

# State consensus for multi-agent systems with switching topologies and time-varying delays

F. XIAO\* and L. WANG

Center for Systems and Control, Department of Mechanics and Space Technologies,  
College of Engineering, Peking University, Beijing 100871, China

(Received 19 February 2006; in final form 22 May 2006)

In this paper, we investigate state consensus problems for discrete-time multi-agent systems with changing communications topologies and bounded time-varying communication delays. The analysis in this paper is based on the properties of non-negative matrices. We first extend the model of networks of dynamic agents to the case with multiple time-delays and prove that if the communication topology, time-delays, and weighting factors are time-invariant, then the necessary and sufficient condition that the multi-agent system solves a consensus problem is that the communication topology, represented by a directed graph, has spanning trees. Then we allow for dynamically changing communication topologies and bounded time-varying communication delays, and present some sufficient conditions for state consensus of system. Finally, as a special case of our model, the problem of asynchronous information exchange is also discussed.

## 1. Introduction

Recently, as a new area of research, consensus problems in networks of dynamic agents have received considerable attention. They are the basic, yet fundamental, problems in distributed coordination of networks of dynamic agents and require that all agents reach consensus on certain quantities of interest. The common value may be attitude in multiple space-craft alignment, heading direction in flocking behaviour, rendezvous of multiple vehicles, or average in distributed computation.

Consensus problems of all agents' states have been studied by many researchers. Vicsek *et al.* (1995) propose a discrete-time model of  $n$  agents all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its neighbours. Jadbabaie *et al.* (2003) provide a theoretical explanation of the consensus behaviour of the Vicsek model, where each agent's set of neighbours changes with time as the system evolves.

The concept of solvability of consensus problems is formally introduced by Olfati-Saber and Murray (2004). Under the assumption that the dynamic of each agent is a simple scalar continuous-time integrator  $\dot{x} = u$ , three consensus problems are discussed in Olfati-Saber and Murray (2004). They are directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. Ren and Beard (2005) extend the results of Jadbabaie *et al.* (2003) and Olfati-Saber and Murray (2004) and present more relaxable conditions for consensus of information under dynamically changing interaction topologies. Moreau (2005) studies the non-linear discrete-time multi-agent systems with time-dependent communication channels, and introduces a novel method based on the notion of convexity. Xiao *et al.* (2005a) characterize the consensus property by the concepts of "leader" and "leaders-followers-decomposition" of multi-agent systems. Recent development also adds some new contents to this subject. For example, asynchronous information consensus (Fang and Antsaklis 2005), dynamic consensus (Spanos *et al.* 2005), consensus problems over random networks (Hatano and Mesbahi 2005), consensus filters (Olfati-Saber and Shamma 2005), and others closely

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\*Corresponding author. Email: fengxiao@pku.edu.cn

related (Chu *et al.* 2005, Mu *et al.* 2005, Shi *et al.* 2004, 2006, Wang *et al.* 2004, Xiao and Wang 2006, Xiao *et al.* 2005b, Xie and Wang 2005). For more details, see the survey of the latest development (Ren *et al.* 2005) and references therein.

The objective of this paper is to investigate consensus problems in the presence of changing communication topologies and time-varying information transmission delays and present conditions that ensure all agents' states to reach consensus asymptotically. There are some works closely related to our results. For example, Olfati-Saber and Murray (2004) study directed networks under the topologies that switch among a finite collection of strongly connected and balanced digraphs, and undirected networks with fixed topology that all communication channels (including self-loops) have a common time-delay. Jadbabaie *et al.* (2003) give the conclusion that consensus can be achieved asymptotically if the union of communication graphs is connected frequently enough as the system evolves. Ren and Beard (2005) extend the results of Jadbabaie *et al.* (2003) to the case of directed information flow, but they also suppose that there exist no communication time-delays. By using a continuous-time model and employing the method of Lyapunov functions, Moreau (2004) gives a sufficient condition that ensures a system that there is a common time-delay in communication between distinct agents to reach consensus asymptotically. Tanner and Christodoulakis (2005) study a discrete-time model with fixed undirected topology that all agents transmit their state information in turn. Consequently, outdated information may be used and the equivalent augmented system becomes a periodical switched system, which can be seen as a multi-agent system with switching topology. In this paper, we study a discrete-time model with switching topologies and allow communication time-delays to be not only different but also time-varying. The analytical tools rely on matrix theory, graph theory, and control theory.

This paper is organized as follows. In §2, we present some basic definitions and results in matrix theory and graph theory. In §3, we formulate the problem. In §4, we establish our main results. Finally, in §5, concluding remarks are stated.

## 2. Preliminaries

In this section, we present some definitions and results in matrix theory and graph theory that will be used in this paper (Horn and Johnson 1985, Godsil and Royal 2001).

Let  $\mathcal{I}_n = \{1, 2, \dots, n\}$ ,  $\mathbb{Z}_+$  be the set of non-negative integers,  $A = [a_{ij}] \in \mathbb{C}^{n \times r}$ , and  $\mathbf{1} = [1, 1, \dots, 1]^T$  with compatible dimensions. We say that  $A \geq 0$  ( $A$  is

non-negative) if all its entries  $a_{ij}$  are non-negative. We say that  $A > 0$  ( $A$  is positive) if all its entries  $a_{ij}$  are positive. Let  $B \in \mathbb{C}^{n \times r}$ . We write  $A \geq B$  if  $A - B \geq 0$ , and  $A > B$  if  $A - B > 0$ . Given a matrix  $A \in \mathbb{C}^{n \times n}$ , the spectral radius of  $A$  is denoted by  $\rho(A)$ . A non-negative matrix  $A \in \mathbb{C}^{n \times n}$  with the property that all its row sums are +1 is said to be a stochastic matrix. Throughout this paper, we let  $\prod_{i=1}^k A_i = A_k A_{k-1} \dots A_1$  denote the left product of matrices. A stochastic matrix  $A$  is called indecomposable and aperiodic (SIA) (or *ergodic*) if there exists  $f \in \mathbb{R}^n$  such that  $\lim_{k \rightarrow \infty} A^k = \mathbf{1}f^T$ .

