagent system; Undirected network; Linear matrix inequality; Average consensus; Switching topology

1. Introduction

In the last few years, distributed coordination of networks of dynamic agents has received a major attention within the control community. This is partly due to broad applications of multiagent systems in many areas including cooperative control of unmanned air vehicles, formation control [4], flocking [3,10,14,18–20], attitude alignment of clusters of satellites [8], and congestion control in communication networks. A critical problem for coordinated control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information in the presence of limited and unreliable information exchange as well as communication delays.

Consensus problems have a long history in the field of computer science, particularly in automata theory and distributed computation [11]. In recent years, a number of researchers have investigated in consensus problems from various perspectives [1,4,7,9,12,13,15–17,21]. In [15], consensus problems are addressed under a variety of assumptions on the network topology (fixed or switching), presence or lack of communication delays, and directed or undirected network information flow. The work in [13] focuses on consensus problems under dynamically changing interaction topologies.

It is well-known that, in general, unmodelled delay effects in a feedback mechanism may destabilize an otherwise stable system. This destabilizing effect of time-delays has been well documented in the literature [6]. In multi-agent systems, time-varying delays may arise naturally, e.g., because of the moving of the agents, the congestion of the communication channels, the asymmetry of interactions, and the finite transmission speed due to the physical characteristics of the medium transmitting the information (e.g., acoustic wave communication between underwater vehicles).

So far, just few works considered consensus problems when communication is affected by time-delays. Two different consensus protocols have been investigated in [13] and [15], respectively. The case when the common constant delay affects only those variables that are actually being communicated between distinct agents in the network was studied in [13]. In [15], the authors studied average consensus problems in undirected networks with a common constant communication delay and fixed topology. A necessary and sufficient consensus condition was established.

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The aim of this paper is to consider average consensus problems in undirected networks with fixed and switching topologies as well as multiple time-varying communication delays. Due to the introduction of switching topologies and time-varying delays, the present methods in [13] and [15] do not apply. In this paper, we shall introduce a linear matrix inequality method to deal with this problem. The linear matrix inequality method has been extensively used in delay system. However, to the best of our knowledge, few people extend this approach to consensus problems. Here, with the help of the linear matrix inequality approach we will prove that the group of dynamic agents can reach average consensus asymptotically for appropriate time-varying delays if the network topology is connected. Our results are presented in terms of feasible linear matrix inequalities, from which the maximal allowable upper bound of time-varying communication delays can be easily obtained by using Matlab's LMI Toolbox [5]. Numerical examples are worked out to illustrate the effectiveness and the sharpness of our theoretical results.

This paper is organized as follows. Section 2 contains the problem formulation, Section 3 is the main results. Some simulation results are presented in Section 4. The conclusion is given in Section 5.

Throughout this paper, the notation * represents the elements below the main diagonal of a symmetric matrix. A^{T} means the transpose of the matrix A. I_n is a $n \times n$ identity matrix. We say $X > Y$ if $X-Y$ is positive definite, where X and Y are symmetric matrices of same dimensions. $\|\cdot\|$ refers to the Euclidean norm for vectors.

2. Problem statement

Let $G = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a weighted undirected graph of order n $(n \geqslant 2)$ with the set of nodes $\mathscr{V} = \{v_1, \ldots, v_n\}$, set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a symmetric weighted adjacency matrix $\mathscr{A} = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indexes belong to a finite index set $\mathscr{I} = \{1, 2, \ldots, n\}$. An edge of G is denoted by $e_{ij} = (v_i, v_j)$. The adjacency elements associated with the edges of the graph are positive, i.e., $e_{ij} \in \mathscr{E}$ if and only if $a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathscr{I}$. The set of *neighbors* of node v_i is denoted by $N_i = \{v_j \in \mathscr{V}: (v_i, v_j) \in \mathscr{E}\}$. An undirected graph is called *connected* if any two distinct nodes of the graph can be connected via a path that follows the edges of the graph. Let $x_i \in R$ denote the value of node v_i. We refer to $G_x = (G, x)$ with $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ as a *network* (or *algebraic graph*) with value $x \in R^n$ and *topology* (or *information flow*) G. The value of a node might represent physical quantities such as attitude, position, temperature, voltage, and so on.

Suppose each node of a graph is a dynamic integrator agent with dynamics

$$\dot{x}_i = u_i, \quad i \in \mathscr{I}. \tag{1}$$

We say a state feedback

$$u_i = k_i(x_{j1}, \ldots, x_{jm_i}) \tag{A}$$

is a *protocol* with topology G if the cluster $J_i = \{v_{j_1}, \ldots, v_{j_{m_i}}\}$ of nodes with $j_1, \ldots, j_{m_i} \in \mathscr{I}$ satisfies the property $J_i \subseteq \{v_i\} \cup N_i$.

Under protocol (A), system (1) reduces to

$$\dot{x}_i = k_i(x_{j1}, \ldots, x_{jm_i}), \quad i \in \mathscr{I}. \tag{2}$$

Let $a = x(0)$ be the initial state of system (2). We say system (2) achieves $\mathscr{X}$-consensus asymptotically if for any $a \in R^n$, $x_i(t) \to \mathscr{X}(a)$ as $t \to \infty$ for each $i \in \mathscr{I}$, where $\mathscr{X}: R^n \to R$ be a function of n variables $x_1, \ldots, x_n$. Particularly, when $\mathscr{X}(x) = \mathrm{Ave}(x(0)) = (\sum_{i=1}^n x_i(0))/n$, we say system (2) achieves average consensus asymptotically. Solving the average consensus problem is a typical example of distributed computation of a linear function $\mathscr{X}(a) = \mathrm{Ave}(a)$ using a network of dynamic systems. This is a more challenging task than reaching a consensus with initial state a, since an extra condition $\lim_{t\to\infty} x_i(t) = \mathrm{Ave}(a)$, $i \in \mathscr{I}$ has to be satisfied, which relates the final state of the system to the initial state a.

