

Asynchronous Consensus in Continuous-Time Multi-Agent Systems With Switching Topology and Time-Varying Delays

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Abstract—The paper studies asynchronous consensus problems of continuous-time multi-agent systems with discontinuous information transmission. The proposed consensus control strategy is implemented based on the state information of each agent's neighbors at some discrete times. The asynchrony means that each agent's update times, at which the agent adjusts its dynamics, are independent of others'. Furthermore, it is assumed that the communication topology among agents is time-dependent and the information transmission is with bounded time-varying delays. If the union of the communication topology across any time interval with some given length contains a spanning tree, the consensus problem is shown to be solvable. The analysis tool developed in this paper is based on nonnegative matrix theory and graph theory. The main contribution of this paper is to provide a valid distributed consensus algorithm that overcomes the difficulties caused by unreliable communication channels, such as intermittent information transmission, switching communication topology, and time-varying communication delays, and therefore has its obvious practical applications. Simulation examples are provided to demonstrate the effectiveness of the theoretical results.

Index Terms—Asynchronous consensus, coordination, intermittent transmission, multi-agent systems, switching topology, time-varying delays.

I. INTRODUCTION

IN recent years, decentralized coordination of multi-agent systems has become an active area of research and attracted the attention of multi-disciplinary researchers in a wide range including system control theory, statistical physics, biology, applied mathematics, and computer science. This is partly due to its broad applications in cooperative control of unmanned aerial vehicles, scheduling of automated highway systems, formation control of satellite clusters, distributed optimization of multiple robotic systems, etc.

In cooperative control of multiple agents, in order to accomplish some complicated tasks or reach their common goals, groups of dynamic agents need to interact with each other and eventually reach an agreement on certain quantities of interest. Such a problem is usually called *consensus* problem, which is one of fundamental research topics in decentralized control.

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Consensus problems have been studied for a long time and their formal investigation can be traced back to 1970's in the field of management science and statistics (See [1] and references therein). In the field of systems and control theory, the pioneering work was done by Borkar and Varaiya [2] and Tsitsiklis and Athans [3]. They considered asynchronous consensus problems with an application in distributed decision-making systems. In [4], a simple but interesting discrete-time model of multiple agents moving in the plane was proposed by Vicsek *et al.* Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. The Vicsek model can be viewed as a special case of a computer model mimicking animal aggregation proposed in [5] for the computer animation industry. By using graph and matrix theories, Jadbabaie *et al.* provided a theoretical explanation of the consensus property of Vicsek model in [6], where each agent's set of neighbors changes with time as the system evolves. A typical continuous-time consensus model was presented by Olfati-Saber and Murray in [7], where the concepts of solvability of consensus problems and consensus protocols were first introduced. Under the assumptions that the dynamics of each agent is a scalar continuous-time integrator and the interaction among agents is unidirectional, Olfati-Saber and Murray used a directed graph to model the communication topology and studied three agreement problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. And it was further assumed that the directed topology is balanced and strongly connected. In [8], Ren and Beard extended the results of [6] and [7] and presented some improved conditions for state agreement under dynamically changing directed topology, which is not necessarily balanced or strongly connected. Along this research line, Xiao and Wang established the relationship between communication topologies and different roles of agents classified by consensus functions [9]. In the past several years, investigation of consensus problems has been developing very fast and several research topics have been addressed, such as agreement over random networks [10], [11], asynchronous information consensus [12], [13], dynamic consensus [14], consensus filters [15], networks with general communication structures [16], and networks with switching topology and time-delays [6]–[8], [17]–[23]. Mathematical machinery and tools employed are diverse as well, such as graph theory, nonnegative matrix theory, characteristic equation theory, Lyapunov's direct method, and set-value Lyapunov theory [17]. For more details, see the surveys [24] and [25]. It should be emphasized that flocking of agents and swarms [26]–[29], and formation control of vehicles [30]–[33] are two

active areas where many useful results obtained in consensus problems have been successfully applied. Furthermore, in [26], the authors provided the first proof of the convergence of Reynolds' rules, using the convergence results of consensus algorithms presented by [7].

In what follows, we summarize some important and closely related works.

In continuous-time systems, none of the existing results shows the stability of the consensus protocol proposed by Olfati-Saber and Murray, in the presence of switching topology and time-varying delays. It was often assumed that time-delays are constant [7], [18], [19] and weighting factors are finite [7], [8]. The weighting factors usually represent confidence or reliability of received information. Therefore, it is more natural to take them from an infinite set.

In the study of discrete-time systems, Tanner and Christodoulakis studied a discrete-time model with fixed undirected topology and assumed that all agents transmit their state information in turn [20]. Consequently, outdated information may be used and the equivalent augmented system becomes a periodically switched system, which can be viewed as a multi-agent system with switching topology. Fang and Antsaklis studied the case with switching topology and time-dependent delays by an asynchronous system with fixed topology. However since the possible topologies are induced by the asynchrony of the studied system, the "switching" topology is not really switching. By using matrix theory, Xiao and Wang derived some sufficient conditions for the solvability of consensus problems for discrete-time systems with switching topology and time-varying delays [21], but it was also assumed that all possible weighting factors are finite. Another related work was done by Angeli and Bliman, who extended the results of Moreau to the case with time-varying delays by set-value Lyapunov theory [22].

In this paper, we propose a distributed asynchronous consensus control strategy that is based on the state information of each agent's neighbors at some discrete time instants. This is partly motivated by the work of Olfati-Saber and Murray and the difficulties encountered in the implementation of the typical continuous-time protocol proposed in [7]. In the real-world networks of agents, we may face the following problems:

- 1) the communication topology is always changing;
- 2) the received information is often with time-delays, and furthermore, the delays may be (randomly) time-varying and unknown;
- 3) the continuity of the state information of each agent's neighbors cannot be ensured, because of unreliable information channels and limited bandwidth.

The proposed consensus control strategy is built upon very mild assumptions. The communication topology is switching, the weighting factors are taken from an infinite set, the communication delays are time-varying, and the information transmission is allowed to be intermittent. Moreover, our control strategy is an asynchronous one, which means that each agent's update actions are independent of others'. It allows each agent to adjust its dynamics independently, which is inherent in distributed control systems. It should be mentioned that asynchronous consensus problems were also studied in [2], [3], and [12], where several asynchronous consensus algorithms were given. By using nonnegative matrix theory and graph theory,

especially the properties of scrambling matrices, we provide some sufficient or necessary conditions for the convergence of the consensus control strategy.

This paper is organized as follows. Some basic definitions and results in matrix and graph theories are presented in Section II. The studied problem is formulated in Section III. The convergence analysis and technical proof are performed in Sections IV and V, respectively. In Section VI, simulation examples are presented. Finally, concluding remarks are stated in Section VII.

II. PRELIMINARIES

This section presents some definitions and results in matrix and graph theories that will be used in this paper [34], [35].

Let $\mathcal{I}_n = \{1, 2, \dots, n\}$ and $\mathbf{1} = [1, 1, \dots, 1]^T$ with compatible dimensions. Matrix A is said to be *nonnegative* if all its entries are nonnegative. We write $A \geq B$ if $A - B$ is nonnegative. A nonnegative square matrix A with the property that all its row sums are +1 is said to be a *stochastic matrix*. A stochastic matrix A is called *indecomposable* and *aperiodic* (SIA) (or *ergodic*) if there exists a column vector ν such that $\lim_{k \rightarrow \infty} A^k = \mathbf{1}\nu^T$. Throughout this paper, let $\prod_{i=1}^k A_i = A_k A_{k-1} \cdots A_1$, denoting the left product of matrices.

Directed graphs will be used to model the communication topologies among agents. A *directed graph* \mathcal{G} consists of a vertex set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ and an edge set $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$, where an edge is an ordered pair of vertices in $\mathcal{V}(\mathcal{G})$. Here, we allow for self-loops, namely, such edges as (v_i, v_i) . The set of *neighbors* of vertex v_i is defined by $\mathcal{N}(\mathcal{G}, v_i) = \{v_j : (v_j, v_i) \in \mathcal{E}(\mathcal{G}), j \neq i\}$. The associated index set is denoted by $\mathcal{N}(\mathcal{G}, i) = \{j : v_j \in \mathcal{N}(\mathcal{G}, v_i)\}$. If (v_i, v_j) is an edge of \mathcal{G} , v_i and v_j are defined as the parent and child vertices, respectively. A *subgraph* \mathcal{G}_s of a directed graph \mathcal{G} is a directed graph such that the vertex set $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and the edge set $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$, \mathcal{G}_s is called a *spanning subgraph*. A *path* in a directed graph \mathcal{G} is a sequence v_{i_1}, \dots, v_{i_k} of vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}(\mathcal{G})$ for $j = 1, \dots, k-1$. A *directed tree* is a directed graph, where every vertex, except one special vertex without any parent, which is called the *root vertex*, has exactly one parent, and the root vertex can be connected to any other vertices through paths. A *spanning tree* of \mathcal{G} is a directed tree that is a spanning subgraph of \mathcal{G} . A directed graph is said to have a spanning tree if a subset of the edges forms a spanning tree. The union of time-dependent directed graph $\mathcal{G}(t)$ with constant vertex set across time interval $[t^0, t^0 + T]$ is a directed graph with the same vertex set as $\mathcal{G}(t)$ and the edge set $\bigcup_{t' \in [t^0, t^0 + T]} \mathcal{E}(\mathcal{G}(t'))$. The union of directed graph $\mathcal{G}(t)$ on the time set $\{s_1, s_2, \dots, s_k\}$ is a directed graph with the same vertex set as $\mathcal{G}(t)$ and the edge set $\bigcup_{i=1}^k \mathcal{E}(\mathcal{G}(s_i))$. A *weighted directed graph* $\mathcal{G}(A)$ is a directed graph \mathcal{G} plus a nonnegative *weight matrix* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff a_{ji} > 0$. And a_{ji} is called the *weight* of edge (v_i, v_j) .

