Active Control and Signal Processing *

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CONTENTS

* Handouts for the course of active control and signal processing that has been taught since 2009, autumn semester. This note is a preliminary version, which will ONLY be used as a reference material for the course study. More reading materials related to the course will be provided later in the corresponding lectures. I will keep working on the handout to refine the contents. Any suggestion/comment/criticisions are welcome.

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I. INTRODUCTION

This course introduces active control and signal processing techniques in particular for vibration and noise control applications. One of the main objectives of this course is to assist students gain insights of practical engineering applications and related control methods and electronic toolkits. Students are assumed to have a background in mathematics, physics, and engineering, and have taken at least one preliminary course of control.

Control is a science developed from engineering, in particular aerospace engineering, and maintains close relationships with electronics and information technologies. However, nowadays a control student normally has little knowledge of any practical engineering system and related physics. In addition, although the student may already know various control techniques in time-domain and frequency-domain, it is more than often that he has no idea of how to implement a control technique in a practical system. Instead, most of the previous experience stay in the numerical simulation stage.

To address the former issue, we select acoustics as the main physical topic in this course. Its control and measurement techniques will be extensively studied during the course. The reasons to select acoustics, rather than any other practical topics, are given below. Firstly, every student should be familiar with sound and therefore already prepare preliminary sense of the topic. Secondly, acoustics has a close connection with fluid and vibration. Generally speaking, the techniques used in vibration control and noise control are quite similar. On the other hand, a large part of noise is flow-induced, and fluid mechanisms research has a long history in our department. Thirdly, acoustic problem is important and can find many practical applications. Acoustic related problem, specifically, noise, is a lasting problem that annoys human beings for generations. Prolonged exposure to excessive levels of acoustic noise can cause permanent hearing loss, safety problems, and lower worker productivity. Finally, nowadays, noise control and measurements employ electronics, in particularly, digital signal processor (DSP) extensively. During this course, students are expected to grasp the most fundamental knowledge of the related electronic equipments and techniques. Through some lab works, we hope to improve the hands-on experience of our students and address the later concern raised in the previous paragraph.

Control technique was formally proposed in 1940s, and the so-called ’control’ actually implicitly denotes feedback control. In other research topics, such as fluid mechanisms, the subject of control includes a much broader scope, which includes passive control and active control. The latter can be categorised into feedforward control and feedback control, also known as open-loop and closed-loop control, respectively. The focus of the course is on active control.

Machinery and engines are major noise sources, and much of this noise occurs in the low frequency range. Passive methods used to reduce noise, such as earmuffs, are not especially effective at low frequencies due to the relatively long wavelength of the sound. However, active noise control (ANC) has proven its effectiveness at these low frequencies. The underlying principle of ANC is to generate a secondary sound wave to destructively interfere with the unwanted noise, thereby reducing the net sound pressure. Active noise control technology can be used in a wide variety of situations, including car or airplane headrests, automobile exhaust mufflers, and refrigerator fans. Headphones that employ ANC would be particularly useful for airport ramp workers, aircraft carrier soldier, ambulance drivers, and many others who operate on or near heavy machinery. Aside from low-frequency noise control, another benefit is that the offending noise, in some cases, can be selectively eliminated, leaving desired sounds such as speech and warning signals to be heard clearly.

This course will describe the implementation of adaptive control and its applications for active noise control. Similar technique should be able to find a usage in active vibration control and active flow control. The potential application fields are broad, which include military, aerospace, automation, etc. In addition, the availability of rapid developing programmable DSPs make the implementation of computationally-intensive adaptive algorithms feasible. The DSP related hardware and software techniques are also introduced in this course. A couple of lab projects have been designed to enable students to grasp fundamental ideas and gain hands-on experiences in embedded system design.