Let τ_{ij} denote the time delay for information communicated from agent j to agent i. By now, two different consensus protocols have been investigated when communication is affected by time-delays. One is

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})].$$

In the simplest case where $\tau_{ij} = \tau$ and the network topology is fixed and undirected, average consensus is achieved if and only if $\tau \in [0, \pi/2\lambda_{max}(L))$, where L is the graph Laplacian matrix of topology G [15]. Another consensus protocol is

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t)].$$

That is, communication delays only affect the information state that is being transmitted. In the case when $\tau_{ij} = \tau$ and the network topology is directed and dynamically changing, A consensus result has been established in [13]. However, for a switching topology, the case where communication delays are different and time-varying remains unknown.

In this paper we consider average consensus problem under the following protocol:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))], \tag{3}$$

where $\tau_{ij}(t)$, $i, j \in \mathscr{I}$, are time-varying communication delays and satisfy $\tau_{ij}(t) = \tau_{ji}(t)$, i.e., the delays in transmission from x_i to x_j and from x_j to x_i coincide.

Under protocol (3), system (1) has the following form

$$\dot{x}_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))]. \tag{4}$$

Rewrite (4) in matrix form as

$$\dot{x}(t) = -\sum_{k=1}^{r} L_k x(t - \tau_k(t)), \tag{5}$$

where $r \leqslant n(n-1)/2$, $\tau_k(\cdot) \in \{\tau_{ij}(\cdot) : i, j = 1, \ldots, n\}$ for $k = 1, \ldots, r$, and $L_k = [l_{kij}]$ is the matrix defined by

$$l_{kij} = \begin{cases} -a_{ij}, & j \neq i, \quad \tau_k(\cdot) = \tau_{ij}(\cdot), \\ 0, & j \neq i, \quad \tau_k(\cdot) \neq \tau_{ij}(\cdot), \\ \sum_{j=1}^{n} l_{kij}, & j = i. \end{cases}$$

Based on the assumptions $\tau_{ij}(\cdot) = \tau_{ji}(\cdot)$ and $\mathscr{A}$ is symmetric, it is easy to see that L_k is symmetric and $\sum_{k=1}^{r} L_k = L = [l_{ij}]$, where L is the graph Laplacian induced by the information flow G and is defined by

$$l_{ij} = \begin{cases} \sum_{j=1, j\neq i}^{n} a_{ij}, & j = i, \\ -a_{ij}, & j \neq i, \end{cases}$$

By definition, each row sum of the Laplacian matrix L and the matrix L_k is zero. Therefore, system (5) always has a continuum of equilibrium points of the form $x^* = \alpha \mathbf{1}$ in which there exists a unique equilibrium point corresponding to every initial value $x(0) \in R^n$, where $\alpha \in R$ and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$.

In the following, we assume that time-varying delays in (5) satisfy

(C1) $0 \leqslant \tau_k(t) \leqslant h_k$, $\dot{\tau}_k(t) \leqslant d_k$ for $t \geqslant 0$ and $k = 1, \ldots, r$, where $h_k > 0$ and $d_k \geqslant 0$, or

(C2) $0 \leqslant \tau_k(t) \leqslant h_k$ for $t \geqslant 0$ and $k = 1, \ldots, r$, where $h_k > 0$. That is, nothing has been known about the derivative of $\tau_k(t)$.

In a network of mobile agents, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between two agents. The opposite situation can arise where new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. We are interested in investigating such a problem: for a network with switching topology, whether it is still possible to reach a consensus or not. In this case, the following hybrid system is considered:

$$\dot{x}(t) = -\sum_{k=1}^{r} L_{ks} x(t - \tau_k(t)), \quad s = \sigma(t) \in \mathscr{I}_0, \tag{6}$$

where L_{ks} is the symmetric matrix defined as above, and $\sum_{k=1}^{r} L_{ks} = L_s$ is the Laplacian of graph $G_s = (\mathscr{V}, \mathscr{E}_s, \mathscr{A}_s)$ that belongs to set Γ. The set Γ is a finite collection of graphs of order n with an index set $\mathscr{I}_0 \subset Z$. The map $\sigma(t) : R \to \mathscr{I}_0$ is a switching signal that determines the network topology.

The following lemmas play an important role in the proof of the main results.

Lemma 1 (*Olfati-Saber and Murray [15]*). *Assume that G with the Laplacian L is a connected undirected graph, then all eigenvalues but one simple eigenvalue at zero of L have positive real-parts.*

Lemma 2. *Assume that G with the Laplacian L is a connected undirected graph, then we have*

$$E^{\mathrm{T}} L E > 0,$$

where

$$E = \begin{bmatrix} I_{n-1} \\ E_0 \end{bmatrix}, \quad E_0 = (-1, -1, \ldots, -1).$$

Proof. By Lemma 1 we have that $\lambda_2(L) > 0$, where

$$0 = \lambda_1(L) < \lambda_2(L) \leqslant \cdots \leqslant \lambda_n(L)$$

are all eigenvalues of L. On the other hand, since L is symmetric, by the basic theory of Linear Algebra we know

$$x^{\mathrm{T}} L x \geqslant \lambda_2(L) x^{\mathrm{T}} x \quad \text{if } \mathbf{1}^{\mathrm{T}} x = 0.$$