III. PROBLEM FORMULATION

Suppose that the system studied in this paper consists of n autonomous agents, e.g., vehicles, robots, etc., labeled 1 through n . All these agents share a common state space \mathbb{R} and interact with each other via local information transmission. Each agent adjusts its current state based upon the information received

from other agents that are defined as neighbors of this agent. Directed graph $\mathcal{G}(t)$ with vertex set $\{v_1, v_2, \dots, v_n\}$ is used to represent the communication topology or information flow. The appearance of parameter t implies that the communication topology may be dynamically changing. Vertex v_i represents agent i . Edge $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(t))$, $i \neq j$, corresponds to an available information channel from agent j to agent i . In the paper, assume that there exist no self-loops in $\mathcal{G}(t)$. The *neighbors* of agent i at time t are those agents whose information is received by agent i at that time. The associated index set of the neighbors is denoted by $\mathcal{N}(t, i)$. Notice that because of the existence of transmission delays, the index set $\mathcal{N}(\mathcal{G}(t), i)$ may not be equal to $\mathcal{N}(t, i)$. This case will be discussed latter.

Let $x_i \in \mathbb{R}$ denote the state of agent i and let $x = [x_1, x_2, \dots, x_n]^T$. The overall system can be represented by the continuous-time model $\dot{x}_i(t) = u_i(t)$, $i \in \mathcal{I}_n$, or by the discrete-time model $x_i(t+1) = u_i(t)$, $i \in \mathcal{I}_n$, where $u_i(t)$ is a local state feedback, called *protocol* [7], to be designed based on the state information received by agent i from its neighbors. If for any initial state, $x(t)$ converges to some equilibrium point x^* (dependent on the initial state) such that $x_i^* = x_j^*$ for all $i, j \in \mathcal{I}_n$, as $t \rightarrow \infty$, then the system, under the proposed protocol, is said to *solve a consensus problem*[7] or *have the consensus property*. The common value of x_i^* is called the *group decision value*.

A. Consensus Control Strategy

For $i \in \mathcal{I}_n$, assume that agent i receives or detects its neighbors' states at *update times* $t_0^i, t_1^i, \dots, t_k^i, \dots$, which can be seen as a real number sequence and are denoted by $\{t_k^i\}^1$. Assume that $\{t_k^i\}$ satisfies the following assumptions:

(A1) there exist positive real numbers $\tilde{\tau}_u$ and $\hat{\tau}_u$ such that for any $k \in \mathbb{N}$, $\tilde{\tau}_u \leq t_{k+1}^i - t_k^i \leq \hat{\tau}_u$;

(A2) $t_0^i = 0$.

The simple reason for the calling of "update times" is that the neighbors' information known by agent i or the dynamics of agent i is updated at those times. The existence of lower and upper bounds $\tilde{\tau}_u, \hat{\tau}_u$ of time intervals between any two consecutive update times is just to guarantee the validity of consensus protocols (1) and (2). Example 1 shows the necessity of upper bound $\hat{\tau}_u$ to some extent. For Assumption (A2), we make

¹Note that it is *not* the set consisting of only one element t_k^i .

it solely for the convenience of our theoretical analysis in the sequel, namely, the main result of our paper is still obtainable without this assumption.

If agent i receives the state information of its neighbors at t_k^i , then agent i is assumed to take the following dynamics, described as shown in (1), shown at the bottom of the page, in time interval $[t_k^i, t_{k+1}^i)$, where $\alpha_{ij}(t_k^i) > 0$ are called *weighting factors*[8], and the right side of (1) is the proposed *protocol* [7].

It is well known that communication time-delays exist extensively in networks. Therefore, it is reasonable to assume that there exist communication time-delays in the information transmission. We assume that the information, received by agent i at update time t_k^i from agent j , is with time-delay τ_{ij}^k , and then the considered system turns into the one represented by (2), also shown at the bottom of the page, where $t \in [t_k^i, t_{k+1}^i)$.

We make the following assumption about system (2) in addition to Assumption (A1, A2):

(A3) $0 \leq \tau_{ij}^k \leq \tau_d = K\tilde{\tau}_u$ for all i, j, k , where $K \in \mathbb{N}$.

Consider system (2). Generally, the time-delays may be unknown for each agent when they receive the information. If the time-delays only satisfy Assumption (A3), the state information of agent j , $x_j(t_k^i - \tau_{ij}^k)$, received by agent i at update time t_k^i , may be outdated, compared with the state information $x_j(t_{k'}^i - \tau_{ij}^{k'})$, $k' < k$, received previously, namely, $t_k^i - \tau_{ij}^k < t_{k'}^i - \tau_{ij}^{k'}$. If the delays are detectable, we can suppose that agent i always uses the most recent data of its neighbors, that is, if there exists $k' < k$ such that $t_k^i - \tau_{ij}^k < t_{k'}^i - \tau_{ij}^{k'}$, then agent i replaces the state information $x_j(t_k^i - \tau_{ij}^k)$ by $x_j(t_{k'}^i - \tau_{ij}^{k'})$ in the time interval $[t_k^i, t_{k+1}^i)$, where $k^* = \arg \max_{k' < k} (t_{k'}^i - \tau_{ij}^{k'})$. This control strategy is called *the-most-recent-data strategy* and can get better convergence rate (see Example 3).

According to different properties of update times, we classify systems (1) and (2) as synchronous or asynchronous systems.

Definition 1 (Synchronous and Asynchronous Systems): System (1) or (2) is said to be *synchronous* if for any $i, j \in \mathcal{I}_n$, $\{t_k^i\} = \{t_k^j\}$, i.e., for any $k \in \mathbb{N}$, $t_k^i = t_k^j$. System (1) or (2) is *asynchronous* if for any $i, j \in \mathcal{I}_n$, $i \neq j$, $\{t_k^i\}$ is independent of $\{t_k^j\}$, i.e., agents may not adjust their dynamics at the same times (See Fig. 1).

In the distributed control of agents, it is difficult for all agents to be synchronous on update actions, thus we mainly discuss the asynchronous consensus property of systems (1) and (2).

$$\dot{x}_i(t) = \begin{cases} \frac{1}{\sum_{j \in \mathcal{N}(t_k^i, i)} \alpha_{ij}(t_k^i)} \times \sum_{j \in \mathcal{N}(t_k^i, i)} \alpha_{ij}(t_k^i) (x_j(t_k^i) - x_i(t)), & \text{if } \mathcal{N}(t_k^i, i) \neq \emptyset; \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$\dot{x}_i(t) = \begin{cases} \frac{1}{\sum_{j \in \mathcal{N}(t_k^i, i)} \alpha_{ij}(t_k^i)} \times \sum_{j \in \mathcal{N}(t_k^i, i)} \alpha_{ij}(t_k^i) (x_j(t_k^i - \tau_{ij}^k) - x_i(t)), & \text{if } \mathcal{N}(t_k^i, i) \neq \emptyset; \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

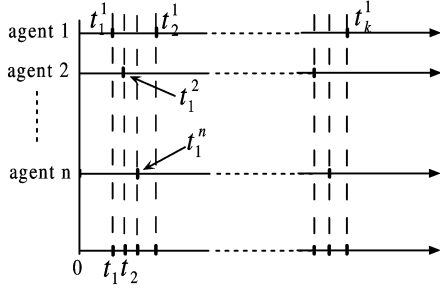


Fig. 1. Update times in asynchronous systems without time-delays.

B. Communication Topology

Since consensus protocols (1) and (2) only depend on discrete state information, we are not concerned with the actual communication topology $\mathcal{G}(t)$ outside those update times. Therefore, a new definition is given next, which is different from the actual communication topology and is also called communication topology.

Definition 2 (Communication Topology $\mathcal{G}^0(t)$): $\mathcal{G}^0(t)$ is a directed graph with the same vertex set as $\mathcal{G}(t)$. For any $i \in \mathcal{I}_n, k \in \mathbb{N}$, if agent i receives the state information of agent j at time $t_k^i, j \neq i$, then $(v_j, v_i) \in \mathcal{G}^0(t)$ for $t \in [t_k^i, t_{k+1}^i)$, and if not, $(v_j, v_i) \notin \mathcal{G}^0(t)$ for $t \in [t_k^i, t_{k+1}^i)$. In addition, assume that there exist no self-loops in $\mathcal{G}^0(t)$.

Proposition 1: $\mathcal{N}(t_k^i, i) = \mathcal{N}(\mathcal{G}^0(t_k^i), i) \subset \mathcal{N}(\mathcal{G}(t_k^i), i)$.

The above proposition follows from the definition of $\mathcal{G}^0(t)$ and the fact that the existence of information channel (v_j, v_i) at t_k^i cannot ensure that agent i can receive the state information of agent j at time t_k^i because of time-delays. Note that $\mathcal{N}(t, i)$ under the proposed protocols is only defined on update times of agent i , because agent i only receives information at those times. By Proposition 1, the definition of agents' neighbors can be extended on the entire time, for instance, let $\mathcal{N}(t, i) = \mathcal{N}(\mathcal{G}^0(t), i)$.

The reason for the introduction of $\mathcal{G}^0(t)$ is that $\mathcal{G}^0(t)$ is more relevant than $\mathcal{G}(t)$ to our control strategies. $\mathcal{G}^0(t)$ is directly connected with the successful information transmission in the network. $\mathcal{E}(\mathcal{G}(t))$ represents all available communication channels, while $\mathcal{E}(\mathcal{G}^0(t))$ stands for the communication channels through which the information has been successfully transmitted.

Here, for simplicity of presentation, we define another matrix $\mathbf{A}(t) = [\mathbf{a}_{ij}(t)]$ in such a way that for $t \in [t_k^i, t_{k+1}^i)$

1) if $\mathcal{N}(t_k^i, i) \neq \emptyset$,

$$\mathbf{a}_{ij}(t) = \begin{cases} \frac{\alpha_{ij}(t_k^i)}{\sum_{s \in \mathcal{N}(t_k^i, i)} \alpha_{is}(t_k^i)}, & j \in \mathcal{N}(t_k^i, i) \\ 0, & j \notin \mathcal{N}(t_k^i, i) \end{cases}$$

2) if $\mathcal{N}(t_k^i, i) = \emptyset$

$$\mathbf{a}_{ij}(t) = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$$

Obviously, $\mathbf{A}(t)$ is stochastic.