The nature of this course is theoretical with a specific engineering application in mind. The rest of this course will be organised as follows. Firstly, related mathematics and physics will be lectured. The sections not only provide preparation knowledge but could help students students to appreciate the beauty of mathematic methods in physical modelling. Secondly, signal processing theory will be summarised, although a preliminary background of it has been assumed in this course. Thirdly, DSP and peripheral hardware will be introduced. Students can conduct lab works thereafter. Fourthly, active control methods will be lectured. And finally, if time permitted, more signal processing methods will be presented. The second and the third parts are more practical. While the first, fourth, and final parts are theoretical. It is also worthwhile to point out that the writing style of this note is not strictly formal with detailed description of academic contents. Only the very fundamental knowledge will be given in this note for brevity. For more information of a topic of interest, we suggest students refer to numerous online literatures that can be easily googled and downloaded. Of course, the classroom lecturing will also complete this note and therefore don't miss any classes!

II. Mathematical Background

The physicist and Nobel Laureate Paul Dirac invented a calculus in the mid 30s of last century. The calculus uses mathematical symbols, which however are not well defined. A rigorous theory of this calculus, so-called generalised function theory, was later developed by Laurant Schwartz.

Physicists, and engineers, often work with an intuitive way similar to that of Dirac, knowing that their manipulations are strictly informal in mathematics but work for practical applications. The intuitive approach is followed in this course as well. A rigorous theory can be referred to the classical textbook written by Nobel.

A generalized function f is a map of $X \to R$ with X a good function space such that f is continuous and linear. If $\phi \in X$ is infinitely continuous differentiable and if ϕ, together with all of its derivatives, vanishes faster at infinity than any power function, the action of f is written

$$f(\phi) =< f, \phi > \int_R f(x)\phi(x)dx. \tag{1}$$

A Dirac's delta function $f = \delta$ is defined by

$$< \delta, \phi >= \int_R \delta(x)\phi(x)dx = \phi(0). \tag{2}$$

The general idea about the integral treats a point mass from a mathematical point. Let ϕ states a mass density. Then $\int_R \delta(x)\phi(x)dx$ states that a point mass is the mass concentrated at a single point. Besides the delta function, other generalised functions include functions with jumps, such as the maximum, or the step function.

Why are those generalised functions meaningful? Since ϕ is assumed to be a good function, then Dirac's delta, the max-function, the step function turn out to be difierentiable in a generalised sense. For short, we can take the derivative of a jump function. How is this possible?

Let f be a generalised function and $\phi \in X$. For the derivative, one can have

$$< \dot{f}, \phi >= - < f, \dot{\phi} >, \tag{3}$$

where $\dot{}$ denotes ordinary differential operation. The equations is satisfied because $< \dot{f}, \phi >= \int_R \dot{f}\phi dx = -\int_R f\dot{\phi}dx + \text{Boundary value} = - < f, \dot{\phi} >$, where the boundary term vanishes since ϕ is a rapidly decreasing function. Hence, we exploit the fact that operations, such as differentiation, Fourier Transform, originally conducted on generalised functions f can be shifted to the functions $\phi \in X$, which can be differentiated.

III. Fundamentals of Flow and Acoustics

A. Physical Background

(1)Fundamentals of fluid knowledge

$$\text{Reynolds number} \quad Re \quad = \frac{U_\infty l}{\nu}$$

(2)Fundamentals of sound knowledge

The speed of sound is 340 m/s. The audible frequency range is 20 Hz to $20,000$ Hz. The maximal sensible frequency to ear is $3,000$ Hz.

Wavelength: ..

Near field: ..

Far field: ..

The important definition of sound pressure level (SPL) is

$$L_p = 10\log_{10}\left(\frac{{p_{\mathrm{rms}}}^2}{{p_{\mathrm{ref}}}^2}\right) = 20\log_{10}\left(\frac{p_{\mathrm{rms}}}{p_{\mathrm{ref}}}\right) \text{ dB}. \tag{4}$$

Question: how to compute SPL from pressure? and vice versa?