Directed graphs will be used to model communication topologies among agents. A directed graph  $\mathcal{G}$  consists of a vertex set  $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$  and an edge set  $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$ , where an edge is an ordered pair of vertices in  $\mathcal{V}(\mathcal{G})$  (we allow for self-loops, i.e., the edges with the same vertices). If  $(v_i, v_j)$  is an edge of  $\mathcal{G}$ ,  $v_i$  is defined as the parent vertex and  $v_j$  is defined as the child vertex. A subgraph  $\mathcal{G}_s$  of a directed graph  $\mathcal{G}$  is a directed graph such that the vertex set  $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$  and the edge set  $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$ . If  $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$ , we call  $\mathcal{G}_s$  a spanning subgraph of  $\mathcal{G}$ . For any  $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$ , if  $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s)$  if and only if  $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ , we call  $\mathcal{G}_s$  an induced subgraph. In this case, we also say that  $\mathcal{G}_s$  is induced by  $\mathcal{V}(\mathcal{G}_s)$ . The set of neighbours of vertex  $v_i$  in  $\mathcal{G}$  is denoted by  $\mathcal{N}(\mathcal{G}, v_i) = \{v_j : (v_j, v_i) \in \mathcal{E}(\mathcal{G}), j \neq i\}$ . The associated index set of neighbours is denoted by  $\mathcal{N}(\mathcal{G}, i) = \{j : v_j \in \mathcal{N}(\mathcal{G}, v_i)\}$ . A (directed) path in a directed graph  $\mathcal{G}$  is a sequence  $v_{i_1}, \dots, v_{i_k}$  of vertices such that  $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}(\mathcal{G})$  for  $s = 1, \dots, k-1$ . A directed graph  $\mathcal{G}$  is strongly connected if between every pair of distinct vertices  $v_i, v_j$  in  $\mathcal{G}$ , there is a directed path that begins at  $v_i$  and ends at  $v_j$  (that is, from  $v_i$  to  $v_j$ ). A directed tree is a directed graph, where every vertex, except one special vertex without any parent, which is called the root vertex, has exactly one parent, and the root vertex can be connected to any other vertices through paths. A spanning tree of  $\mathcal{G}$  is a directed tree that is a spanning subgraph of  $\mathcal{G}$ . We say that a graph has (or contains) a spanning tree if a subset of the edges forms a spanning tree. A weighted directed graph  $\mathcal{G}(A)$  is a directed graph  $\mathcal{G}$  plus a nonnegative weight matrix  $A = [a_{ij}] \in \mathbb{C}^{n \times n}$  such that  $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \Leftrightarrow a_{ji} > 0$ . The union of a group of directed graphs  $\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \dots, \mathcal{G}_{i_k}$  with a common vertex set  $\mathcal{V}$  is a directed graph with vertex set  $\mathcal{V}$  and with the edge set given by the union of the edge sets of  $\mathcal{G}_{i_j}, j = 1, \dots, k$ .

The following lemma provides us a basic tool to deal with consensus problems of discrete-time multi-agent systems. And we denote the set of matrices that satisfy the conditions of the following lemma by  $\mathcal{S}$ .

**Lemma 1:** *Let  $A$  be a stochastic matrix. If  $\mathcal{G}(A)$  has a spanning tree with the property that the root vertex of the spanning tree has self-loop in  $\mathcal{G}(A)$ , then  $A$  is SIA.*

**Proof:** See the appendix.  $\square$

### 3. Problem formulation

In this section, we formulate the problem to be studied and present some basic definitions.

Suppose that the system to be studied consists of  $n$  autonomous agents, e.g., birds, robots, etc., labelled 1 through  $n$ . All these agents share a common state space  $\mathbb{R}$ . At each time, each agent updates its current state based upon the information received from other agents. We use directed graphs to model communication topologies, and these directed graphs have a common vertex set  $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ . Agent  $i$  is represented by vertex  $v_i$ . Edge  $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(t))$  corresponds to an available information channel from agent  $j$  to agent  $i$  at time  $t$ , where  $\mathcal{G}(t)$  is the communication topology at time  $t$ . And the neighbours of agent  $i$  at time  $t$  correspond to the set of neighbours  $\mathcal{N}(\mathcal{G}(t), v_i)$ . Let  $\tilde{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m\}$  denote the set of all possible communication topologies.

Suppose that  $x_i \in \mathbb{R}$  denotes the state of vertex  $v_i$  (agent  $i$ ). Let  $x = [x_1, x_2, \dots, x_n]^T$ . Then the whole system can be represented by the discrete-time model  $x(t+1) = u(t)$ , where  $u(t)$  is a state feedback. If for any initial state,  $x(t)$  converges to some equilibrium point  $x^*$  (dependent on the initial state) such that  $x_i^* = x_j^*$  for all  $i, j \in \mathcal{I}_n$ , as  $t \rightarrow \infty$ , then we say that this system solves a consensus problem (Olfati-Saber and Murray 2004) (or has consensus property).