For any $\bar{x} \in R^{n-1}$ and $\bar{x} \neq 0$, by the definition of E, we can easily get $\mathbf{1}^{\mathrm{T}}(E\bar{x}) = 0$. Thus, for any $\bar{x} \in R^{n-1}$ and $\bar{x} \neq 0$, we have

$$\begin{aligned} \bar{x}^{\mathrm{T}} E^{\mathrm{T}} L E \bar{x} &= (E\bar{x})^{\mathrm{T}} L (E\bar{x}) \\ &\geqslant \lambda_2(L)(E\bar{x})^{\mathrm{T}}(E\bar{x}) \\ &> 0. \end{aligned}$$

This implies that $E^{\mathrm{T}} L E > 0$. The proof of Lemma 2 is complete. □

Lemma 3 (*Schur complement, Boyd et al. [2]*). *Let M, P, Q be given matrices such that $Q > 0$. Then*

$$\begin{bmatrix} P & M \\ M^{\mathrm{T}} & -Q \end{bmatrix} < 0 \iff P + MQ^{-1}M^{\mathrm{T}} < 0.$$

Lemma 4. *For any real differentiable vector function $x(t) \in R^n$ and any $n \times n$ constant matrix $W = W^{\mathrm{T}} > 0$, we have the following inequality*

$$\begin{aligned} &h_k^{-1}[x(t) - x(t - \tau_k(t))]^{\mathrm{T}} W [x(t) - x(t - \tau_k(t))] \\ &\leqslant \int_{t-\tau_k(t)}^{t} \dot{x}^{\mathrm{T}}(s) W \dot{x}(s)\, \mathrm{d}s, \quad t \geqslant 0, \end{aligned} \tag{7}$$

where $\tau_k(t)$ satisfies (C1) or (C2).

Proof. Using Schur complement, we get

$$\begin{pmatrix} \dot{x}^{\mathrm{T}}(s) W \dot{x}(s) & \dot{x}^{\mathrm{T}}(s) \\ \dot{x}(s) & W^{-1} \end{pmatrix} \geqslant 0, \quad s \geqslant -h_k.$$

Integrating the above inequality from $t - \tau_k(t)$ to t and noting that $\tau_k(t) \leqslant h_k$ for $t \geqslant 0$, we have

$$\begin{pmatrix} \int_{t-\tau_k(t)}^{t} \dot{x}^{\mathrm{T}}(s) W \dot{x}(s)\, \mathrm{d}s & x^{\mathrm{T}}(t) - x^{\mathrm{T}}(t - \tau_k(t)) \\ x(t) - x(t - \tau_k(t)) & h_k W^{-1} \end{pmatrix} \geqslant 0.$$

Using Schur complement again, we get (7). □

3. Main results

In this section, we will consider the average consensus problem in two cases: networks with fixed topology and networks with switching topology. Throughout this section, we assume that $x_i(t)=0$ for any $t<0$ and $i\in\mathscr{I}$. By introducing a linear matrix inequality approach, we prove system (5) or (6) achieves average consensus asymptotically for appropriate upper bounds of communication delays $\tau_k(t)$ if its network topology is connected. Some feasible linear matrix inequalities are also established to determine the maximal allowable upper bound of delays that guarantees the average consensus of the system.

3.1. Networks with fixed topology

Consider system (5) with fixed topology G. Notice that $\mathbf{1}^{\mathrm{T}}L=0$ and $\mathbf{1}^{\mathrm{T}}L_k=0$ for $k=1,\ldots,r$. Thus, $\alpha=\mathrm{Ave}(x)$ is an invariant quantity. The invariance of $\mathrm{Ave}(x)$ allows the following decomposition of x:

$$x=\alpha\mathbf{1}+\delta, \tag{8}$$

where $\delta=(\delta_1,\ldots,\delta_n)^{\mathrm{T}}\in R^n$ satisfies $\mathbf{1}^{\mathrm{T}}\delta=0$. Here, we refer to δ as the disagreement vector. The vector δ is orthogonal to $\mathbf{1}$. Moreover, δ evolves according to the (group) disagreement dynamics given by

$$\dot{\delta}(t)=-\sum_{k=1}^{r}L_k\delta(t-\tau_k(t)). \tag{9}$$

In the following, we assume $x_i(t)=0$ for any $t<0$.

Theorem 1. *Assume that* (C1) *holds and the topology of G is connected. Then for any $0\leqslant d_k<1$, $k=1,\ldots,r$, there exist appropriate $h_k>0$ such that system* (5) *achieves average consensus asymptotically. Particularly, the allowable h_k can be obtained from the following feasible linear matrix inequality*:

$$\Omega=\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13}\\ * & \Omega_{22} & \Omega_{23}\\ * & * & \Omega_{33}\end{bmatrix}<0, \tag{10}$$

where

$$\Omega_{11}=E^{\mathrm{T}}\Big(-2L+\sum_{k=1}^{r}d_kP_k\Big)E,$$

$$\Omega_{12}=[E^{\mathrm{T}}(L_1+(1-d_1)P_1)E,\ldots,E^{\mathrm{T}}(L_r+(1-d_r)P_r)E],$$

$$\Omega_{13}=-[E^{\mathrm{T}}LQ_1,\ldots,E^{\mathrm{T}}LQ_r],$$

$$\Omega_{22}=\mathrm{diag}\{E^{\mathrm{T}}((d_1-1)P_1-h_1^{-1}Q_1)E,\ldots,E^{\mathrm{T}}((d_r-1)P_r-h_r^{-1}Q_r)E\}$$

$$\Omega_{23}=\begin{bmatrix}E^{\mathrm{T}}L_1Q_1 & \cdots & E^{\mathrm{T}}L_1Q_r\\ \vdots & \ddots & \vdots\\ E^{\mathrm{T}}L_rQ_1 & \cdots & E^{\mathrm{T}}L_rQ_r\end{bmatrix},$$

$$\Omega_{33}=\mathrm{diag}\{-h_1^{-1}Q_1,\ldots,-h_r^{-1}Q_r\},$$

E is defined as in Lemma 2, *$P_k>0$ and $Q_k>0$ are matrices of appropriate dimensions.*