B. Navier-Stokes Equations

The first principles behind the governing equations that describe fluid physics are: conservation of mass, conservation of momentum, and conservation of energy, etc. These lead to a set of partial, nonlinear differential equations plus appropriate boundary conditions and initial conditions.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) &= 0, &(5)\\ \rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left[\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \delta_{ki}\lambda\frac{\partial u_i}{\partial x_i}\right] &(6)\\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k}. \end{aligned}$$

The Einstein notation (Einstein summation convention) is used for the clarity of the equations. The energy equation follows the assumption of homentropic, i.e. the entropy s of the fluid is uniform and stationary throughout the fluid, namely $s =$contant. We may assume that the pressure and density are related by an equation of the form

$$p = p(\rho, s). \tag{7}$$

The details of symbols are below.

p	–	thermodynamic pressure, $N.m^2$.
μ and λ	–	1st and 2nd coefficients of viscosity, kg/ms.
σ_{ik}	–	viscous stress tensor.
δ_{ki}	–	Kronecker delta, $\delta_{ki} = 1$ if $k = i$, 0 otherwise.
T	–	temperature, K.
R	–	gas constant, Nm/kgK,
		= (Boltzman constant/mass of molecule = k/m).
Stokes hypothesis	–	$\lambda + \frac{2}{3}\mu = 0$.
State Equation	–	$p = \rho RT$.

C. *Euler Equations*

Outside boundary the fluid at low mach number and high Reynolds number could be invisicd.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \tag{8}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i \tag{9}$$

$$\rho c_v\left(\frac{\partial T}{\partial t} + u_k \frac{\partial T}{\partial x_k}\right) = -p\frac{\partial u_k}{\partial x_k} \tag{10}$$

D. *Acoustic Equation*

The acoustic disturbance is assumed small compared to the mean flow (recall the sound pressure at 140 dB). That is, $p = p_0 + p', u = u_0 + u'$, etc. Furthermore, let us focus on the simplest case with a stationary flow, where (1) $u_0 = 0$, (2) p_0 and ρ_0 are constant, and (3) $p' = c^2\rho'$. The effect due to gravity can be omitted as well. As a result, Euler equations can be linearised to

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'_k}{\partial x_k} = 0, \tag{11}$$

$$\rho_0 \frac{\partial u'_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0. \tag{12}$$

It is straight to obtain

$$\frac{\partial^2 p'}{\partial t^2} = c^2 \nabla^2 p', \tag{13}$$

where ∇^2 is the Laplacian operator, p' is the sound pressure disturbance.

The simplest case is one-dimensional wave equation,

$$\partial^2 p'/\partial t^2 = c^2 \partial p'/\partial x^2. \tag{14}$$

Following the method of functional separation of variables, the solution of the wave equation can be written in the form of $p'(x, y, z, t) = p'(x, y, z)e^{-j\omega t}$. For the one-dimensional case, in free space, following the cartesian coordinates , $p'(x) =$

$Ae^{-jkx} + Be^{jkx}$. Moreover, for the three-dimensional case the $p'(x) = Ae^{-jkr} + Be^{jkr}$. A and B depend on boundary conditions, and k is the wavenumber. (Present examples of monopole and dipole sources here).

E. *Green's Function*

(The prerequisite knowledge is generalized function.)

If there is a sound source $f(x, y, z, t)$ the wave equation takes inhomogeneous form,

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \nabla^2 p' = f(x, t). \tag{15}$$

For convenience of notation the Einsetin summation convention is used, that is, x denotes (x, y, z). The equation can be further written in a simplified form

$$L\, p' = f. \tag{16}$$

The main idea of Green's function is that given a $G(x, s)$ satisfying

$$L\, G(x, s) = \delta(x - s). \tag{17}$$

The solution $p'(x)$ can be found by

$$p'(x) = \int G(x, s) f(s) ds. \tag{18}$$

More specifically, for an impulse source appears at x_0 (location) and τ (time), the inhomogeneous equation is

$$L\, G(\mathbf{x}, \mathbf{x_0}, t, \tau) = \delta(\mathbf{x} - \mathbf{x_0})\delta(t - \tau). \tag{19}$$

The corresponding Green's function in a free space is

$$G(\mathbf{x}, \mathbf{x_0}, t, \tau) = \frac{1}{4\pi\, r}\delta(t - \tau - \frac{r}{c_0}), \tag{20}$$

where $r = |\mathbf{x} - \mathbf{x_0}|$.