In this paper, we study the model presented by Ren and Beard (2005) and take communication time-delays into account. We suppose that each agent takes the following dynamics, (cf. Ren and Beard (2005)),

$$x_i(t+1) = \frac{1}{\sum_{j=1}^n \alpha_{ij}(t) G_{ij}(t)} \sum_{j=1}^n \alpha_{ij}(t) G_{ij}(t) x_j(t - \tau_{ij}(t)), \quad (1)$$

where  $i, j \in \mathcal{I}_n$ ,  $\alpha_{ij}(t) > 0$  is a weighting factor chosen from any finite set  $\tilde{\alpha}$ ,  $G_{ii}(t) \equiv 1$ ,  $G_{ij}(t)$ ,  $i \neq j$ , equals one if agent  $i$  obtains the state information of agent  $j$  at time  $t$ , and equals zero otherwise,  $\tau_{ij}(t) \in \mathbb{Z}_+$ ,  $\tau_{ij}(t) \leq \tau_{\max}$  is the transmission time-delay of information from  $v_j$  to  $v_i$ ,  $\tau_{ii}(t) \equiv 0$ , and  $x_j(t - \tau_{ij}(t))$ ,  $j \neq i$ , is the state information of agent  $j$  obtained by agent  $i$  at time  $t$  if  $G_{ij}(t) = 1$ .

In system (1),  $\tau_{\max} \in \mathbb{Z}_+$  is the maximal communication time-delay, and we allow the weighting factor  $\alpha_{ij}(t)$  to be dynamically changing to represent possible time-varying relative confidence of each agent's information state or relative reliability of different information exchange links between agents. We assume that delays affect only the information that is actually being transmitted from one agent to another, i.e.,  $\tau_{ii}(t) \equiv 0$  for all  $i \in \mathcal{I}_n$ .  $G_{ii}(t) \equiv 1$  implies that each agent always can get its own state value, i.e., for any

$j \in \{1, 2, \dots, m\}$ ,  $i \in \mathcal{I}_n$ ,  $(v_i, v_j) \in \mathcal{G}_j$ . We can see that that is a reasonable assumption.

The existence of communication channel  $(v_j, v_i)$  at time  $t_a, t_a + 1, \dots, t_b$  does not imply that  $G_{ij}(t) = 1$ ,  $t_a \leq t \leq t_b$ , since communication time-delays may destroy the continuity of information. At each time, agent  $j$  sends its state information through communication channel  $(v_j, v_i)$ . Because of time-varying delays, some information may reach at agent  $i$  simultaneously, and there also may be some times, at which no information reach at agent  $i$  at all. If at time  $t$ , the states of agent  $j$  received by agent  $i$  are more than one, then agent  $i$  randomly choose one of them to use and throw away others. However, if the topology and delays are time-invariant, then information obtained by agent  $i$  will be continuous after the first  $\tau_{\max}$  time steps.

Due to unreliable transmission or limited communication/sensing range, the communication topology may be dynamically changing. If communication channel  $(v_j, v_i)$  fails at time  $t$ , then the data being transmitted on the way are all lost. If  $(v_j, v_i)$  creates at time  $t$ , then agent  $j$  starts to transmit its current state information to agent  $i$  through communication channel  $(v_j, v_i)$ .

### 4. Main results

In this section, we present our main results.

In order to facilitate our analysis, we introduce some notations. For any  $n \times n$  non-negative matrix  $A = [a_{ij}]$ , let  $\Lambda(A) = \{B: b_{ij} \text{ equals } a_{ij} \text{ or } 0\}$ , and let  $\Pi(A)$  denote the set of matrix

$$\left[ \begin{array}{cccccc} \text{diag}(A) + A_0 & A_1 & \cdots & A_{\tau_{\max}-1} & A_{\tau_{\max}} & \\ I & 0 & \cdots & 0 & 0 & \\ 0 & I & \cdots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & I & 0 & \end{array} \right]_{(\tau_{\max}+1)n \times (\tau_{\max}+1)n},$$

where  $\text{diag}(A)$  is a diagonal matrix with the same diagonal entries as  $A$ ,  $A_0, \dots, A_{\tau_{\max}} \in \Lambda(A)$  such that  $\text{diag}(A) + A_0 + A_1 + \dots + A_{\tau_{\max}} = A$ , and  $I$  is the identity matrix with compatible dimensions.

Let  $D(t) = [d_{ij}(t)] \in \mathbb{R}^{n \times n}$ , where  $d_{ij}(t) = \alpha_{ij}(t) G_{ij}(t) / \sum_{j=1}^n \alpha_{ij}(t) G_{ij}(t)$ . Obviously,  $D(t)$  is stochastic and system (1) becomes

$$x_i(t+1) = \sum_{j=1}^n d_{ij}(t) x_j(t - \tau_{ij}(t)). \quad (2)$$

**Remark 1:** By the assumptions in previous section,  $\mathcal{E}(\mathcal{G}(D(t))) \subset \mathcal{E}(\mathcal{G}(t))$ ,  $t \in \mathbb{Z}_+$ . But that  $\mathcal{E}(\mathcal{G}(D(t))) = \mathcal{E}(\mathcal{G}(t))$  does not hold generally because of the existence of time-varying delays.