Proof. We first prove that (10) is always feasible for any $1>d_k\geqslant 0$ under the assumption of Theorem 1. That is, there exist appropriate $P_k>0$, $Q_k>0$ and $h_k>0$ such that (10) holds if the topology of G is connected. Since the topology of G with Laplacian L is connected, by Lemma 2, we have $E^{\mathrm{T}}LE>0$. Choosing $P_k=\varepsilon I_n$ and $Q_k=I_n$ for $\varepsilon>0$ and $k=1,\ldots,r$, and using the Schur complement (Lemma 3), we get that (10) is equivalent to

$$\begin{bmatrix}-2E^{\mathrm{T}}LE & E^{\mathrm{T}}L_1E & \cdots & E^{\mathrm{T}}L_rE\\ * & -h_1^{-1}E^{\mathrm{T}}E & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & -h_r^{-1}E^{\mathrm{T}}E\end{bmatrix}$$
$$+\varepsilon\begin{bmatrix}\sum_{k=1}^{r}d_kE^{\mathrm{T}}E & (1-d_1)E^{\mathrm{T}}E & \cdots & (1-d_r)E^{\mathrm{T}}E\\ * & (d_1-1)E^{\mathrm{T}}E & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & (d_r-1)E^{\mathrm{T}}E\end{bmatrix}$$
$$+\sum_{k=1}^{r}h_kF^{\mathrm{T}}F<0, \tag{11}$$

where

$$F^{\mathrm{T}}=[-LE\ L_1E\cdots L_rE]^{\mathrm{T}}.$$

Noting that $E^{\mathrm{T}}LE>0$, for any $d_k\geqslant 0$, by choosing h_0 and ε_0 sufficiently small, we can easily see that (11) holds for $h_k\leqslant h_0$ and $\varepsilon\leqslant\varepsilon_0$. Hence, (10) is always feasible for any $0\leqslant d_k<1$ under the assumption of Theorem 1.

Next, we prove that, for any $0\leqslant d_k<1$, system (5) achieves average consensus asymptotically for $0\leqslant\tau_k(t)\leqslant h_k$, where h_k, $k=1,\ldots,r$, are determined by (10). By the decomposition (8), it suffices to prove that the zero solution of system (9) is asymptotically stable if (10) holds. Now, construct the Lyapunov function as the following:

$$V(t)=\delta^{\mathrm{T}}(t)\delta(t)+\sum_{k=1}^{r}\int_{t-\tau_k(t)}^{t}\delta^{\mathrm{T}}(s)P_k\delta(s)\,\mathrm{d}s$$
$$+\sum_{k=1}^{r}\int_{t-h_k}^{t}(s-t+h_k)\dot{\delta}^{\mathrm{T}}(s)Q_k\dot{\delta}(s)\,\mathrm{d}s, \tag{12}$$

where P_k, Q_k and h_k satisfy (10). Rewrite system (9) as the following equivalent form

$$\dot{\delta}(t)=-L\delta(t)+\sum_{k=1}^{r}L_k\eta_k(t), \tag{13}$$

where $\eta_k(t)=\delta(t)-\delta(t-\tau_k(t))$. Along the trajectory of the solution of system (13), by (C1) we have

$$\dot V(t)\leqslant -2\delta^{\mathrm T}(t)L\delta(t)+2\sum_{k=1}^{r}\delta^{\mathrm T}(t)L_k\eta_k(t)+\sum_{k=1}^{r}\delta^{\mathrm T}(t)P_k\delta(t)$$
$$-\sum_{k=1}^{r}(1-d_k)\delta^{\mathrm T}(t-\tau_k(t))P_k\delta(t-\tau_k(t))$$
$$+\sum_{k=1}^{r}h_k\dot\delta^{\mathrm T}(t)Q_k\dot\delta(t)$$
$$-\sum_{k=1}^{r}\int_{t-\tau_k(t)}^{t}\dot\delta^{\mathrm T}(s)Q_k\dot\delta(s)\,\mathrm ds$$
$$=\delta^{\mathrm T}(t)(-2L+\sum_{k=1}^{r}d_kP_k)\delta(t)+\sum_{k=1}^{r}(d_k-1)\eta_k^{\mathrm T}(t)P_k\eta_k(t)$$
$$+2\sum_{k=1}^{r}\delta^{\mathrm T}(t)[L_k+(1-d_k)P_k]\eta_k(t)$$
$$+\sum_{k=1}^{r}h_k\dot\delta^{\mathrm T}(t)Q_k\dot\delta(t)-\sum_{k=1}^{r}\int_{t-\tau_k(t)}^{t}\dot\delta^{\mathrm T}(s)Q\dot\delta(s)\,\mathrm ds.$$

By (13) and Lemma 4, we have

$$\dot V(t)\leqslant y^{\mathrm T}(t)\hat\Omega y(t), \tag{14}$$

where $y^{\mathrm T}(t)=(\delta^{\mathrm T}(t),\eta_1^{\mathrm T}(t),\ldots,\eta_r^{\mathrm T}(t))$, and

$$\hat\Omega=\begin{bmatrix}\hat\Omega_{11} & \hat\Omega_{12}\\ * & \hat\Omega_{22}\end{bmatrix}+\sum_{k=1}^{r}h_k[-L\ L_1\cdots L_r]^{\mathrm T}Q_k[-L\ L_1\cdots L_r]$$

with

$$\hat\Omega_{11}=-2L+\sum_{k=1}^{r}d_kP_k,$$

$$\hat\Omega_{12}=[L_1+(1-d_1)P_1\cdots L_r+(1-d_r)P_r],$$

$$\hat\Omega_{22}=\mathrm{diag}\{(d_1-1)P_1-h_1^{-1}Q_1,\ldots,(d_r-1)P_r-h_r^{-1}Q_r\}.$$