Now given the acoustic source has the form of Q

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \int \int_{-\infty}^{\infty} Q(\mathbf{y}, \tau) G(\mathbf{x}, \mathbf{y}, t, \tau) d^3\, \mathbf{y} d\tau. \tag{21}$$

For the particular case in the free space:

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \int \int_{-\infty}^{\infty} \frac{Q(\mathbf{y}, \tau)}{|\mathbf{x} - \mathbf{y}|} \delta\left(t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0}\right) d^3\, \mathbf{y} d\tau = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{Q(\mathbf{y}, t - |\mathbf{x} - \mathbf{y}|/c_0)}{|\mathbf{x} - \mathbf{y}|} d^3\, \mathbf{y} d\tau. \tag{22}$$

F. *Acoustic Analogy*

Multiplying Eq. (5) by u_i and add to Eq. (6), we have:

$$\frac{\partial(\rho u_i)}{\partial t} = -\frac{\partial \pi_{ik}}{\partial x_k}, \tag{23}$$

where $\pi_{ik} = \rho u_i u_k + (p - p_0)\delta_{ik} - \sigma_{ik}$.

The continuity equation can be written to:

$$\frac{\partial(\rho - \rho_0)}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0. \tag{24}$$

Eliminate ρu_i by $\frac{1}{\partial t}$(Eq. (24))+ $\frac{1}{\partial x_k}$(Eq. (23)), and minus $\nabla^2[c_0^2(\rho - \rho_0)]$:

$$(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2)[c_0^2(\rho - \rho_0)] = \frac{\partial^2 T_{ik}}{\partial x_i \partial x_k}, \tag{25}$$

where Lighthill's stress tensor $T_{ij} = \pi_{ij} - [c_0^2(\rho - \rho_0)]\delta_{ij} = \rho u_i u_k + (p - p_0)\delta_{ij} - \sigma_{ij} - [c_0^2(\rho - \rho_0)]\delta_{ij}$. If we restrict ourself to a finite region of unsteady flow, and T_{ij} is assumed negligible outside the region. That is, viscous effect is omitted, hence σ_{ik} is removed. The sound speed outside the unsteady region is considered constant and equals c_0. The sound perturbations everywhere almost satisfy $(p - p_0) - [c_0^2(\rho - \rho_0)]$ (the error is M^2, M Mach number). As a result, $T_{ij} \simeq \rho u_i u_k$. Moreover, outside the unsteady flow region, $\rho u_i u_k$ is negligible again, and $T_{ij} \simeq 0$. Therefore, only a small region with unsteady flows can be considered as a sound source area.

G. *Surface Effects*

A surface immersed in the fluid could be defined by

$$S(\mathbf{x}, t) = 0, \tag{26}$$

while $S(\mathbf{x}, t) > 0$ denotes the area outside the surface, and vice versa. Provided an observer moves along with the motion of the surface, we can have

$$\frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} = 0. \tag{27}$$

The generalised mass and momentum equations which account for the presence of the surface are derived. It is worthwhile to mention that the fluctuation variables are of interest here, that is $(\rho - \rho_0)H(S) = \rho' H(S)$. The Navier-Stokes equations can be rewritten to

$$\begin{aligned}
\frac{\partial[(\rho - \rho_0)H(S)]}{\partial t} + \frac{\partial}{\partial x_k}[\rho u_k H(S)] &= (\rho - \rho_0)\frac{\partial H(S)}{dt} + \rho u_k \frac{\partial H(S)}{\partial x_k} \\
&= (\rho - \rho_0)\delta(S)\frac{\partial S}{dt} + \rho u_k \delta(S)\frac{\partial S}{\partial x_k} \\
&= \underbrace{(\rho_0 v_i + \rho(u_k - v_i))\frac{\partial S}{\partial x_k}}_{Q}\delta(S), \qquad (28) \\
\frac{\partial[\rho u_i H(S)]}{\partial t} + \frac{\partial}{\partial x_k}[(\rho u_i u_k + p_{ik})H(S)] &= \underbrace{p_{ik} + \rho u_i(u_k - v_k)\frac{\partial S}{\partial x_k}}_{F_i}\delta(S). \qquad (29)
\end{aligned}$$