Proposition 2: For any $t \geq 0$

$$\mathcal{N}(\mathcal{G}(\mathbf{A}(t)), i) = \mathcal{N}(\mathcal{G}^0(t), i).$$

And if we ignore the weight of each edge and the self-loops in $\mathcal{G}(\mathbf{A}(t))$, then $\mathcal{G}(\mathbf{A}(t))$ and $\mathcal{G}^0(t)$ represent the same directed graph.

Since $\mathcal{G}(t)$ may be dynamically changing, we should investigate all possible directed graphs. Because of the finiteness of vertices, there are at most $2^{n(n-1)}$ different kinds of directed graphs, representing the topology. Let $\Gamma_{\mathcal{G}}$ denote the set of all those directed graphs. Assume that the weighting factors satisfy the assumption:

(A4) there exist real numbers $\hat{\alpha} \geq \check{\alpha} > 0$, such that $\check{\alpha} \leq$

$$\alpha_{ij}(t) \leq \hat{\alpha} \text{ for all } i, j, t.$$

Consequently all possible $\mathbf{A}(t)$ constitute a compact set², denoted by $\Gamma_{\mathbf{A}}$, and if $\mathbf{a}_{ij}(t) \neq 0$, then $\frac{\check{\alpha}}{(n-1)\hat{\alpha}} < \mathbf{a}_{ij}(t) \leq 1$.

C. Counter-Example

Finally, we construct a counter-example to show the necessity of the existence of upper bound $\hat{\tau}_u$ in Assumption (A1).

Example 1 (Counter-Example): Consider the synchronous case without time-delays and let $n = 2$ and $\mathcal{E}(\mathcal{G}(t)) \equiv \{(v_1, v_2), (v_2, v_1)\}$. We drop the parameter t and assume that for any $i \in \mathcal{I}_n, j \in \mathcal{N}(\mathcal{G}, i)$, agent i can always obtain the state information of agent j at its update times. Then, $\mathcal{N}(\mathcal{G}, i) = \mathcal{N}(\mathcal{G}^0, i)$. If update times are $t_1^1 = \ln 8, t_2^1 = t_1^1 + \ln 16, \dots, t_{k+1}^1 = t_k^1 + \ln 2^{k+3}, \dots, i = 1, 2$, weighting factors are all equal, and the initial state $x(0) = [1, -1]^T$, then

$$\lim_{k \rightarrow \infty} |t_{k+1} - t_k| = \lim_{k \rightarrow \infty} \ln 2^{k+3} = \infty$$

and

$$x_1(t_0) - x_2(t_0) = 2$$

$$x_2(t_1) - x_1(t_1) = 2 \left(1 - \frac{1}{4}\right)$$

$$x_1(t_2) - x_2(t_2) = 2 \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{8}\right)$$

⋮

where $t_k = t_k^1 = t_k^2, k \in \mathbb{N}$.

For any $k \geq 1$, we have

$$\begin{aligned} & |x_1(t_k) - x_2(t_k)| \\ &= 2 \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{8}\right) \cdots \left(1 - \frac{1}{2^{k+1}}\right) \\ &\geq 2 \left(1 - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^{k+1}}\right). \end{aligned}$$

Therefore, $\lim_{k \rightarrow \infty} |x_1(t_k) - x_2(t_k)| \geq 2(1 - 0.5) = 1$. Thus, states of agent 1 and agent 2 will never reach a consensus.

IV. CONVERGENCE ANALYSIS

With the above preparation, we now present the main result of this paper.

Theorem 1: If there exists $T \geq 0$ such that for all $t^0 \geq 0$, the union of graph $\mathcal{G}^0(t)$ across time interval $[t^0, t^0 + T]$ contains a spanning tree, then the system (1) satisfying Assumption (A1, A2, A4) or the time-delayed system (2) satisfying Assumption (A1–A4) solves a consensus problem. Moreover, if $\mathcal{G}(t)$ is time-

²The set of all $r \times s$ matrices can be viewed as the metric space \mathbb{R}^{rs} and compact sets are equivalent to bounded closed sets.

invariant, then the solvability of the consensus problem implies that $\mathcal{G}(t)$ contains a spanning tree.

To show the correctness of the above theorem, we need to investigate further the dynamical property of the studied system under the proposed protocols. Our approach is to transform the continuous-time systems into their discrete-time counterparts, which possess the same consensus property as the original systems. The obtained discrete-time systems have their special structures and the proof of their consensus property is postponed to the next section.

Some notations are used in the sequel. If the studied system is free of time-delays, let $\{t_k\} = \{t_k^i, i \in \mathcal{I}_n, k \in \mathbb{N}\}$, and if the system is with time-delays, let $\{t_k\} = \{t_k^i, t_k^i - \tau_{ij}^k : t_k^i - \tau_{ij}^k \geq 0, i \in \mathcal{I}_n, j \in \mathcal{N}(t_k^i, i), k \in \mathbb{N}\}$, where $\{t_k\}$ represents the set of time sequence $t_0, t_1, \dots, t_k, \dots$, such that $t_0 = 0$ and $t_{k+1} > t_k$ (see Fig. 1). Let $\tau_k = t_{k+1} - t_k, k \in \mathbb{N}$. And for any $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, let

$$\Lambda(A) = \left\{ B = [b_{ij}] \in \mathbb{R}^{n \times n} : \text{for any } i, j \in \mathcal{I}_n, \right. \\ \left. b_{ij} = a_{ij}, \text{ or } b_{ij} = 0 \right\}.$$

Obviously $\Lambda(A)$ is a finite set. Let $\Pi(m, t)$ denote the set of matrices

$$\begin{bmatrix} e^{-h}I + (1 - e^{-h})A_1 & (1 - e^{-h})A_2 & \cdots \\ I & 0 & \cdots \\ 0 & I & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \\ (1 - e^{-h})A_{m-1} & (1 - e^{-h})A_m \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ I & 0 \end{bmatrix}_{mn \times mn} \quad (3)$$

such that $0 < h \leq \hat{\tau}_u, A_1, \dots, A_m \in \Lambda(\mathbf{A}(t))$, and $A_1 + \dots + A_m = \mathbf{A}(t)$, where m is a positive integer, I is the $n \times n$ identity matrix, and $\mathbf{A}(t)$ is defined in the previous section. The above matrix (3) is uniquely determined by h, m , and A_1, \dots, A_m , and is denoted by $\pi(h, A_1, \dots, A_m)^3$. Because \mathbf{A} is stochastic, so is $\pi(h, A_1, \dots, A_m)$.

A. Asynchronous Consensus Without Time-Delays

Consider the asynchronous case of system (1). For $i \in \mathcal{I}_n$ and update time t_k , there exists $s \in \mathbb{N}$ such that $t_s^i \leq t_k < t_{k+1} \leq t_{s+1}^i$. Solving (1), we have that

$$x_i(t_{k+1}) = e^{-\tau_k} x_i(t_k) + (1 - e^{-\tau_k}) \sum_{j=1}^n \mathbf{a}_{ij}(t_s^i) x_j(t_s^i) \quad (4)$$

³It may happen that $\pi(h, A_1, \dots, A_m) = \pi(h', A_1', \dots, A_m')$, while $h \neq h'$. In this paper, we assume that if we write $\pi(h, A_1, \dots, A_m) \in \Pi(m, t)$, then h, A_1, \dots, A_m have the properties given for the elements of $\Pi(m, t)$ in the definition, namely, $0 < h \leq \hat{\tau}_u, A_1, \dots, A_m \in \Lambda(\mathbf{A}(t))$, and $A_1 + \dots + A_m = \mathbf{A}(t)$. The same assumption is taken for the matrix sets $\Pi(\tilde{m}), \Pi_0, \Pi_1, \Pi_2, \Pi_3$, etc.

where $\mathbf{a}_{ij}(t_s^i) = \mathbf{a}_{ij}(t_k)$ by the definition of matrix $\mathbf{A}(t)$.

In order to represent equation (4) in matrix form, the range of t_s^i in $\{t_k\}$ needs to be estimated. From Assumption (A1), we have the following fact

Lemma 1: Consider the asynchronous case of system (1) satisfying Assumption (A1, A2, A4). For any $i \in \mathcal{I}_n, k \in \mathbb{N}$, the number of elements in set $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ is not greater than $(\lfloor \hat{\tau}_u / \check{\tau}_u \rfloor + 1)(n - 1) + 1$, denoted by \check{m} , where $\lfloor \hat{\tau}_u / \check{\tau}_u \rfloor$ is the maximum integer not greater than $\hat{\tau}_u / \check{\tau}_u$.