**4.1 The case with fixed topology and time-invariant delays**

As a first step toward the general case, we first assume that the communication topology  $\mathcal{G}(t) \in \bar{\mathcal{G}}$  is fixed, which is denoted by  $\mathcal{G}$ , the communication delays are time-invariant and weighting factors  $\alpha_{ij}(t)$  are constant. As a result,  $D(t)$ ,  $t \geq \tau_{\max}$ , is also time-invariant, denoted by  $D$ . Furthermore,  $\mathcal{G}(D) = \mathcal{G}$  when we ignore the weight of each edge.

Let  $y(t)$  denote  $[x(t)^T, x(t-1)^T, \dots, x(t-\tau_{\max})^T]^T$ . Then there exists a matrix  $\Xi \in \Pi(D)$  such that for any  $t \geq \tau_{\max}$ , system (1) can be equivalently represented by

$$y(t+1) = \Xi y(t). \tag{3}$$

In order to introduce the main result of this subsection, we need the following lemma.

**Lemma 2:** Let  $A_0, A_1, \dots, A_{\tau_{\max}}$  be  $n \times n$  non-negative matrices and let

$$\begin{aligned} M_1 &= \begin{bmatrix} A_0 & A_1 & \cdots & A_{\tau_{\max}-1} & A_{\tau_{\max}} \\ I & & & & \\ & I & & 0 & \\ & & \ddots & & \\ & 0 & & I & \end{bmatrix}, \\ M_2 &= \begin{bmatrix} A_0 & \cdots & A_{\tau_{\max}-2} & A_{\tau_{\max}-1} & A_{\tau_{\max}} \\ I & & & & \\ I & & & 0 & \\ & \ddots & & & \\ 0 & & I & & \end{bmatrix}, \\ \dots M_{\tau_{\max}} &= \begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_{\tau_{\max}} \\ I & & & & \\ I & & 0 & & \\ \vdots & & & & \\ I & & & & \end{bmatrix}. \end{aligned} \tag{4}$$

If  $\mathcal{G}(A_0 + A_1 + \dots + A_{\tau_{\max}})$  has spanning trees, then  $\mathcal{G}(M_1), \mathcal{G}(M_2), \dots, \mathcal{G}(M_{\tau_{\max}})$  also have spanning trees. Furthermore, if  $A_0 + A_1 + \dots + A_{\tau_{\max}}$  is stochastic and there exists  $\mu > 0$  such that  $A_0 \geq \mu I$ , then for any  $i \in \{1, 2, \dots, \tau_{\max}\}$ ,  $M_i$  is SIA.

**Proof:** Let the vertices in  $\mathcal{G}(A_0 + A_1 + \dots + A_{\tau_{\max}})$  be  $v_1, v_2, \dots, v_n$ , and let the vertices in  $\mathcal{G}(M_i)$  be  $u_1, u_2, \dots, u_{(\tau_{\max}+1)n}$ . It is easy to see that for any  $j \in \mathcal{I}_n$ , there exist paths from  $u_j$  to  $u_{j+n}, u_{j+2n}, \dots, u_{j+\tau_{\max}n}$ . The reason is that  $(u_j, u_{j+n}), (u_{j+n}, u_{j+2n}), \dots, (u_{j+(\tau_{\max}-1)n}, u_{j+\tau_{\max}n}) \in \mathcal{G}(M_1)$ ,  $(u_j, u_{j+n}), (u_j, u_{j+2n}), (u_{j+n}, u_{j+3n}), \dots, (u_{j+(\tau_{\max}-2)n}, u_{j+\tau_{\max}n}) \in \mathcal{G}(M_2), \dots, (u_j, u_{j+n}), (u_j, u_{j+2n}), \dots, (u_j, u_{j+\tau_{\max}n}) \in \mathcal{G}(M_{\tau_{\max}})$ . If there exists an edge  $(v_j, v_k)$  in  $\mathcal{G}(A_0 + A_1 + \dots + A_{\tau_{\max}})$ , then there exists  $0 \leq l \leq \tau_{\max}$  such that  $(u_{j+ln}, u_k) \in \mathcal{E}(\mathcal{G}(M_i))$ . Therefore,

if  $\mathcal{G}(A_0 + A_1 + \dots + A_{\tau_{\max}})$  has a spanning tree and  $v_j$  is the root vertex, then  $\mathcal{G}(M_i)$  also has a spanning tree and  $u_j$  is the root.

If  $A_0 + A_1 + \dots + A_{\tau_{\max}}$  is stochastic,  $M_i$  is also stochastic. If  $A_0 \geq \mu I$ , then  $M_i$  satisfies the conditions of Lemma 1, i.e.,  $M_i \in \mathcal{S}$ . Therefore  $M_i$  is SIA.  $\square$

**Theorem 1:** With a time-invariant communication topology, time-invariant communication delays, and constant weighting factors, system (1) solves a consensus problem if and only if the communication topology has spanning trees.

**Proof:** We consider system (3).

*Necessity.* If the communication topology has not any spanning tree, then there will be several subsystems, among which there is no information transmission, and thus system (1) will not solve any consensus problem.

*Sufficiency.* Obvious  $D$  is a stochastic matrix with positive diagonal entries. The fact that graph  $\mathcal{G}$  has spanning trees implies that directed graph  $\mathcal{G}(D)$  also has spanning trees. By Lemma 2,  $\Xi$  is SIA, and there exists  $f \in \mathbb{R}^{(\tau_{\max}+1)n}$ ,  $f \geq 0$ , such that  $\lim_{k \rightarrow \infty} \Xi^k = \mathbf{1}f^T$ , which implies that system (3) solves a consensus problem. And hence system (1) under the assumptions of this theorem solves a consensus problem.  $\square$

**4.2 The general case**

In this subsection, we investigate the general case, i.e., the case with time-varying communication topologies, time-varying delays, and time-varying weighting factors.