Noting that $\mathbf 1^{\mathrm T}\delta\equiv 0$, we can rewrite $\delta=E\tilde\delta$, where $\tilde\delta=(\delta_1,\ldots,\delta_{n-1})^{\mathrm T}$, and E is defined as in Lemma 2. Thus, by (14), we have

$$\dot V(t)\leqslant \tilde y^{\mathrm T}(t)W^{\mathrm T}\hat\Omega W\tilde y(t),$$

where $\tilde y(t)=(\tilde\delta(t),\tilde\eta_1(t),\ldots,\tilde\eta_r(t))$, $\tilde\eta_k(t)=\tilde\delta(t)-\tilde\delta(t-\tau_k(t))$ for $k=1,\ldots,r$, and $W=\mathrm{diag}\{E,\ldots,E\}$ is an $n(1+r)\times(n-1)(1+r)$-dimensional matrix. Using Schur complement (Lemma 3), we see that (10) implies that $W^{\mathrm T}\hat\Omega W<0$. Therefore, there exists a positive constant $\beta>0$ such that

$$\dot V(t)\leqslant -\beta\|\tilde\delta(t)\|^2.$$

On the other hand, using the basic inequality $a^2+b^2\geqslant 2ab$ for $a,b\in R$ and $\delta=E\tilde\delta$, we can easily get $\|\tilde\delta(t)\|^2\geqslant\|\delta(t)\|^2/n$. Thus, we have

$$\dot V(t)\leqslant -(\beta/n)\|\delta(t)\|^2.$$

This implies that the zero solution of system (9) is asymptotically stable by Theorem 2.1 in [6, Chapter 5]. Hence, for any $1>d_k\geqslant 0$, system (5) achieves average consensus asymptotically for $0\leqslant\tau_k(t)\leqslant h_k$ where h_k is determined by (10). The proof of Theorem 1 is complete. □

Remark 1. For any $0\leqslant d_k<0$, the maximal allowable h_k guaranteeing average consensus in Theorem 1 can be obtained from the following optimization problem:

Maximize	h_k
s.t.	$0\leqslant d_k<1,\quad P_k>0,\quad Q_k>0,\quad$ and $\quad$(10).

This optimization problem can be solved by using the GEVP solver in Matlab's Control Systems Toolbox [12].

When $d_k\geqslant 1$ or nothing has been known about the derivative of $\tau_k(t)$, we may construct the following Lyapnunov function as

$$V(t)=\delta^{\mathrm T}(t)\delta(t)+\sum_{k=1}^{r}\int_{t-h_k}^{t}(s-t+h_k)\dot\delta^{\mathrm T}(s)Q_k\dot\delta(s)\,\mathrm ds.$$

Similar to the proof of Theorem 1, we can easily obtain the following corollary:

Corollary 2. *Assume that* (C2) *holds and the topology of G is connected. Then there exist appropriate $h_k>0$, $k=1,\ldots,r$, such that system* (5) *achieves average consensus asymptotically. Particularly, the allowable h_k can be obtained from the following feasible linear matrix inequality*:

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13}\\ * & \Phi_{22} & \Phi_{23}\\ * & * & \Phi_{33}\end{bmatrix}<0, \tag{15}$$

where

$$\Phi_{11}=-2E^{\mathrm T}LE,$$

$$\Phi_{12}=[E^{\mathrm T}L_1E\cdots E^{\mathrm T}L_rE],$$

$$\Phi_{13}=\Omega_{13},$$

$$\Phi_{22}=\mathrm{diag}\{-h_1^{-1}E^{\mathrm T}Q_1E,\ldots,-h_r^{-1}E^{\mathrm T}Q_rE\},$$

$$\Phi_{23}=\Omega_{23},$$

$$\Phi_{33}=\Omega_{33},$$

E is defined as in Lemma 2, *Ω_{13}, Ω_{23} and Ω_{33} are the same as in Theorem* 1, *and $Q_k>0$ are matrices of appropriate dimensions.*

3.2. Networks with switching topology

Consider system (6) with the switching topology $\{G_s : s=\sigma(t)\in\mathscr I_0\}$, where $\mathscr I_0\subset Z$ is a finite index set, and $\sigma(t)$ is a switching signal that determines the network topology. Under arbitrary switching signal, $\alpha=\mathrm{Ave}(x)$ is also an invariant quantity. This allows the decomposition of any solution $x(t)$ of system (6) in the form (8). Therefore, the disagreement switching system induced by (6) takes the following form:

$$\dot{\delta}(t) = -\sum_{k=1}^{r} L_{ks}\delta(t-\tau_k(t)), \quad s=\sigma(t)\in\mathscr{I}_0. \tag{16}$$