Proof: By Assumption (A1), we have that $t_{k+1}^i - t_k^i \leq \hat{\tau}_u$. For any $j \in \mathcal{I}_n, j \neq i$, agent j updates the state information of its neighbors at most $\lfloor \hat{\tau}_u / \check{\tau}_u \rfloor + 1$ times in time interval (t_k^i, t_{k+1}^i) . And there are $n - 1$ possible j . Taking t_k^i into account, we have that the number of elements in $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ is not greater than $(\lfloor \hat{\tau}_u / \check{\tau}_u \rfloor + 1)(n - 1) + 1$. ■

Consider equation (4). By Lemma 1, $t_s^i \geq t_{k-\check{m}+1}$ (Otherwise, $\{t_j : t_j \in [t_s^i, t_{s+1}^i]\} \supset \{t_{k-\check{m}}, t_{k-\check{m}+1}, \dots, t_k\}$, whose number of elements is larger than \check{m} , contradicting Lemma 1). And by the definition of τ_k and Assumption (A1), $0 < \tau_k \leq \hat{\tau}_u$. Let

$$y(k) = [x(t_k)^T, x(t_{k-1})^T, \dots, x(t_{k-\check{m}+1})^T]^T$$

where $k \geq \check{m} - 1$. From the above discussion, there exists a matrix $\pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\check{m}}(t_k)) \in \Pi(\check{m}, t_k)$, which is defined by (3), such that

$$y(k+1) = \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\check{m}}(t_k))y(k). \quad (5)$$

By the expression of equation (4), if $t_s^i = t_{k'}$, then the i th row of matrix $\mathbf{A}_{k-k'+1}(t_k)$ is equal to the i th row of matrix $\mathbf{A}(t)$ and the i th rows of other matrices in $\{\mathbf{A}_1(t_k), \dots, \mathbf{A}_{\check{m}}(t_k)\}$ are equal to zeros. For simplicity, we use $\pi(\tau_k, \cdot)$ to denote the matrix $\pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\check{m}}(t_k))$ if it is self-evident from the context. Clearly, the set of all possible state matrix $\pi(\tau_k, \cdot)$ of system (5) is a subset of $\Pi(\check{m}, t_k)$, and furthermore it is not compact because τ_k may take any value in $(0, \hat{\tau}_u]$ by the asynchrony of different agents' update actions. $\Pi(\check{m}, t_k)$ is also not compact by its definition.

Since the state matrix $\pi(\tau_k, \cdot)$ is stochastic, system (5) can be viewed as a discrete-time multi-agent system with $\check{m}n$ agents and the associated communication topology is $\mathcal{G}(\pi(\tau_k, \cdot))$. The next proposition claims that the study of system (5) is sufficient for the investigation of system (1).

Proposition 3: System (1) solves a consensus problem if and only if system (5) solves a consensus problem.

Proof: The necessity follows from the definition of state variable $y(k)$.

To prove the sufficiency, assume that system (5) solves a consensus problem. Let $a \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} y(k) = \mathbf{1}a$. For any $i \in \mathcal{I}_n, t > 0$, there exists $k \in \mathbb{N}$, such that $t_k^i < t \leq t_{k+1}^i$, and

$$x_i(t) - x_i(t_k^i) \\ = \left(1 - e^{-(t-t_k^i)}\right) \left(\sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) x_j(t_k^i) - x_i(t_k^i) \right).$$

Since $t_k^i \rightarrow \infty$ as $t \rightarrow \infty$, $\lim_{t \rightarrow \infty} (x_j(t_k^i) - x_i(t_k^i)) = a - a = 0$, and thus

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) x_j(t_k^i) - x_i(t_k^i) \right) \\ = \lim_{t \rightarrow \infty} \sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) (x_j(t_k^i) - x_i(t_k^i)) = 0 \end{aligned}$$

which yields that

$$\lim_{t \rightarrow \infty} (x_i(t) - x_i(t_k^i)) = 0.$$

Therefore, for any $i \in \mathcal{I}_n$, $\lim_{t \rightarrow \infty} x_i(t) = a$, and system (1) solves a consensus problem. ■

To prove the first part of Theorem 1 about system (1), it is only necessary to prove system (4) or system (5) solves a consensus problem. System (4) can be viewed as a discrete-time multi-agent system consisting of n agents with time-delays bounded by $\tilde{m} - 1$. Angeli and Bliman studied this kind of problems in [22]. We first claim that system (4) cannot be covered by the systems studied in [22]. Because τ_k may take any value in $(0, \hat{\tau}_u]$, the set of all possible states of agent i at t_{k+1} is

$$e_i(t_k) = \left\{ \begin{aligned} &e^{-h} x_i(t_k) + (1 - e^{-h}) \sum_{j=1}^n a_{ij} x_j(t_s^i) : \\ &h \in (0, \hat{\tau}_u], A = [a_{ij}] \in \Gamma_A \end{aligned} \right\}$$

where $t_s^i \leq t_k < t_{s+1}^i$

By setting $h = 0$, we obtain that the closure of $e_i(t_k)$, denoted by $\text{cle}_i(t_k)$, which is the smallest compact set containing $e_i(t_k)$, always contains $x_i(t_k)$. Taking the notion $e_i(\mathcal{A}(t_k))(\cdot)$ from [17] and [22], $x_i(t_k) \in \text{cle}_i(t_k) \subset e_i(\mathcal{A}(t_k))(\hat{x})$, which does not satisfy Assumption A of [22].

The augmented system (5) can be viewed as a discrete-time multi-agent system consisting $\tilde{m}n$ agents, without time-delays, which is also not covered by the models of [17] and [22]. Denote those agents' states by $y_i(k)$, $i = 1, 2, \dots, \tilde{m}n$. The diagonal entries of state matrix $\pi(\tau_k, \cdot)$ are not all non-zeros, which leads to that system (5) does not satisfy the *strict* convexity assumption, namely, Assumption 1(3) of [17]. The strict convexity means that $y_i(k+1)$ is a strict convex combination of $y_i(k)$ and the neighbors' states, and therefore the diagonal entries of $\pi(\tau_k, \cdot)$ should be all non-zeros, which is violated by system (5). On the other hand, the closure of all possible state matrices contains the matrix

$$\begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{\tilde{m}n \times \tilde{m}n}.$$

Therefore, for any $i \in \mathcal{I}_n$, $y_i(k)$ always belongs to $e_i(\mathcal{A}(k))(y)$, which violates Assumption 1(2) of [17] and Assumption A of [22].

In contrast with the set-value Lyapunov theory employed in [17] and [22], nonnegative matrix theory is used to show the consensus property of system (5). The technical proof is presented in the next section.

B. Asynchronous Consensus With Time-Delays

For agent i and any $k \in \mathbb{N}$, there exists s such that $t_s^i \leq t_k < t_{k+1} \leq t_{s+1}^i$, and solving (2) gives

$$x_i(t_{k+1}) = e^{-\tau_k} x_i(t_k) + (1 - e^{-\tau_k}) \sum_{j=1}^n \mathbf{a}_{ij}(t_k) x_j(t_s^i - \tau_{ij}^s) \quad (6)$$

where $\tau_{ii}^s = 0$.

To represent it in matrix form, the range of $t_s^i - \tau_{ij}^s$ in $\{t_k\}$ should be estimated.

Lemma 2: Consider the asynchronous case of system (2) satisfying Assumption (A1–A4). For any $i \in \mathcal{I}_n$, $k \in \mathbb{N}$, the number of elements in set $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ is not greater than $\tilde{m}n(K(n-1) + 1)$.

Proof: Obviously, there cannot be infinite elements in $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ by Assumption (A1) and (A3). We work out one upper bound of the number of elements in this set.

For any i , let

$$\{t_k^{(i)}\} = \{t_k^i - \tau_{ij}^k : t_k^i - \tau_{ij}^k \geq 0, \quad k \in \mathbb{N}, j \in \mathcal{N}(t_k^i, i)\} \cup \{t_k^i\}.$$

Given $k \in \mathbb{N}$, $i \in \mathcal{I}_n$, by Assumption (A3),

$$\begin{aligned} \{t_k^{(i)}\} \cap (t_k^i, t_{k+1}^i) \subset &\left\{ t_{k+1}^i - \tau_{ij}^{k+1}, j \in \mathcal{N}(t_{k+1}^i, i), \right. \\ &t_{k+2}^i - \tau_{ij}^{k+2}, j \in \mathcal{N}(t_{k+2}^i, i), \dots, \\ &\left. t_{k+K}^i - \tau_{ij}^{k+K}, j \in \mathcal{N}(t_{k+K}^i, i) \right\}. \end{aligned}$$

Because $|\mathcal{N}(t_{k'}^i, i)| \leq n - 1$ for any k' , where $|\mathcal{N}(t_{k'}^i, i)|$ denotes the number of elements in set $\mathcal{N}(t_{k'}^i, i)$

$$\left| \{t_k^{(i)}\} \cap [t_k^i, t_{k+1}^i] \right| \leq K(n-1) + 1. \quad (7)$$

Let $i \in \mathcal{I}_n$, $k \in \mathbb{N}$ be fixed. By Lemma 1,

$$\left| \left\{ t_{k'}^j : t_{k'}^j \in [t_k^i, t_{k+1}^i], \quad k' \in \mathbb{N}, j \in \mathcal{I}_n \right\} \right| \leq \tilde{m} + 1.$$

Let its elements be $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{m'+1}$ such that $\bar{t}_{k'} < \bar{t}_{k'+1}$, $k' = 1, 2, \dots, m'$. Obviously, $m' \leq \tilde{m}$. For any $k' \in \{1, 2, \dots, m'\}$, $j \in \mathcal{I}_n$, there exists k'' such that $t_{k''}^j \leq \bar{t}_{k'} < \bar{t}_{k'+1} \leq t_{k''+1}^j$. By inequality (7)

$$\begin{aligned} \left| [\bar{t}_{k'}, \bar{t}_{k'+1}] \cap \{t_k^{(j)}\} \right| &\leq \left| [t_{k''}^j, t_{k''+1}^j] \cap \{t_k^{(j)}\} \right| \\ &\leq K(n-1) + 1. \end{aligned}$$

Since $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\} = \bigcup_{k'=1}^{m'} (\bar{t}_{k'}, \bar{t}_{k'+1}) \cap (\bigcup_{j=1}^m \{t_k^{(j)}\})$, we have

$$|\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}| \leq \check{m}n(K(n-1) + 1).$$

Let $\check{m}n = \check{m}n(K(n-1) + 1)$ and $\hat{m}n = (K+1)\check{m}n$. Consider equation (6). By Lemma 2 and Assumption (A3), if $s \geq K$, $k \geq \hat{m}n$, $t_s^i - \tau_{ij}^s \geq t_{k-\hat{m}n+1}$. Otherwise, $[[t_s^i - \tau_{ij}^s, t_{s+1}^i] \cap \{t_k\}] > [[t_{k-\hat{m}n+1}, t_k] \cap \{t_k\}] = \hat{m}n$. On the other hand, by Assumption (A3), $\tau_{ij}^s \leq K\check{\tau}_u$ and thus $[[t_s^i - \tau_{ij}^s, t_{s+1}^i] \cap \{t_k\}] \leq [[t_s^i - K\check{\tau}_u, t_{s+1}^i] \cap \{t_k\}] \leq [[t_{s-K}^i, t_{s+1}^i] \cap \{t_k\}]$ [by Assumption (A1)] $\leq (K+1)\check{m}n = \hat{m}n$ (by Lemma 2), which is a contradiction.