With the same arguments as previous subsection, there exists matrix  $\Xi(t) \in \Pi(D(t))$  such that for any  $t \geq \tau_{\max}$ ,

$$y(t+1) = \Xi(t)y(t). \tag{5}$$

We will study the consensus property of system (1) by investigating augmented system (5). System (5) is a switching system without time-delays. It looks similar to system (1) without time-delays. However, we cannot use the method provided by Jadbabaie *et al.* (2003) directly, since the diagonal entries of  $\Xi(t)$  are not all non-zero (see Jadbabaie *et al.* (2003, Lemma 2)). We first present some lemmas.

**Lemma 3:** For any  $t \in \mathbb{Z}_+$ ,  $D(t)$  is a stochastic matrix with positive diagonal entries. Let  $\tilde{\mathcal{D}} = \{D(t) : t \in \mathbb{Z}_+\}$ . Then  $\tilde{\mathcal{D}}$  is a finite set.

**Proof:** It is obvious from the fact that  $\bar{\alpha}$  and  $\bar{\mathcal{G}}$  are finite sets.  $\square$

**Lemma 4:** For any  $t \in \mathbb{Z}_+$ ,  $\Xi(t)$  is a stochastic matrix. Let  $\bar{\Xi} = \{\Xi(t) : t \in \mathbb{Z}_+\}$ . Then  $\bar{\Xi}$  is a finite set.

**Proof:** The first property follows from the fact that  $D(t)$  is stochastic and the second is a consequence of the fact that  $\tilde{\mathcal{D}}$  is finite.  $\square$

**Lemma 5** (Wolfowitz 1963): *Let  $P_1, P_2, \dots, P_k$  be a finite set of SIA matrices with the property that for each sequence  $P_{i_1}, P_{i_2}, \dots, P_{i_j}$  of positive length, the matrix product  $P_{i_j}, P_{i_{j-1}}, \dots, P_{i_1}$  is SIA. Then, for each infinite sequence  $P_{i_1}, P_{i_2}, \dots$ , there exists a column vector  $f$  such that*

$$\prod_{j=1}^{\infty} P_{i_j} = \mathbf{1}f^T.$$

**Lemma 6:** *Let  $t_a, t_b \in \mathbb{Z}_+$  such that  $t_a < t_b$ , and the communication topology be  $\mathcal{G}_l \in \bar{\mathcal{G}}$  in the time interval between  $t_a$  and  $t_b$ . If  $t_b \geq t_a + \tau_{\max}$ , then  $\mathcal{G}(\sum_{t=t_a}^{t_b} D(t))$  is equivalent to  $\mathcal{G}_l$  when we ignore the weight of each edge.*

**Proof:** Since the maximal communication time-delay is  $\tau_{\max}$ , for any  $i \in \mathcal{I}_n, j \in \mathcal{N}(\mathcal{G}_l, i)$ , agent  $i$  can obtain the state information  $x_j(t_a)$  at some time  $t', t_a \leq t' \leq t_b$ . That implies that  $d_{ij}(t') > 0$ . Therefore the lemma holds.  $\square$

**Lemma 7:** *Let  $\{z_1, z_2, \dots, z_q\}$  be any finite subset of  $\mathbb{Z}_+$ . If  $\mathcal{G}(\sum_{k=1}^q D(z_k))$  has spanning trees, then  $\prod_{k=1}^q \Xi(z_k)$  is SIA.*

**Proof:** Let

$$\Xi(t) = \begin{bmatrix} \text{diag}(D(t)) + A_0(t) & A_1(t) & \cdots & A_{\tau_{\max}-1}(t) & A_{\tau_{\max}}(t) \\ I & & & & \\ & I & & & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & & & I & \end{bmatrix}.$$

Since  $\bar{\mathcal{D}}$  is finite, there exists  $0 < \mu \leq 1$  such that  $\text{diag}(D(t)) \geq \mu I$  for any  $t \in \mathbb{Z}_+$ , and we have that

$$\begin{aligned} \Xi(z_2)\Xi(z_1) &\geq \mu^2 \begin{bmatrix} I + A_0(z_2) & A_1(z_2) & \cdots & A_{\tau_{\max}-1}(z_2) & A_{\tau_{\max}}(z_2) \\ I & & & & \\ & I & & & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & & & I & \end{bmatrix} \\ &\times \begin{bmatrix} I + A_0(z_1) & A_1(z_1) & \cdots & A_{\tau_{\max}-1}(z_1) & A_{\tau_{\max}}(z_1) \\ I & & & & \\ & I & & & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & & & I & \end{bmatrix} \\ &\geq u^2 \begin{bmatrix} I + A_0(z_2) + A_0(z_1) + A_1(z_2) & \cdots & A_{\tau_{\max}-2}(z_1) + A_{\tau_{\max}-1}(z_2) & A_{\tau_{\max}-1}(z_1) + A_{\tau_{\max}}(z_2) & A_{\tau_{\max}}(z_1) \\ I & & & & \\ I & & & & \mathbf{0} \\ & \ddots & & & \\ \mathbf{0} & & & I & \end{bmatrix}. \end{aligned}$$

By induction, we can prove that  $\prod_{k=1}^q \Xi(z_k) \geq \mu^q F$ , where  $F$  satisfies the following properties:

- (i) if  $q \leq \tau_{\max} - 1$ , then  $F$  has the form of  $M_q$ , which is defined in Lemma 2;
- (ii) if  $q \geq \tau_{\max}$ , then  $F$  has the form of  $M_{\tau_{\max}}$ ;
- (iii) let the first  $n$  rows of  $F$  be  $[E_0, E_1, \dots, E_{\tau_{\max}}]$  and then  $E_0 + E_1 + \dots + E_{\tau_{\max}} = I + \sum_{k=1}^q \sum_{i=0}^{\tau_{\max}} A_i(z_k)$  and  $E_0 \geq I$ .

