Inverted pendulum

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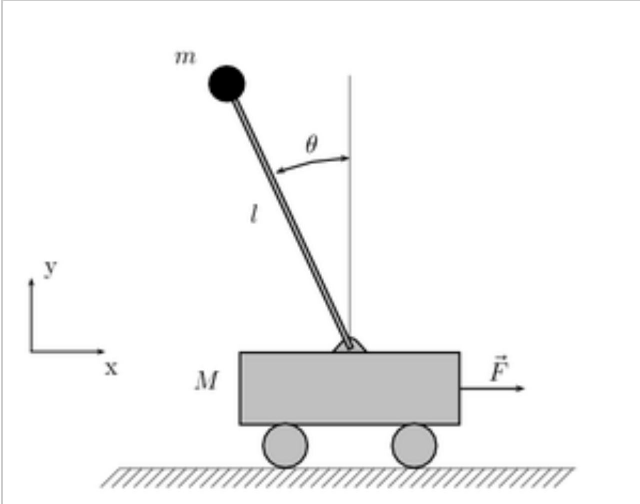
An **inverted pendulum** is a pendulum which has its mass above its pivot point. It is often implemented with the pivot point mounted on a cart that can move horizontally and may be called a **cart and pole**. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright, either by applying a torque at the pivot point or by moving the pivot point horizontally as part of a feedback system.



A schematic drawing of the inverted pendulum on a cart. The rod is considered massless. The mass of the cart and the pointmass at the end of the rod are denoted by M and m. The rod has a length l.

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Overview

The inverted pendulum is a classic problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms (PID controllers, neural networks, fuzzy control, genetic algorithms, etc.). Variations on this problem include multiple links, allowing the motion of the cart to be commanded while maintaining the pendulum, and balancing the cart-pendulum system on a see-saw. The inverted pendulum is related to rocket or missile guidance, where thrust is actuated at the bottom of a tall vehicle. The understanding of a similar problem is built in the technology of Segway, a self-balancing transportation device. The largest implemented use are on huge lifting cranes on shipyards. When moving the shipping containers back and forth, the cranes move the box accordingly so that it never swings or sways. It always stays perfectly positioned under the operator even when moving or stopping quickly.

Another way that an inverted pendulum may be stabilized, without any feedback or control mechanism, is by oscillating the support rapidly up and down. If the oscillation is sufficiently strong (in terms of its acceleration and amplitude) then the inverted pendulum can recover from perturbations in a strikingly counterintuitive manner. If the driving point moves in simple harmonic motion, the pendulum's motion is described by the Mathieu equation.

In practice, the inverted pendulum is frequently made of an aluminum strip, mounted on a ball-bearing pivot; the oscillatory force is conveniently applied with a jigsaw.

Equations of motion

Stationary pivot point

The equation of motion is similar to that for a uninverted pendulum except that the sign of the angular position as measured from the vertical unstable equilibrium position:

$$\ddot{\theta} - \frac{g}{\ell}\sin\theta = 0$$

When added to both sides, it will have the same sign as the angular acceleration term:

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

Thus, the inverted pendulum will accelerate away from the vertical unstable equilibrium in the direction initially displaced, and the acceleration is inversely proportional to the length. Tall pendulums fall more slowly than short ones.

Pendulum on a cart

The equations of motion can be derived easily using Lagrange's equations. Referring to the drawing where x(t) is the position of the cart, $\theta(t)$ is the angle of the pendulum with respect to the vertical direction and the acting forces are gravity and an external force in the x-direction, the Lagrangian $L = T - V$, where T is the kinetic energy in the system and V the potential energy, so the written out expression for L is:

$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{\mathrm{d}}{\mathrm{d}t}(\ell\cos\theta)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(x - \ell\sin\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\dot{x}\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}\left(M+m\right)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}} - \frac{\partial L}{\partial x} = F$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion of the inverted pendulum:

$$\left(M+m\right)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F$$
$$ml(-g\sin\theta - \ddot{x}\cos\theta + l\ddot{\theta}) = 0$$

These equations are nonlinear, but since the goal of a control system would be to keep the pendulum upright the equations can be linearized around $\theta \approx 0$.