Let

$$z(k) = [x(t_k)^T, x(t_{k-1})^T, \dots, x(t_{k-\hat{m}n+1})^T]^T$$

and let $k_0 = \max\{\hat{m}n, \arg \min_k \{t_k : t_k \geq \max_i t_k^i\}\}$. Then for any $k \geq k_0$, there exists a matrix $\pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}n}(t_k)) \in \Pi(\hat{m}n, t_k)$ such that

$$z(k+1) = \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}n}(t_k))z(k). \quad (8)$$

With the same arguments as Proposition 3, we have

Proposition 4: System (2) solves a consensus problem if and only if system (8) solves a consensus problem.

Systems (6) and (8) have similar structures to systems (4) and (5) respectively. For the same reason as mentioned in the previous subsection, Theorem 1 about system (2) cannot be derived by the work of [17] and [22] either.

V. TECHNICAL PROOF

This section presents a complete proof of Theorem 1. We only prove the asynchronous case with time-delays, and the case without time-delays holds accordingly.

We first list some definitions and lemmas.

A. Prerequisite Definitions and Lemmas

The following notations are taken from [36] and the lemmas are useful in drawing the main conclusion.

Let $A = [a_{ij}]$ be $r \times r$ stochastic matrices and let

$$\delta(A) = \max_j \max_{i_1, i_2} |a_{i_1 j} - a_{i_2 j}|.$$

Thus $\delta(A)$ measures how different the rows of A are. If the rows of A are identical, $\delta(A) = 0$ and vice versa. A and B are said to be of the same type, denoted by $A \sim B$, if they have zero elements and positive elements in the same places. Let $\mathbf{n}(r)$ be the number of different types of all $r \times r$ SIA matrices. Define

$$\lambda(A) = 1 - \min_{i_1, i_2} \sum_j \min\{a_{i_1 j}, a_{i_2 j}\}.$$

If $\lambda(A) < 1$, A is called a *scrambling matrix*.

Lemma 3 ([36], Lemma 3): Let A_1, A_2 be stochastic matrices. If A_2 is an SIA matrix and $A_1 A_2 \sim A_1$, then A_1 is a scrambling matrix.

Lemma 4 ([36], Lemma 1): If one or more matrices in a product of stochastic matrices is scrambling, so is the product.

Lemma 5 ([36], Lemma 2): For any stochastic matrices $A_1, A_2, \dots, A_k, k > 0$,

$$\delta(A_1 A_2 \cdots A_k) \leq \prod_{i=1}^k \lambda(A_i).$$

The next lemma generalizes the result of Lemma 4 in [36].

Lemma 6: Let A_1, A_2, \dots, A_k (repetitions permitted) be $r \times r$ SIA matrices with the property that for any $1 \leq k_1 < k_2 \leq k$, $\prod_{i=k_1}^{k_2} A_i$ is SIA. If $k > \mathbf{n}(r)$, then $\prod_{i=1}^k A_i$ is a scrambling matrix.

Proof: Since $k > \mathbf{n}(r)$, there exist $k_1 < k_2$, such that $\prod_{i=k_1}^{k_2} A_i \sim \prod_{i=k_2}^{k_2} A_i$. It follows from Lemma 3 and $\prod_{i=k_1}^{k_2-1} A_i$ being SIA that $\prod_{i=k_2}^{k_2} A_i$ is a scrambling matrix. Thus $\prod_{i=1}^k A_i$ is also a scrambling matrix by Lemma 4. ■

Lemma 7 ([21], Lemma 1): Let A be a stochastic matrix. If $\mathcal{G}(A)$ contains a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(A)$, then A is SIA.

Lemma 8: Let $A_1, \dots, A_{\hat{m}n}$ be $n \times n$ nonnegative matrices, let

$$D = \begin{bmatrix} A_1 & A_2 & \cdots & A_{\hat{m}n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{\hat{m}n \times \hat{m}n}$$

let

$$M_0 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{\hat{m}n \times \hat{m}n}$$

and let $M_k = D + M_0^k$ for any $k \in \{1, 2, \dots, \hat{m}n - 1\}$. Then if $\mathcal{G}(\sum_{i=1}^{\hat{m}n} A_i)$ contains a spanning tree, then $\mathcal{G}(M_k)$ contains a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(M_k)$.

Proof: Let $\mathcal{G}(M_k), \mathcal{G}(M_0^k)$, and $\mathcal{G}(D)$ be with the same vertex set $\{u_1, u_2, \dots, u_{\hat{m}n}\}$ and let $\mathcal{V}(\mathcal{G}(\sum_{i=1}^{\hat{m}n} A_i)) = \{v_1, v_2, \dots, v_n\}$. Apparently $\mathcal{E}(\mathcal{G}(M_k)) = \mathcal{E}(\mathcal{G}(M_0^k)) \cup \mathcal{E}(\mathcal{G}(D))$. We now investigate the edge sets $\mathcal{E}(\mathcal{G}(M_0^k))$ and $\mathcal{E}(\mathcal{G}(D))$. It can be observed that for any $i \in \mathcal{I}_n$

$$\left\{ (u_i, u_i), (u_i, u_{n+i}), (u_i, u_{2n+i}), \dots, (u_i, u_{kn+i}), \right. \\ \left. (u_{n+i}, u_{(k+1)n+i}), (u_{2n+i}, u_{(k+2)n+i}), \dots, \right. \\ \left. (u_{(\hat{m}n-k-1)n+i}, u_{(\hat{m}n-1)n+i}) \right\} \subset \mathcal{E}(\mathcal{G}(M_0^k)).$$

Therefore, there exist paths connecting u_i to $u_{n+i}, \dots, u_{(\hat{m}n-1)n+i}$ in $\mathcal{E}(\mathcal{G}(M_0^k))$ (see Fig. 2). If there exists an edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(\sum_{s=1}^{\hat{m}n} A_s))$, then there exists $0 \leq r \leq \hat{m}n - 1$ such that $(u_{i+rn}, u_j) \in \mathcal{E}(\mathcal{G}(D))$, and therefore there exists a path from u_i to u_j in $\mathcal{G}(M_k)$. It follows from the above discussion that if $\mathcal{G}(\sum_{s=1}^{\hat{m}n} A_s)$ contains a

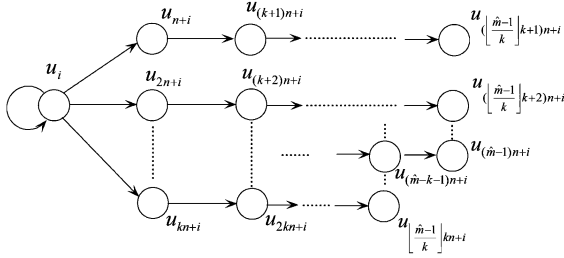


Fig. 2. Paths starting from u_i in $\mathcal{G}(M_0^k)$. If $\lfloor (\hat{m}-1)/k \rfloor k = \hat{m}-1$, then $u_{\lfloor (\hat{m}-1)/k \rfloor kn+i}$ and $u_{(\hat{m}-1)n+i}$ are the same vertex.

spanning tree with root vertex v_i , then $\mathcal{G}(M_k)$ also contains a spanning tree with root vertex u_i , and $(u_i, u_i) \in \mathcal{E}(M_k)$. ■

B. Properties of State Matrices

This subsection characterizes the properties of state matrices of system (8) by studying two compact sets. The first one contains all possible state matrices and the second one contains all possible products of a given number of state matrices at consecutive time-steps.

The next lemma gives an equivalent formulation of the condition given in Theorem 1.

Lemma 9: The existence of $T \geq 0$ such that for all $t^0 \geq 0$, the union of graph $\mathcal{G}^0(t)$ across time interval $[t^0, t^0 + T]$ contains a spanning tree, is equivalent to the condition that there exist a positive integer ϵ and a positive real number $\hat{\tau}_h$ with the following property:

for any $\mathcal{U} = \{t_k, t_{k+1}, \dots, t_{k+\epsilon-1}\}$, $k \in \mathbb{N}$, there exists a subset of \mathcal{U} , denoted by \mathcal{U}_h , such that the union of $\mathcal{G}^0(t)$ on \mathcal{U}_h contains a spanning tree and for any $t_s \in \mathcal{U}_h$, $t_{s+1} - t_s \geq \hat{\tau}_h$.

Proof: The sufficiency is rather straightforward and only the necessity is proved.

By Lemma 2, there exists $p \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $t_{k+p} - t_k \geq T$, for instance, let $p = \hat{m}(\lfloor T/\hat{\tau}_u \rfloor + 2)$. The reason is as follows. For any given $i \in \mathcal{I}_n$, there exists k' such that $t_{k'}^i \leq t_k < t_{k'+1}^i$. Then $t_{k'+p/\hat{m}}^i - t_{k'+1}^i \geq \hat{\tau}_u(p/\hat{m} - 1) = \hat{\tau}_u(\lfloor T/\hat{\tau}_u \rfloor + 1) > T$. We claim that $t_{k+p} \geq t_{k'+p/\hat{m}}^i$. If not, $[[t_{k'}^i, t_{k'+p/\hat{m}}^i] \cap \{t_k\}] \geq [[t_k, t_{k+p}] \cap \{t_k\}] = p + 1$. But by Lemma 2, $[[t_{k'}^i, t_{k'+p/\hat{m}}^i] \cap \{t_k\}] \leq (p/\hat{m})\hat{m} = p$, which is a contradiction. Therefore $t_{k+p} \geq t_{k'+p/\hat{m}}^i$. Hence $t_{k+p} - t_k \geq t_{k'+p/\hat{m}}^i - t_k > t_{k'+p/\hat{m}}^i - t_{k'+1}^i > T$.