Therefore, if  $\mathcal{G}(\sum_{k=1}^q D(z_k))$  has spanning trees, then  $\mathcal{G}(E_0 + \dots + E_{\tau_{\max}})$  also has spanning trees. From the proof of Lemma 2.  $\prod_{k=1}^q \Xi(z_k) \in \mathcal{S}$  and by Lemma 1  $\prod_{k=1}^q \Xi(z_k)$  is SIA.  $\square$

The following theorem is one of our main results.

**Theorem 2:** *If there exists an infinite sequence of time  $t_0, t_1, t_2, \dots$ , where  $t_0 = 0, 0 < t_{k+1} - t_k \leq T, k, T \in \mathbb{Z}_+$ , with the property that for any  $k \in \mathbb{Z}_+$  the union of graphs  $\mathcal{G}(D(t_k)), \mathcal{G}(D(t_k + 1)), \dots, \mathcal{G}(D(t_{k+1} - 1))$  has spanning trees, then system (1) solves a consensus problem.*

**Proof:** We consider the equivalent system (5). Let  $\Phi(t, t) = I, t \geq 0$ , and let  $\Phi(t_b, t_a) = \prod_{t=t_a}^{t_b-1} \Xi(t)$ , where  $t_a, t_b \in \mathbb{Z}_+$  and  $t_a < t_b$ . Since the union of graph  $\mathcal{G}(D(t_k)), \dots, \mathcal{G}(D(t_{k+1} - 1))$  is equivalent to  $\mathcal{G}(\sum_{t=t_k}^{t_{k+1}-1} D(t))$ . For any  $k \in \mathbb{Z}_+$ , by Lemma 7,  $\Phi(t_{k+1}, t_k)$  is SIA. Moreover, the set of possible  $\Phi(t_{k+1}, t_k), k \in \mathbb{Z}_+$ , must be finite because each  $\Phi(t_{k+1}, t_k)$  is a product of at most  $T$  matrices from  $\bar{\Xi}$  which is a finite set. By Lemma 7, the set of possible  $\Phi(t_{k+1}, t_k), k \in \mathbb{Z}_+$ , also satisfies the condition of Lemma 5. Therefore,

$$\lim_{k \rightarrow \infty} \Phi(t_k, 0) = \mathbf{1}f^T, \tag{6}$$

where  $f \in \mathbb{R}^{(\tau_{\max}+1)n}$  and  $f \geq 0$ .

For each  $t \geq 0$ , let  $k_t$  be the largest non-negative integer such that  $t_{k_t} \leq t$ . Then,  $\Phi(t, 0) = \Phi(t, t_{k_t})\Phi(t_{k_t}, 0)$  and  $\Phi(t, t_{k_t})\mathbf{1} = \mathbf{1}$ . So

$$\Phi(t, 0) - \mathbf{1}f^T = \Phi(t, t_{k_t})(\Phi(t_{k_t}, 0) - \mathbf{1}f^T).$$

Since the set of all possible  $\Phi(t, t_{k_t})$  is finite, by (6),

$$\lim_{t \rightarrow \infty} \Phi(t, 0) = \mathbf{1}f^T. \quad \square$$

The above theorem gives us a sufficient condition that system (1) solves a consensus problem, but there is a slight difference when it is compared with the conditions provided by Ren and Beard (2005, Theorem 3.10) which is for the case without communication time-delays and is not valid for our model, see Example 1. For any time interval  $t_a, t_a + 1, \dots, t_b$ , because of the existence of communication time-delays, that the union of  $\mathcal{G}(t_a), \mathcal{G}(t_a + 1), \dots, \mathcal{G}(t_b)$  has spanning trees can not guarantee that the union of  $\mathcal{G}(D(t_a)), \mathcal{G}(D(t_a + 1)), \dots, \mathcal{G}(D(t_b))$  has spanning trees.

**Example 1:** Let  $n = 2$ , and let  $\bar{\mathcal{G}} = \{\mathcal{G}(A_1), \mathcal{G}(A_2)\}$ , where

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

If communication time-delays between agent 1 and agent 2 under topologies  $\mathcal{G}(A_1)$  and  $\mathcal{G}(A_2)$  are all larger than 1, and  $\mathcal{G}(A_1)$  and  $\mathcal{G}(A_2)$  switch in turn at each time step, then the information of agent 1 can not reach at agent 2, and vice versa. Therefore, if the initial values of agent 1 and agent 2 are not equal, then they can not reach consensus asymptotically although this system meets the conditions of Ren and Beard (2005, Theorem 3.10).