Theorem 3. *Assume that* (C1) *holds and the topology of* G_s *is connected for each* $s\in\mathscr{I}_0$. *Then for any* $1>d_k\geqslant 0$, $k=1,\ldots,r$, *there exist appropriate* $h_k>0$ *such that for any switching signal* $\sigma(\cdot)$, *system* (6) *achieves average consensus asymptotically. Particularly, the allowable* h_k *can be obtained from the following feasible linear matrix inequalities*:

$$\Omega_s = \begin{bmatrix} \Omega_{11s} & \Omega_{12s} & \Omega_{13s} \\ * & \Omega_{22s} & \Omega_{23s} \\ * & * & \Omega_{33s} \end{bmatrix} < 0, \quad s\in\mathscr{I}_0, \tag{17}$$

where

$$\Omega_{11s} = E^{\mathrm{T}}(-2L_s + \sum_{k=1}^{r} d_k P_k)E,$$

$$\Omega_{12s} = [E^{\mathrm{T}}(L_{1s}+(1-d_1)P_1)E, \ldots, E^{\mathrm{T}}(L_{rs}+(1-d_r)P_r)E],$$

$$\Omega_{13s} = -[E^{\mathrm{T}}L_sQ_1, \ldots, E^{\mathrm{T}}L_sQ_r],$$

$$\Omega_{22s} = \mathrm{diag}\{E^{\mathrm{T}}((d_1-1)P_1 - h_1^{-1}Q_1)E, \ldots, E^{\mathrm{T}} ((d_r-1)P_r - h_r^{-1}Q_r)E\},$$

$$\Omega_{23s} = \begin{bmatrix} E^{\mathrm{T}}L_{1s}Q_1 & \cdots & E^{\mathrm{T}}L_{1s}Q_r \\ \vdots & \ddots & \vdots \\ E^{\mathrm{T}}L_{rs}Q_1 & \cdots & E^{\mathrm{T}}L_{rs}Q_r \end{bmatrix},$$

$$\Omega_{33s} = \mathrm{diag}\{-h_1^{-1}Q_1, \ldots, -h_r^{-1}Q_r\},$$

E is defined as in Lemma 2, $P_k>0$ *and* $Q_k>0$ *are matrices of appropriate dimensions*.

Proof. We first prove that (17) is always feasible for any $1>d_k\geqslant 0$ under the assumption of Theorem 3. Since the topology of G_s with Laplacian L_s is connected, by Lemma 2 we have that $E^{\mathrm{T}}L_sE>0$ for each $s\in\mathscr{I}_0$. Set $P_k=\varepsilon I_n$ ($\varepsilon>0$) and $Q_k=I_n$, similar to the proof of Theorem 1 we have that, for any $1>d_k\geqslant 0$, there exist sufficiently small h_{0s} and ε_{0s} such that $\Omega_s<0$ for $h_k\leqslant h_{0s}$ and $\varepsilon\leqslant\varepsilon_{0s}$. Thus, for any $1>d_k\geqslant 0$, (17) is feasible under the assumption of Theorem 2. One of its feasible solutions is $P_k=\varepsilon I_n$ with $\varepsilon=\min_{s\in\mathscr{I}_0}\{\varepsilon_{0s}\}$, $Q_k=I_n$ and $h_k=\min_{s\in\mathscr{I}_0}\{h_{0s}\}$for $k=1,\ldots,r$.

Now, we prove that, for any $1>d_k\geqslant 0$, system (6) achieves average consensus asymptotically for any switching signal $\sigma(t)$ and time delay $0\leqslant\tau_k(t)\leqslant h_k$, where h_k, $k=1,\ldots,r$, are determined by (17). It suffices to prove that the zero solution of switching system (16) is asymptotically stable for any switching signal $\sigma(t)$ if (17) holds. Let $V(t)$ defined as in (12) be the common Lyapunov function of (16). Assume that the sth subsystem is activated at time t, i.e., $\sigma(t)=s$. Consider the right-hand side derivative of $V(t)$, i.e., $D_+V(t)$, along the trajectories of (16). Similar to the analysis in Theorem 1, (17) implies that there exists $\beta_s>0$ such that $D_+V(t)\leqslant-\beta_s\|\delta(t)\|^2$. Let $\beta=\min\{\beta_s : s\in\mathscr{I}_0\}$. Then we have $D_+V(t)\leqslant-\beta\|\delta(t)\|^2$ for any switching signal $\sigma(t)$. By Theorem 2.1 in [6, Chapter 5], we have that the zero solution of (16) is asymptotically stable for any switching signal. Thus, for any $1>d_k\geqslant 0$, system (6) achieves average consensus asymptotically for $0\leqslant\tau_k(t)\leqslant h_k$ where h_k is determined by (17). The proof of Theorem 3 is complete. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 4. *Assume that* (C2) *holds and the topology of* G_s *is connected for* $s\in\mathscr{I}_0$. *Then there exist appropriate* $h_k>0$, $k=1,\ldots,r$, *such that for any switching signal* $\sigma(\cdot)$, *system* (6) *achieves average consensus asymptotically. Particularly, the allowable* h_k *can be obtained from the following feasible linear matrix inequalities*:

$$\Phi_s = \begin{bmatrix} \Phi_{11s} & \Phi_{12s} & \Phi_{13s} \\ * & \Phi_{22s} & \Phi_{23s} \\ * & * & \Phi_{33s} \end{bmatrix} < 0, \quad s\in\mathscr{I}_0, \tag{18}$$

where

$$\Phi_{11s} = -2E^{\mathrm{T}}L_sE,$$

$$\Phi_{12s} = [E^{\mathrm{T}}L_{1s}E \cdots E^{\mathrm{T}}L_{rs}E],$$

$$\Phi_{13s} = \Omega_{13s},$$

$$\Phi_{22s} = \mathrm{diag}\{-h_1^{-1}E^{\mathrm{T}}Q_1E, \ldots, -h_r^{-1}E^{\mathrm{T}}Q_rE\},$$

$$\Phi_{23s} = \Omega_{23s},$$

$$\Phi_{33s} = \Omega_{33s},$$

E is defined as in Lemma 2, Ω_{13s}, Ω_{23s} *and* Ω_{33s} *are the same as in Theorem* 3, *and* $Q_k>0$ *are matrices of appropriate dimensions to be determined*.