Let $\epsilon = p + 2\hat{m}$ and consider \mathcal{U} . By the above arguments, $t_{k+\hat{m}+p-1} - t_{k+\hat{m}-1} \geq T$. Therefore the union graph $\mathcal{G}^0(t)$ on $\{t_{k+\hat{m}-1}, \dots, t_{k+\hat{m}+p-1}\}$ contains a spanning tree. Let the edge set of the spanning tree be \mathcal{E} . If $(v_j, v_i) \in \mathcal{E}$, there exists k' such that $k + \hat{m} - 1 \leq k' \leq k + \hat{m} + p - 1$ and $(v_j, v_i) \in \mathcal{G}^0(t_{k'})$. For $t_{k'}$, there exists k'' such that $t_{k''}^i \leq t_{k'} < t_{k''+1}^i$, and (v_j, v_i) is an edge of the graph $\mathcal{G}^0(t)$ for any $t \in [t_{k''}^i, t_{k''+1}^i]$.

We claim that $t_{k''}^i \geq t_k$ and $t_{k''+1}^i \leq t_{k+\epsilon-1}$. In fact, if $t_{k''}^i < t_k$, then $[[t_{k''}^i, t_{k''+1}^i] \cap \{t_k\}] > [[t_k, t_{k'}] \cap \{t_k\}] \geq [[t_k, t_{k+\hat{m}-1}] \cap \{t_k\}] = \hat{m}$, which contradicts Lemma 2. And if $t_{k''+1}^i > t_{k+\epsilon-1}$, then $[[t_{k''}^i, t_{k''+1}^i] \cap \{t_k\}] \geq [[t_{k'}, t_{k+\epsilon-1}] \cap \{t_k\}] \geq [[t_{k+\hat{m}+p-1}, t_{k+\epsilon-1}] \cap \{t_k\}] = \hat{m} + 1$, which also contradicts Lemma 2.

Since $\{t_k\} \cap [t_{k''}^i, t_{k''+1}^i)$ has at most \hat{m} elements, there exists $t_s \in \{t_k\} \cap [t_{k''}^i, t_{k''+1}^i) \subset \mathcal{U}$ such that $t_{s+1} - t_s \geq (t_{k''+1}^i -$

$t_{k''}^i)/\hat{m} \geq \hat{\tau}_u/\hat{m}$. Let $\hat{\tau}_h = \hat{\tau}_u/\hat{m}$. If (v_j, v_i) takes very possible edge in \mathcal{E} , we obtain all possible t_s . Let the set of them be \mathcal{U}_h . Then \mathcal{U}_h has the aforementioned property and the necessity is proved. ■

Next, we introduce the two compact sets.

The first compact set $\Pi(\hat{m})$ includes all possible state matrices of system (8), which is defined by

$$\begin{aligned} \Pi(\hat{m}) = & \left\{ \pi(h, A_1, \dots, A_{\hat{m}}) : \right. \\ & 0 \leq h \leq \hat{\tau}_u, \text{ and there exists some} \\ & \mathbf{A}' \in \Gamma_{\mathbf{A}}, \text{ such that } A_1, \dots, A_{\hat{m}} \in \Lambda(\mathbf{A}'), \\ & \left. \text{and } A_1 + \dots + A_{\hat{m}} = \mathbf{A}' \right\}. \end{aligned}$$

The second compact set Π_0 includes all possible products of ϵ state matrices at consecutive time-steps of system (8), defined by

$$\begin{aligned} \Pi_0 = & \left\{ \prod_{i=1}^{\epsilon} \pi(h_i, A_{i1}, A_{i2}, \dots, A_{i\hat{m}}) : \right. \\ & \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and there exists a subset of} \\ & \{1, 2, \dots, \epsilon\}, \text{ denoted by } \mathcal{H}, \text{ such that for any} \\ & s \in \mathcal{H}, \hat{\tau}_h \leq h_s \leq \hat{\tau}_u \text{ and } \mathcal{G} \left(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij} \right) \\ & \left. \text{contains a spanning tree} \right\} \end{aligned}$$

where ϵ and $\hat{\tau}_h$ are defined in Lemma 9.

Lemma 10: $\Pi(\hat{m})$ and Π_0 are compact sets, and for any $k \geq k_0$, $\Pi(\hat{m}, t_k) \subset \Pi(\hat{m})$ and $\prod_{i=k}^{k+\epsilon-1} \pi(\tau_i, \mathbf{A}_i(t_i), \mathbf{A}_2(t_i), \dots, \mathbf{A}_{\hat{m}}(t_i)) \in \Pi_0$.

Proof: Since $\Gamma_{\mathbf{A}}$ is compact and, given any \mathbf{A}' , all possible choices of $A_1, \dots, A_{\hat{m}}$ are finite, we have that $\Pi(\hat{m})$ is a compact set. And for any $k \geq k_0$, by the definition of $\Pi(\hat{m}, t_k)$, $\Pi(\hat{m}, t_k) \subset \Pi(\hat{m})$.

The compactness of Π_0 follows from the following facts

- 1) $\Pi(\hat{m})$ is a compact set;
- 2) all possible choices of \mathcal{H} are finite;
- 3) for any $s \in \mathcal{H}$, $h_s \in [\hat{\tau}_h, \hat{\tau}_u]$, which is a compact set;
- 4) all possible choices of the spanning tree are finite;
- 5) given the spanning tree and \mathcal{H} ,

$$\begin{aligned} \Pi_1 = & \left\{ \prod_{i=1}^{\epsilon} \pi(h_i, A_{i1}, \dots, A_{i\hat{m}}) : \right. \\ & \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and for any } s \in \mathcal{H}, \\ & h_s \in [\hat{\tau}_h, \hat{\tau}_u] \text{ and } \mathcal{G} \left(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij} \right) \\ & \left. \text{contains the specified spanning tree} \right\} \end{aligned}$$

is compact.

We only prove the fact 5).

Let $|\mathcal{H}| = q, q \leq \epsilon$. Since the product of ϵ matrices is continuous, to prove Π_1 is compact, it suffices to prove that

$$\Pi_2 = \left\{ \begin{array}{l} [\pi(h_1, A_{11}, \dots, A_{1\hat{m}}), \dots, \pi(h_\epsilon, A_{\epsilon 1}, \dots, A_{\epsilon \hat{m}})] : \\ \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and for any } s \in \mathcal{H}, \\ h_s \in [\hat{\tau}_h, \hat{\tau}_u], \text{ and } \mathcal{G} \left(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij} \right) \\ \text{contains the specified spanning tree} \end{array} \right\}$$

is compact. Since $\pi(h_i, \cdot) \in \Pi(\hat{m})$, which is compact, it suffices to prove that

$$\Pi_3 = \left\{ \begin{array}{l} [\pi(h_1, A_{11}, \dots, A_{1\hat{m}}), \dots, \pi(h_q, A_{q1}, \dots, A_{q\hat{m}})] : \\ \pi(h_i, \cdot) \in \Pi(\hat{m}), h_i \in [\hat{\tau}_h, \hat{\tau}_u], i = 1, 2, \dots, q, \\ \text{and } \mathcal{G} \left(\sum_{i=1}^q \sum_{j=1}^{\hat{m}} A_{ij} \right) \\ \text{contains the specified spanning tree} \end{array} \right\}$$

is compact.

Let $B^{(p)} = [b_{ij}^{(p)}] \in \Pi_3, p = 1, 2, \dots$, be a sequence of matrices, and $\lim_{p \rightarrow \infty} B^{(p)} = B = [b_{ij}]$. Since Π_3 is a bounded set, it suffices to prove that $B \in \Pi_3$.

It is clear that $\lim_{p \rightarrow \infty} b_{ij}^{(p)} = b_{ij}$. For any $i, j \in \mathcal{I}_n$, if $b_{ij}^{(p)} \neq 0$, then by the definition of $\Gamma_{\mathbf{A}}, b_{ij}^{(p)} > (1 - e^{-\hat{\tau}_h})\hat{\alpha}/((n-1)\hat{\alpha})$. Therefore, if $b_{ij} > 0$, then there exists $P_{ij} \in \mathbb{N}$ such that for any $p > P_{ij}, b_{ij}^{(p)} > 0$; if $b_{ij} = 0$, there exists $P_{ij} \in \mathbb{N}$ such that for any $p > P_{ij}, b_{ij}^{(p)} = 0$. Let $P = \max_{i,j} P_{ij}$, and then for any $p > P, B^{(p)} \sim B$.

Let $B = [B_1, B_2, \dots, B_q], B_i \in \mathbb{R}^{\hat{m}n \times \hat{m}n}, 1 \leq i \leq q$. From the compactness of $\Pi(\hat{m}), B_i \in \Pi(\hat{m})$ and has the form of $\pi(h_{b_i}, B_{i1}, \dots, B_{i\hat{m}})$ such that $h_{b_i} \in [\hat{\tau}_h, \hat{\tau}_u]$. And from $B^{(p)} \sim B, \mathcal{G}(\sum_{i=1}^q \sum_{j=1}^{\hat{m}} B_{ij})$ contains the spanning tree. To conclude, $B \in \Pi_3$.

Since for any $i \geq k, \pi(\tau_i, \mathbf{A}_1(t_i), \dots, \mathbf{A}_{\hat{m}}(t_i)) \in \Pi(\hat{m}, t_i) \subset \Pi(\hat{m})$, by Lemma 9, we have that $\prod_{i=k}^{k+\epsilon-1} \pi(\tau_i, \mathbf{A}_1(t_i), \mathbf{A}_2(t_i), \dots, \mathbf{A}_{\hat{m}}(t_i)) \in \Pi_0$. ■

Lemma 11: For any $\pi_0 \in \Pi_0, \pi_0$ is SIA. And for any $k > \mathbf{n}(n\hat{m})$, if $\pi_1, \dots, \pi_k \in \Pi_0$, then $\prod_{i=1}^k \pi_i$ is a scrambling matrix and there exists a $\hat{\lambda}(k) \in [0, 1)$, such that $\lambda(\prod_{i=1}^k \pi_i) \leq \hat{\lambda}(k)$.