The following theorem gives us a sufficient condition from the point of communication topologies' properties that ensure the consensus property of system (1). We first make the following assumption:

(A) For any  $\mathcal{G}_i \in \bar{\mathcal{G}}$ , if it takes effect at some time, then it must last for at least  $\tau_{\max} + 1$  time steps, that is, if  $\mathcal{G}(t) = \mathcal{G}_i$  and  $\mathcal{G}(t - 1) \neq \mathcal{G}_i$ , then  $\mathcal{G}(t) = \mathcal{G}(t + 1) = \dots = \mathcal{G}(t + \tau_{\max}) = \mathcal{G}_i$ .

**Theorem 3:** *If system (1) satisfies assumption (A) and there exists an infinite sequence of time  $t_0, t_1, t_2, \dots$ , where  $t_0 = 0, 0 < t_{k+1} - t_k \leq T, k, T \in \mathbb{Z}_+$ , with the property that the union of graphs  $\mathcal{G}(t_k), \mathcal{G}(t_k + 1), \dots, \mathcal{G}(t_{k+1} - 1)$ ,  $k \in \mathbb{Z}_+$ , has spanning trees, then system (1) solves a consensus problem.*

**Proof:** We reconstruct a time sequence that satisfies the conditions in Theorem 2 by recursion.

Let  $k_0 = l_0 = 0$  and let  $k_1$  be the smallest nonnegative integer such that  $t_{k_1} - t_{l_0+1} \geq \tau_{\max}$ .

If  $k_i$  has already been defined, we let  $l_i$  be the smallest non-negative integer such that  $t_{l_i} - t_{k_i} \geq \tau_{\max}$  and let  $k_{i+1}$  be the smallest non-negative integer such that  $t_{k_{i+1}} - t_{l_i+1} \geq \tau_{\max}$ .

Obviously  $\{t_{k_i}\}$  is a subsequence of  $\{t_k\}$ , and for any  $i > 0, k_i < l_i < k_{i+1}$ . Since for any  $i$ , the union of graphs  $\mathcal{G}(t_{l_i}), \mathcal{G}(t_{l_i} + 1), \dots, \mathcal{G}(t_{l_i+1} - 1)$  has spanning trees, by assumption (A) and Lemma 6, we have that the union of graphs  $\mathcal{G}(D(t_{k_i})), \mathcal{G}(D(t_{k_i} + 1)), \dots, \mathcal{G}(D(t_{k_{i+1}} - 1))$  has spanning trees. Moreover, for any  $i \in \mathbb{Z}_+$ ,  $t_{l_i} - t_{k_i} \leq \tau_{\max} + T - 1$  and  $t_{k_{i+1}} - t_{l_i+1} \leq \tau_{\max} + T - 1$ . Therefore we have that for any  $i \in \mathbb{Z}_+$ ,  $t_{k_{i+1}} - t_{k_i} \leq 2\tau_{\max} + 3T - 2$ . By Theorem 2, system (1) solves a consensus problem.  $\square$

**Corollary 1:** *With a time-invariant communication topology, bounded time-varying communication delays, and time-varying weighting factors, system (1) solves a consensus problem if and only if the communication topology has spanning trees.*

**Proof:** The necessary part is obvious and we only prove the sufficiency.

Suppose that the communication topology has spanning trees, and let  $t_0 = 0, t_1 = 1 + \tau_{\max}, t_2 = 2(1 + \tau_{\max}), \dots$ . Obviously this system satisfies assumption (A), and by Theorem 3, system (1) solves a consensus problem.  $\square$

As a special case of Theorem 2, although the conditions required in Theorem 3 are relatively stringent, it gives us a direct way to judge the consensus property of system (1) from the properties of communication topologies. Furthermore, we can relax its requirements in several ways. For example, for any  $\mathcal{G}_i \in \bar{\mathcal{G}}, i \in \{1, 2, \dots, m\}$ , instead of assumption (A), we can assume that the maximal communication time-delay is  $\tau_i$  and if  $\mathcal{G}_i$  takes effect, it lasts for at least  $\tau_i + 1$  time steps.

### 4.3 Applications in asynchronous consensus problems

In this subsection, we will show that our analysis approach can find its broad applications in asynchronous consensus problems. We only consider the model under asynchronous update scheme with fixed topology, and our results can be easily applied to the case with time-varying topologies. The following model is taken

from Fang and Antsaklis (2005),

$$x_i(t+1) = \begin{cases} \sum_{j=1}^n b_{ij}x_j(t - \tau_{ij}(t)) & \text{if } t \in S(t) \\ x_i(t) & \text{otherwise,} \end{cases} \quad (7)$$

where  $b_{ij} \geq 0$ ,  $\tau_{ij}(t) \in \mathbb{Z}_+$ ,  $i, j \in \mathcal{I}_n$ ,  $S(t)$  are non-empty subsets of  $\mathcal{I}_n$  and called updating sets, and  $B = [b_{ij}]$  is stochastic.

We assume that system (7) satisfies the following assumptions

- (B1) There exists a nonnegative integer  $\tau_{\max}$  such that  $0 \leq \tau_{ij}(t) \leq \tau_{\max}$  for any  $i, j \in \mathcal{I}_n$ ,  $t \in \mathbb{Z}_+$ ;
- (B2) The updating sets  $S(t)$  satisfy

$$\exists b \geq 0, \bigcup_{t=i}^{i+b} S(t) = \mathcal{I}_n, \quad \text{for any } i \in \mathbb{Z}_+;$$

- (B3) For any  $i \in \mathcal{I}_n$ ,  $b_{ii} > 0$  and  $\tau_{ii}(t) \equiv 0$ .