Pendulum with oscillatory base

The equation of motion for a pendulum with an oscillatory base is derived the same way as with the pendulum on the cart, using the Lagrangian.
The position of the point mass is now given by:

$$\left(\ell\sin\theta, y + \ell\cos\theta\right)$$

and the velocity is found by taking the first derivative of the position:

$$v^2 = \dot{y}^2 - 2\ell\dot{\theta}\dot{y}\sin\theta + \ell^2\dot{\theta}^2.$$

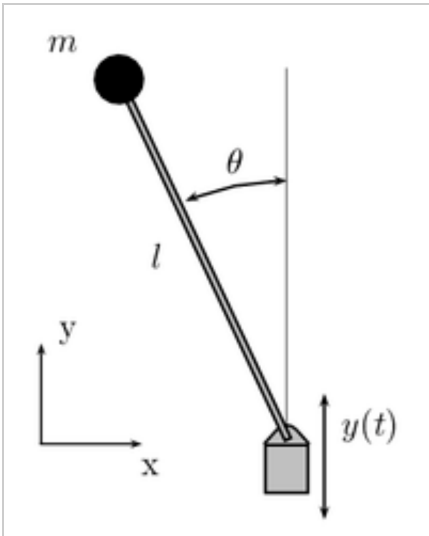
The Lagrangian for this system can be written as:

$$L = \frac{1}{2}m\left(\dot{y}^2 - 2\ell\dot{\theta}\dot{y}\sin\theta + \ell^2\dot{\theta}^2\right) - mg\left(y + \ell\cos\theta\right)$$

and the equation of motion follows from:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

resulting in:



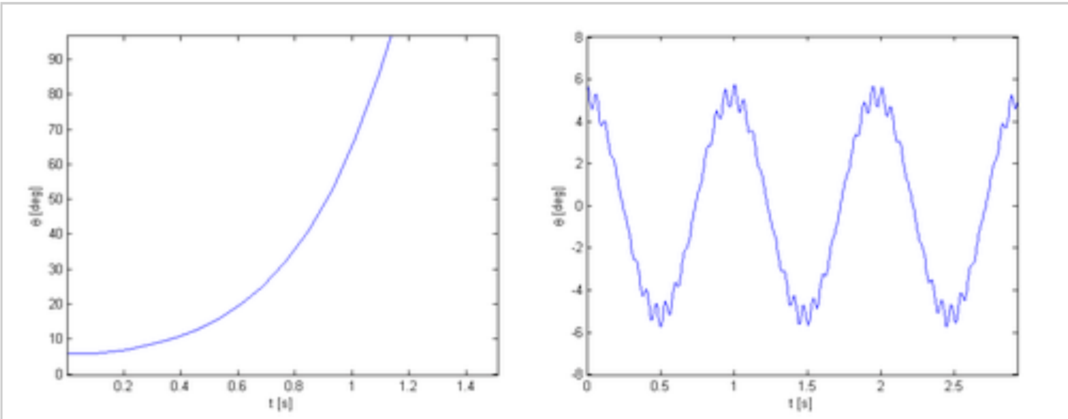
A schematic drawing of the inverted pendulum on an oscillatory base. The rod is considered massless. The pointmass at the end of the rod is denoted by m. The rod has a length l.

$$\ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta.$$

If y represents a simple harmonic motion, $y = a\sin\omega t$, the following differential equation is:

$$\ddot{\theta} - \frac{g}{\ell}\sin\theta = -\frac{a}{\ell}\omega^2\sin\omega t\sin\theta.$$

A solution for this equation will show that the pendulum stays upright for fast oscillations. The first plot shows that when y is a slow oscillation, the pendulum quickly falls over when disturbed from the upright position. The angle θ exceeds 90° after a short time, which means the pendulum has fallen on the ground.



Plots for the inverted pendulum on an oscillatory base. The first plot shows the response of the pendulum on a slow oscillation, the second the response on a fast oscillation

If y is a fast oscillation the pendulum can be kept stable around the vertical position. The second plot shows that when disturbed from the vertical position, the pendulum now starts an oscillation around the vertical position ($\theta = 0$). The deviation from the vertical position stays small, and the pendulum doesn't fall over.

Applications

The inverted pendulum was a central component in the design of several early Seismometers.[1]

See also

- Self-balancing unicycle
- Segway
- Double inverted pendulum
- Inertia wheel pendulum
- Furuta pendulum

References

1. ^ http://earthquake.usgs.gov/learn/topics/seismology/history/part12.php

- D. Liberzon *Switching in Systems and Control* (2003 Springer) pp. 89ff

Further reading

- Franklin; et al. (2005). Feedback Control of Dynamic Systems, 5, Prentice Hall. ISBN 0-13-149930-0

External links

- YouTube - inverted pendulum (http://www.youtube.com/watch?v=MWJHcI7UcuE)
- Vest - obrnjeno nihalo (http://www.vest.si/2007/10/12/obrnjeno-nihalo/) (in Slovene, and animated)
- A dynamical simulation of an inverse pendulum on an oscillatory base (http://mw.concord.org/modeler1.3/mirror/mechanics/inversependulum.html)
- Inverted pendulum modeling with several control solutions in Matlab (http://www.me.cmu.edu/ctms/controls/ctms/examples/pend/invpen.htm)