Proof: Let $\pi_0 = \sum_{i=1}^{\epsilon} \pi(h_i, A_{i1}, \dots, A_{i\hat{m}})$ and \mathcal{H} be the associated subset of $\{1, 2, \dots, \epsilon\}$ defined for the elements of Π_0 . Let M_0 be the same as in Lemma 8 and let

$$D_i = \begin{bmatrix} A_{i1} & A_{i2} & \cdots & A_{i\hat{m}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \prod_{i=1}^{\epsilon} \pi(h_i, \cdot) &\geq \prod_{i=1}^{\epsilon} (e^{-h_i} M_0 + (1 - e^{-h_i}) D_i) \\ &\geq e^{-\epsilon \hat{\tau}_u} M_0^\epsilon + e^{-(\epsilon-1)\hat{\tau}_u} \\ &\quad \times \sum_{i=1}^{\epsilon} (1 - e^{-h_i}) M_0^{i-1} D_i M_0^{\epsilon-i} \\ &\geq e^{-\epsilon \hat{\tau}_u} M_0^\epsilon + e^{-(\epsilon-1)\hat{\tau}_u} \sum_{i \in \mathcal{H}} (1 - e^{-h_i}) D_i M_0^{\epsilon-i} \\ &\geq e^{-\epsilon \hat{\tau}_u} M_0^\epsilon + e^{-(\epsilon-1)\hat{\tau}_u} (1 - e^{-\hat{\tau}_h}) \sum_{i \in \mathcal{H}} D_i M_0^{\epsilon-i} \\ &\geq \min \left\{ e^{-\epsilon \hat{\tau}_u}, e^{-(\epsilon-1)\hat{\tau}_u} (1 - e^{-\hat{\tau}_h}) \right\} \\ &\quad \times \left(M_0^\epsilon + \sum_{i \in \mathcal{H}} D_i M_0^{\epsilon-i} \right). \end{aligned}$$

The second inequality follows from $0 \leq h_i \leq \hat{\tau}_u$, the third follows from $M_0^{\epsilon-1} D_i \geq D_i$, and the fourth follows from $h_i \geq \hat{\tau}_h, i \in \mathcal{H}$.

Let the first n rows of $D_i M_0^{\epsilon-i}$ be $[B_{i1}, B_{i2}, \dots, B_{i\hat{m}}]$, where $B_{ij} \in \mathbb{R}^{n \times n}, j = 1, 2, \dots, \hat{m}$. We have $\sum_{j=1}^{\hat{m}} B_{ij} = \sum_{j=1}^{\hat{m}} A_{ij}$ and thus $\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} B_{ij} = \sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij}$. Since $\mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij})$ contains a spanning tree, $\mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} B_{ij})$ also contains a spanning tree. Let $N \in \mathbb{R}^{\hat{m}n \times \hat{m}n}$ be with the same first n rows as $\sum_{i \in \mathcal{H}} D_i M_0^{\epsilon-i}$ and all other rows are zeros. Then $\prod_{i=1}^{\epsilon} \pi(h_i, \cdot) \geq \min \{ e^{-\epsilon \hat{\tau}_u}, e^{-(\epsilon-1)\hat{\tau}_u} (1 - e^{-\hat{\tau}_h}) \} (M_0^\epsilon + N)$, and N has the same property as the matrix D in Lemma 8. By Lemma 8 and that $M_0^k = M_0^{\hat{m}-1}$ if $k \geq \hat{m} - 1$, we have that $\mathcal{G}(M_0^\epsilon + N)$ contains a spanning tree with the property that the root vertex of the spanning tree has a self-loop, and so is $\mathcal{G}(\pi_0)$. Since π_0 is stochastic, by Lemma 7, π_0 is SIA.

With the same arguments, we can conclude that for any $1 \leq k_1 < k_2 \leq k, \prod_{i=k_1}^{k_2} \pi_i$ is SIA (We only need to replace ϵ by $(k_2 - k_1 + 1)\epsilon$ in the above arguments and let \mathcal{H} be the index set associated to any π_i , defined for the elements of Π_0). By Lemma 6, for any $k > \mathbf{n}(n\hat{m}), \prod_{i=1}^k \pi_i$ is a scrambling matrix. Let

$$\hat{\lambda}(k) = \max_{\pi_i \in \Pi_0} \lambda \left(\prod_{i=1}^k \pi_i \right).$$

Since Π_0 is a compact set and $\lambda(\cdot)$ is continuous, $\hat{\lambda}(k)$ exists and $\hat{\lambda}(k) < 1$. Obviously $\lambda(\prod_{i=1}^k \pi_i) \leq \hat{\lambda}(k)$. ■

C. Proof of Theorem 1

The second part of Theorem 1 is obvious. The subsection only provides the proof of the first part covering system (2).

For any $k \in \mathbb{N}$, let

$$\pi_k = \prod_{s=k\epsilon+k_0}^{(k+1)\epsilon+k_0-1} \pi(\tau_s, \mathbf{A}_1(t_s), \mathbf{A}_2(t_s), \dots, \mathbf{A}_{\hat{m}}(t_s)).$$

By Lemma 10, $\pi_k \in \Pi_0$.

Let $p = \mathbf{n}(\hat{m}\mathbf{n}) + 1$. For any $q > p\epsilon$, there exist $s \in \mathbb{N}$ and $0 \leq q' < p\epsilon$, such that $q = sp\epsilon + q'$. By Lemma 5 and Lemma 11,

$$\begin{aligned} \delta \left(\prod_{k=k_0}^{q+k_0-1} \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) \right) \\ \leq \prod_{i=0}^{s-1} \lambda \left(\prod_{j=ip}^{(i+1)p-1} \pi_j \right) \leq (\hat{\lambda}(p))^s. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{q \rightarrow \infty} \delta \left(\prod_{k=k_0}^{q+k_0-1} \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) \right) \\ = \lim_{s \rightarrow \infty} (\hat{\lambda}(p))^s = 0 \end{aligned}$$

which implies that there exists a column vector $\nu \in \mathbb{R}^{\hat{m}\mathbf{n}}$ such that

$$\lim_{q \rightarrow \infty} \prod_{k=k_0}^{q+k_0-1} \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) = \mathbf{1}\nu^T.$$

And thus system (8) solves a consensus problem. Therefore, Theorem 1 holds.

Remark: By employing the same arguments, we can show that under the same condition as in Theorem 1, the following control strategy solves a consensus problem, as shown at the bottom of the page, where $t \in [t_k^i, t_{k+1}^i)$.

VI. SIMULATIONS

In this section, we construct some examples to illustrate the effectiveness of the theoretical results.

Example 2 (Fixed Topology Without Time-Delays): Suppose that the system consists of 4 agents. Let $\mathcal{E}(\mathcal{G}^0(t)) \equiv \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_2, v_1), (v_3, v_1)\}$ (see Fig. 3), all weighting factors be chosen randomly from $[0.1, 0.9]$, $\check{\tau}_u = 0.2$ and $\hat{\tau}_u = 0.9$. And suppose that there do not exist communication time-delays and each agent can get all its neighbors' states at its update times. For any $i \in \mathcal{I}_n$, $t_{k+1}^i - t_k^i$ is evenly distributed between 0.2 and 0.9. Since the communication topology has a spanning tree, the consensus is achievable under

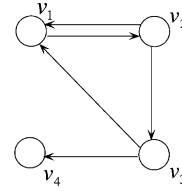


Fig. 3. Communication topology $\mathcal{G}^0(t)$ in Example 2.

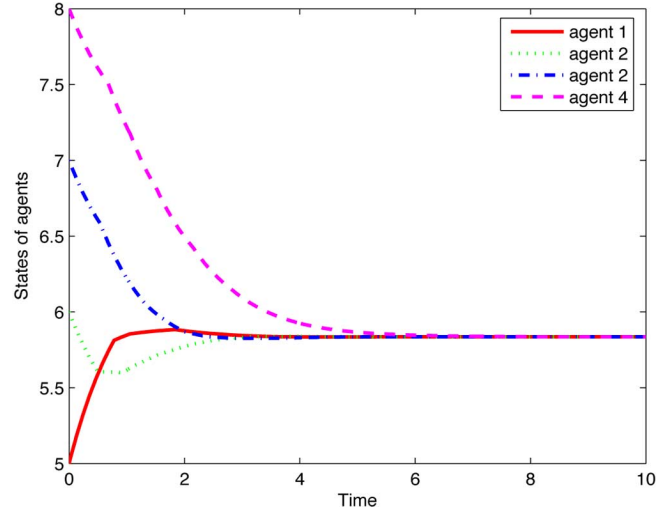


Fig. 4. State trajectories of agents of the system in Example 2 without time-delays.

asynchronous consensus control strategy (1). Let initial state $x(0) = [5, 6, 7, 8]^T$. In the simulation experiment, the update times $t_k^i, i = 1, 2, 3, 4$, are randomly generated and independent of each other. The state trajectories of agents are shown in Fig. 4.

It can be shown that the group decision values of synchronous systems are determined solely by the communication topologies and initial states. As a remarkable difference from the synchronous case, the group decision value of the asynchronous system is dependent on the update times. We repeat the simulation experiment 100 times independently, and the group decision values are shown in Fig. 5.