Then we have

**Theorem 4** (c.f. Fang and Antsaklis (2005, Theorem 2)): *If system (7) satisfies assumptions B1–B3, then system (7) solves a consensus problem if and only if  $\mathcal{G}(B)$  has spanning trees.*

**Proof:** System (7) can be seen as a special case of system (2) and this result follows from Theorem 2.  $\square$

### 5. Conclusion

We have considered problems of state consensus under dynamically changing communication topologies and time-varying communication delays. For the case with fixed topology, we proved that the necessary and sufficient condition that the states of agents reach consensus asymptotically is that the communication topology has spanning trees. We also proposed two sufficient conditions for state consensus under dynamically changing communication topologies and time-varying communication delays. Finally, we showed the applications of our results in asynchronous consensus problems.

### Acknowledgments

This work was supported by NSFC (10372002 and 60528007) and National 973 Program (2002CB312200).

### Appendix A

**Definition A1:** A non-negative matrix  $A \in \mathbb{C}^{n \times n}$  is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus.

**Lemma A1** (Horn and Johnson 1985, p. 511, Corollary 8.4.8; p. 522, Problem 5): *Let  $A \in \mathbb{C}^{n \times n}$  be non-negative and irreducible. If at least one main diagonal entry is positive, then  $A$  is primitive.*

**Lemma A2** (Horn and Johnson, 1985, p. 497, Lemma 8.2.7): *Let  $A \in \mathbb{C}^{n \times n}$  be given, let  $\lambda \in \mathbb{C}$  be given, and suppose  $x$  and  $y$  are vectors such that*

- (i)  $Ax = \lambda x$ ;
- (ii)  $A^T y = \lambda y$ ;
- (iii)  $x^T y = 1$ ;
- (iv)  $\lambda$  is an eigenvalue of  $A$  with geometric multiplicity 1;
- (v)  $|\lambda| = \rho(A) > 0$ ; and
- (vi)  $\lambda$  is the only eigenvalue of  $A$  with modulus  $\rho(A)$ .

Define  $L = xy^T$ . Then  $(\lambda^{-1}A)^k = L + (\lambda^{-1}A - L)^k \rightarrow L$  as  $k \rightarrow \infty$ .

**Lemma A3** (Ren and Beard 2005, Lemma 3.4): *Let  $A$  be a stochastic matrix.  $\mathcal{G}(A)$  has a spanning tree if and only if the eigenvalue 1 of  $A$  has algebraic multiplicity equal to one.*

**Lemma A4** (Horn and Johnson 1985, p. 503, Theorem 8.3.1): *If  $A \in \mathbb{C}^{n \times n}$  and  $A \geq 0$ , then  $\rho(A)$  is an eigenvalue of  $A$  and there is a non-negative vector  $f \geq 0$ ,  $f \neq 0$ , such that  $Af = \rho(A)f$ .*

**Proof of Lemma 1:** *We assume that there exists a spanning tree with vertex  $v_{l_1}$  as its root and  $(v_{l_1}, v_{l_i}) \in \mathcal{E}(\mathcal{G}(A))$ . Suppose that subgraph  $\mathcal{G}_s$  induced by  $v_{l_1}, v_{l_2}, \dots, v_{l_s} (1 \leq s \leq n)$  is the maximal induced subgraph that is strongly connected. Let the vertices in  $\mathcal{V}(\mathcal{G}(A)) - \{v_{l_1}, v_{l_2}, \dots, v_{l_s}\}$  be  $v_{l_{s+1}}, \dots, v_{l_n}$ . Then there exists a permutation matrix  $P$  such that*

$$\begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} = P \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}.$$

Therefore,

$$PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$\begin{aligned}
 A_{11} &= \begin{bmatrix} a_{l_1, l_1} & \cdots & a_{l_1, l_s} \\ \vdots & \ddots & \vdots \\ a_{l_s, l_1} & \cdots & a_{l_s, l_s} \end{bmatrix}, \\
 A_{12} &= \begin{bmatrix} a_{l_1, l_{s+1}} & \cdots & a_{l_1, l_n} \\ \vdots & \ddots & \vdots \\ a_{l_s, l_{s+1}} & \cdots & a_{l_s, l_n} \end{bmatrix}, \\
 A_{21} &= \begin{bmatrix} a_{l_{s+1}, l_1} & \cdots & a_{l_{s+1}, l_s} \\ \vdots & \ddots & \vdots \\ a_{l_n, l_1} & \cdots & a_{l_n, l_s} \end{bmatrix} \text{ and} \\
 A_{22} &= \begin{bmatrix} a_{l_{s+1}, l_{s+1}} & \cdots & a_{l_{s+1}, l_n} \\ \vdots & \ddots & \vdots \\ a_{l_n, l_{s+1}} & \cdots & a_{l_n, l_n} \end{bmatrix}.
 \end{aligned}$$

By the assumption that  $\mathcal{G}_s$  is maximal,  $A_{12} = 0$ . By Lemma A1,  $A_{11}$  is primitive. Since 1 is an eigenvalue of  $A_{11}$ , by Lemma A3, 1 is not an eigenvalue of  $A_{22}$ . By Lemma A4 and Geršgorin disk theorem,  $\rho(A_{22}) < 1$ . Therefore, 1 is the only eigenvalue of maximum modulus. Let  $f^T A = f^T$  such that  $f^T \mathbf{1} = 1$ . By Lemma A3 and A2,  $\lim_{k \rightarrow \infty} A^k = \mathbf{1}f^T$ .

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