4. Examples and simulation results

The following six undirected graphs with 0–1 weights will be needed in the analysis of this section.

Example 1. Consider an undirected network with fixed topology G_a in Fig. 1. For simplicity, we assume that $\tau_{ij}(t)=\tau_1(t)$. That is $r=1$. It is easy to see that G_a is a connected graph. By Theorem 1, we have that system (5) with fixed topology G_a achieves average consensus asymptotically for appropriate $h_1>0$. On the other hand, using the Matlab's LMI toolbox to solve (10) and (15) we get the following estimates on h_1 for different d_1:

(1) For $d_1=0$, i.e., $\tau_1(t)\equiv h_1$, we have $h_1\leqslant 0.353$. On the other hand, based on the result in [15], we know that this

system achieves average consensus asymptotically if and only if $h_1 < \pi/2\lambda = 0.395$, where $\lambda = 4$ is the largest eigenvalue of the Laplacian L associated with the topology G_a. It is evident that our estimate on h_1 is very close to its critical value. The simulation result also reveals this fact (see Fig. 2).

(2) We have $h_1 \leqslant 0.303$ for $d_1 = 0.5$; $h_1 \leqslant 0.261$ for $d_1 = 0.9$, and $h_1 \leqslant 0.249$ when nothing has been known about the derivative of the time-varying delay $\tau_1(t)$.

Remark 2. The state trajectories of the system associated with topologies G_b and G_c are shown in Fig. 3. By comparing, it is clear that the allowable delay increases as the number of the different communication links increases. This may be reasonable since (10) in this case is equivalent to

$$\begin{bmatrix} -2E^{\mathrm{T}}LE & E^{\mathrm{T}}LE \\ * & -h_1^{-1}E^{\mathrm{T}}Q_1E \end{bmatrix} + \begin{bmatrix} d_1E^{\mathrm{T}}P_1E & (1-d_1)E^{\mathrm{T}}P_1E \\ * & (d_1-1)E^{\mathrm{T}}P_1E \end{bmatrix} + h_1 \begin{bmatrix} -E^{\mathrm{T}}L \\ E^{\mathrm{T}}L \end{bmatrix} Q_1[-LE \;\; LE] < 0 \qquad (19)$$

and the algebraic connectivity (or $\lambda_2(L)$) increases as the number of the different communication links increases [15]. We believe that the delay bound increases as the algebraic connectivity increases. We hope to kindle reader's interest in further research on this problem.

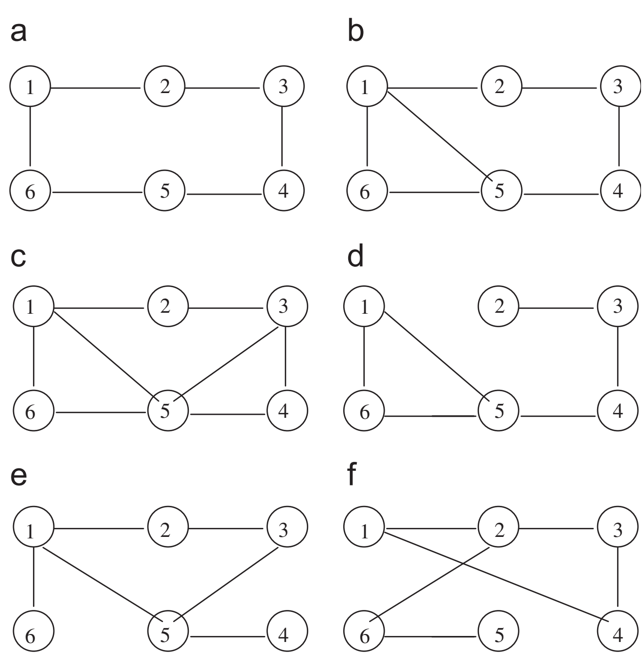

Fig. 1. Six examples of undirected connected graphs.

Remark 3. In Example 1 we find an interesting phenomenon that the allowable delay decreases as the derivative of the delay increases. This becomes apparent since the second part of (19) decreases as d_1 increases, i.e.,

$$\begin{bmatrix} d_1E^{\mathrm{T}}P_1E & (1-d_1)E^{\mathrm{T}}P_1E \\ * & (d_1-1)E^{\mathrm{T}}P_1E \end{bmatrix} < \begin{bmatrix} d_2E^{\mathrm{T}}P_1E & (1-d_2)E^{\mathrm{T}}P_1E \\ * & (d_2-1)E^{\mathrm{T}}P_1E \end{bmatrix}$$

when $d_1 < d_2$.

Example 2. Consider an undirected network with the switching topology $\{G_d, G_e, G_f\}$. In this case, some of the existing communication links fail and some of them are created due to the moving of the agents. Here, we assume also that $\tau_{ij}(t) = \tau_1(t)$. We can easily see that topologies G_e, G_d and G_f are all connected. Thus, by Theorem 3, we have that there exists an appropriate $h_1 > 0$ such that system (6) associated with the switching topology G_e, G_d and G_f achieves average consensus asymptotically for arbitrary switching signal $\sigma(t)$. By solving (17) and (18) we get the following estimates on h_1 for different d_1: $h_1 \leqslant 0.254$ for $d_1 = 0$; $h_1 \leqslant 0.226$ for $d_1 = 0.5$; $h_1 \leqslant 0.208$ for $d_1 = 0.9$, $h_1 \leqslant 0.205$ when nothing has been known about the derivative of the time-varying delay $\tau_1(t)$. In Fig. 4, we present two simulation results under random switching signal for the case of constant delay which indicate that for