Example 3 (Fixed Topology With Time-Delays): We again consider the system in Example 2 and suppose that there exist communication time-delays bounded by τ_d . Let initial state $x(t) = [5, 6, 7, 8], t \in [-\tau_d, 0]$. Figs. 6 and 7 show the state trajectories of agents with maximum communication time-delay $\tau_d = 2(K = 10)$ and $\tau_d = 10(K = 50)$ respectively, where the time-delays are randomly generated. We can see that the system with $\tau_d = 2$ converges faster than the system with $\tau_d = 10$. If we adopt the-most-recent-data strategy, better convergence rate can be achieved. Figs. 8 and 9 show the state trajectories of

$$\dot{x}_i(t) = \begin{cases} \sum_{j \in \mathcal{N}(t_k^i, i)} \alpha_{ij}(t_k^i) (x_j(t_k^i - \tau_{ij}^k) - x_i(t)), & \text{if } \mathcal{N}(t_k^i, i) \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}$$

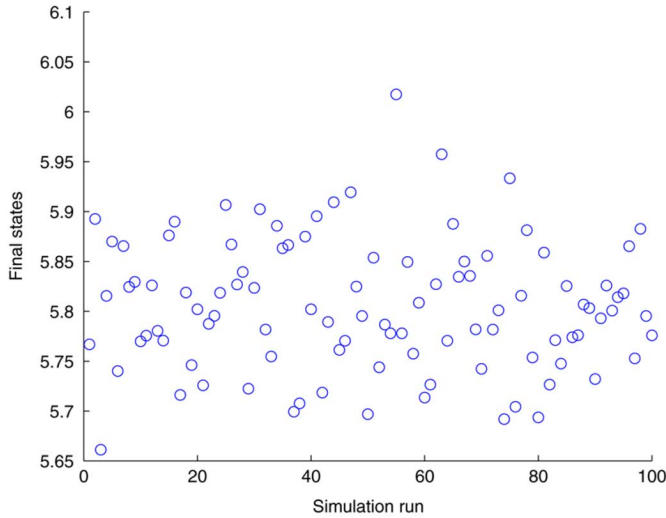


Fig. 5. Group decision values of different experiments in Example 2.

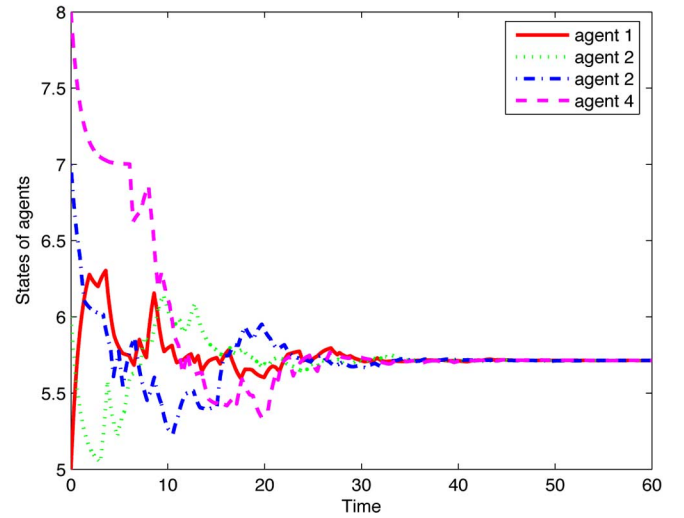


Fig. 7. State trajectories of agents of the system with the topology shown in Fig. 3 and time-delays bounded by $\tau_d = 10$.

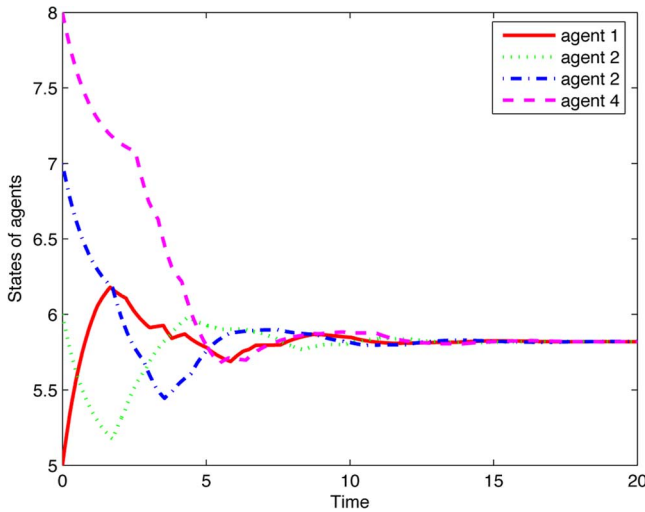


Fig. 6. State trajectories of agents of the system with the topology shown in Fig. 3 and time-delays bounded by $\tau_d = 2$.

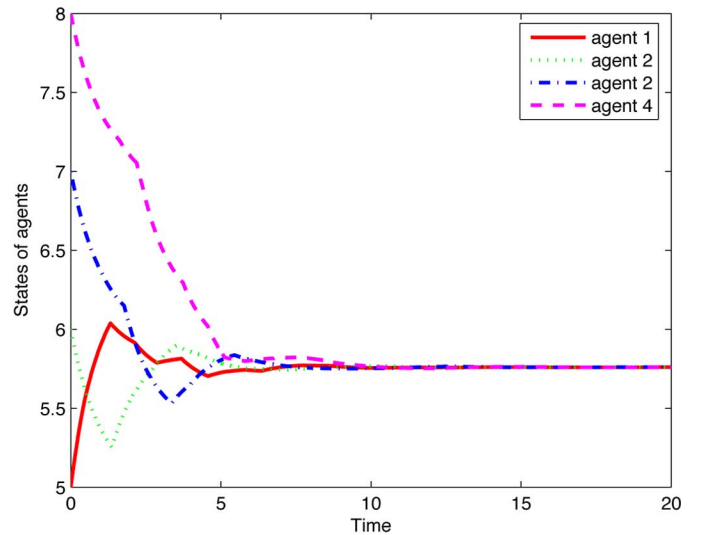


Fig. 8. State trajectories of agents by the-most-recent-data strategy with the topology shown in Fig. 3 and time-delays bounded by $\tau_d = 2$.

agents by the-most-recent-data strategy with randomly generated time-delays bounded by $\tau_d = 2$ and $\tau_d = 10$, respectively.

Example 4 (Switching Topology With Time-Delays): We still consider the asynchronous system consisting of 4 agents. Each agent's update times are randomly generated and update intervals are evenly distributed between 0.2 and 0.9. Suppose that the time-delays are bounded by $\tau_d = 2$ and random, and initial state $x(t) = [5, 6, 7, 8], t \in [-\tau_d, 0]$.

We assume that all weighting factors are randomly chosen from $[0.1, 0.9]$ and

- 1) agent 1 can get the state of agent 2 at update times $t_{4k}^1, k \in \mathbb{N}$, and can get the state of agent 3 at update times $t_{4k+2}^1, k \in \mathbb{N}$;
- 2) agent 2 can get the state of agent 1 at update times $t_{4k+1}^2, k \in \mathbb{N}$;
- 3) agent 3 can get the state of agent 2 at update times $t_{4k+2}^3, k \in \mathbb{N}$;
- 4) agent 4 can get the state of agent 3 at update times $t_{4k+3}^4, k \in \mathbb{N}$.

By Theorem 1, this system solves a consensus problem. The state trajectories of agents are shown in Fig. 10.

VII. CONCLUSION

We presented an asynchronous consensus control strategy for continuous-time multi-agent systems with switching topology and time-varying delays, which is implemented based on the discrete state information of each agent's neighbors. We first transformed the continuous-time system into its equivalent discrete-time system, and then by employing the tools from non-negative matrix theory and graph theory, we performed the convergence analysis. It is important to note that the equivalent discrete-time system has its special structures and is not covered by the existing models. The introduction of communication topology $\mathcal{G}^0(t)$ facilitated our analysis and established a connection between the actual communication topology and the proposed control strategy. Examples were provided to demonstrate the effectiveness of our theoretical results.

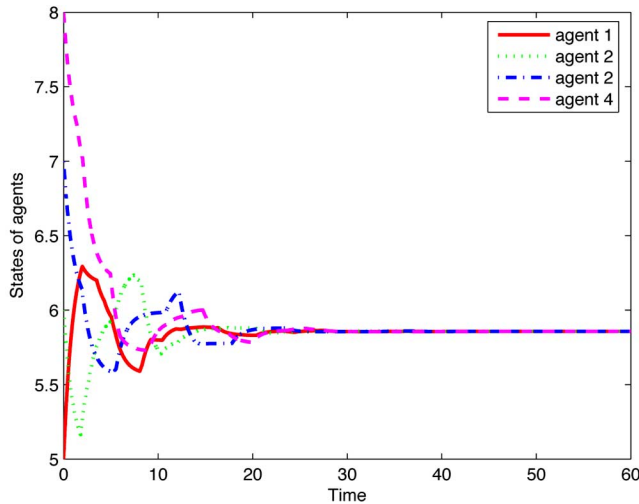


Fig. 9. State trajectories of agents by the-most-recent-data strategy with the topology shown in Fig. 3 and time-delays bounded by $\tau_d = 10$.

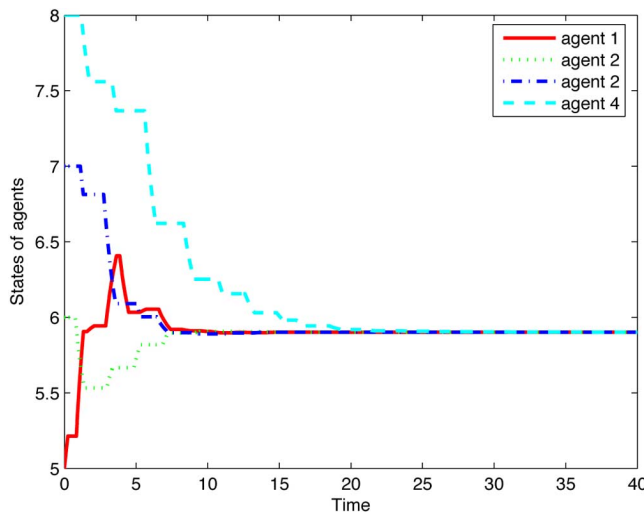
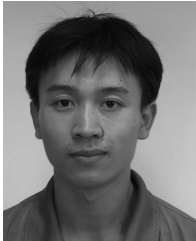


Fig. 10. State trajectories of agents of the system in Example 4 with switching topology.

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