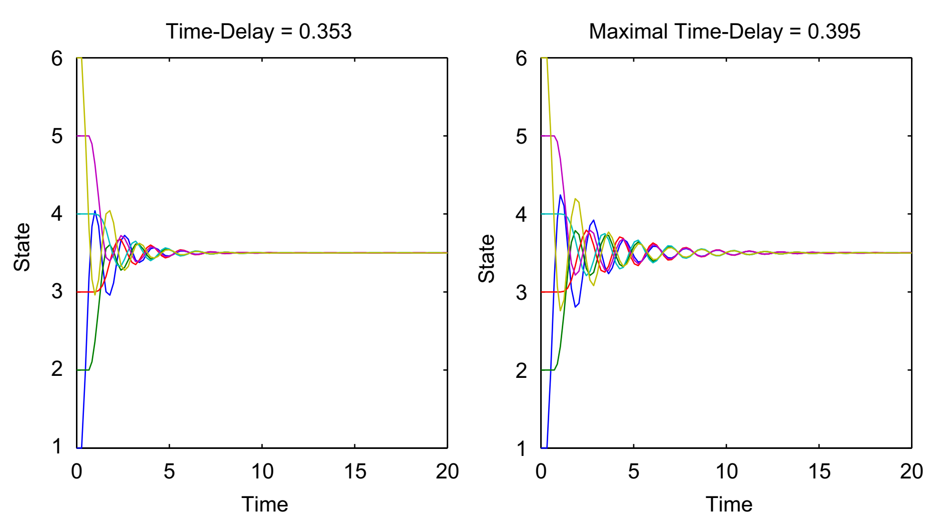

Fig. 2. Consensus problem with communication time-delays on graph G_a given in Fig. 1.

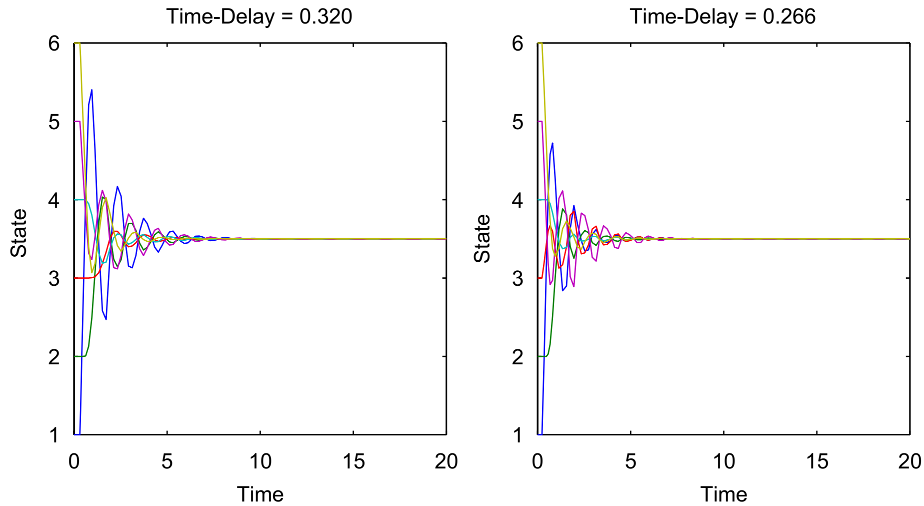

Fig. 3. Consensus problem with communication time-delays on graphs G_b and G_c given in Fig. 1.

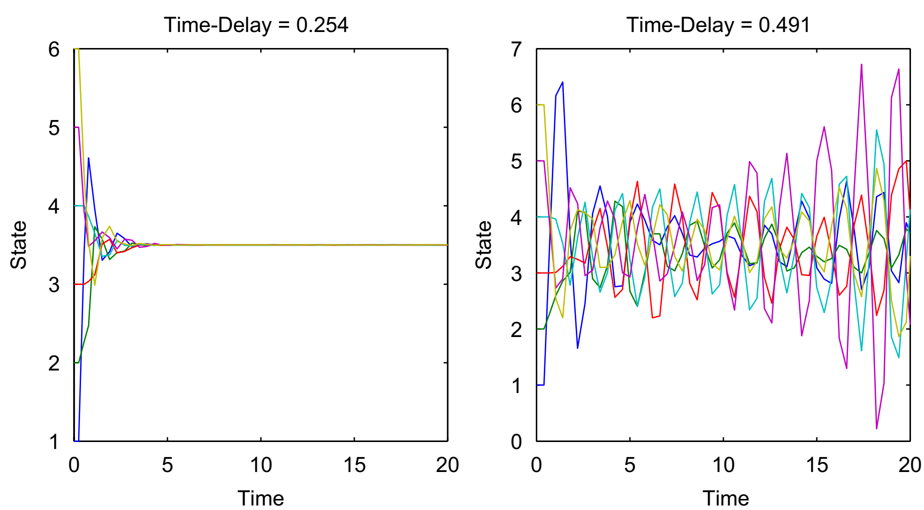

Fig. 4. Consensus problem with communication time-delays on switching topologies $\{G_d, G_e, G_f\}$ given in Fig. 1.

the obtained $h_1 = 0.254$, the asymptotic average consensus is ensured and the asymptotic average consensus disappears for the value of h_1 beyond 0.491.

5. Conclusion

This paper has considered the average consensus problems in undirected networks of dynamic agents with fixed and switching topologies as well as multiple time-varying communication delays. By introducing a linear matrix inequality method, we proved that all the nodes in the network can reach average consensus asymptotically for an appropriate upper bound of communication delays if the network topology is connected. Some feasible linear matrix inequalities have also been established to get the maximal allowable upper bound of the time-varying communication delays. Some simulation results have been presented to demonstrate the effectiveness and the sharpness of our theoretical results.

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