Following the way that forms Lighthill's wave equation previously, we can have

$$(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2)[(\rho - \rho_0)H(S)] = \frac{\partial^2}{\partial x_i \partial x_k}[T_{ik}H(S)] - \frac{\partial}{\partial x_i}[F_i\delta(S)] + \frac{\partial}{\partial t}[Q\delta(S)]. \tag{30}$$

For many applications the surface S is impermeable, that means the normal velocity of **v** and **u** should be equal. In addition, the normal unit **n** of the surface S is $(\partial S/\partial x_i)/|\nabla S|$, we can have

$$\rho(u_i - v_i)\frac{\partial S}{\partial x_i} = \rho|\nabla S|n_i(u_i - v_i) = 0, \tag{31}$$

$$\rho u_i(u_i - v_i)\frac{\partial S}{\partial x_i} = \rho u_i|\nabla S|n_i(u_i - v_i) = 0, \tag{32}$$

Only $\rho_0 v_i$ survives in Q, and p_{ij} survives in F.

The farfield noise related to the monopole source Q can be described by

$$\begin{aligned} p - p_0 &= \frac{1}{4\pi x}\frac{\partial}{\partial t}\int_V [Q\delta(S)](\mathbf{y}, \tau) \times \delta(t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0})d\mathbf{y}d\tau \\ &= \frac{1}{4\pi x}\frac{\partial}{\partial t}\int_S \rho_0 v_n(\mathbf{y}^*, \tau)\delta(t - \tau - \frac{|\mathbf{x} - \mathbf{y}^*|}{c_0})d\mathbf{y}^* d\tau \\ &= \frac{1}{4\pi x}\frac{\partial}{\partial t}\int_S \rho_0 v_n(\mathbf{y}^*, t - \tau - \frac{|\mathbf{x} - \mathbf{y}^*|}{c_0})d\mathbf{y}^* \\ &= \frac{1}{4\pi x}\frac{\partial}{\partial t}\int_S \rho_0 v_n(\mathbf{y}^*, t - \tau - \frac{x}{c_0})d\mathbf{y}^*, \end{aligned} \tag{33}$$

where $x \to \infty$, $\partial/\partial t \sim u/l$, $d\mathbf{y} \sim b^2$, u is the characteristic velocity of turbulence, b is the characteristic length of the surface. Hence, $p = p_0$ at x is on the order of v^2, where v is the characteristic speed of turbulence. Remind previously in Lighthill's analogy without the presence of the surface, $p = p_0$ at x is on the order of v^4.

IV. FUNDAMENTALS OF SIGNAL PROCESSING

* Reference material: “Introduction to Fourier analysis and generalised functions” by M. J. Lighthill

A. Convolution

The definition of the convolution is:

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau)\, d\tau, \tag{34}$$

where $*$ denotes the operation of convolution.

The discrete version is:

$$(f * g)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m] \cdot g[n-m] = \sum_{m=-\infty}^{\infty} f[n-m] \cdot g[m]. \tag{35}$$

B. Autocorrelation

The definition of autocorrelation below is proposed from the angle of signal processing.

$$R_{ff}(t) = (f \star f)(t) = f^*(-t) * f(t) = \int_{-\infty}^{\infty} f^*(\tau) \cdot f(t+\tau)\, d\tau. \tag{36}$$

where f^* is the complex conjugate of f, $\star$ denotes correlation and $*$ represents convolution.

C. Cross-correlation

The definition of cross-correlation below is proposed from the angle of signal processing.

$$R_{fg}(t) = (f \star g)(t) = f^*(-t) * g(t) = \int_{-\infty}^{\infty} f^*(\tau) \cdot g(t+\tau)\, d\tau. \tag{37}$$

For discrete functions, the definition is

$$(f \star g)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f^*[m]\ g[n+m]. \tag{38}$$

D. Fourier Series

For a T-periodic function $f(t)$ that is integrable on $[-T/2, T/2]$, there is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)], \tag{39}$$

where

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t)\, dt, \quad n \geqslant 0, \tag{40}$$

$$b_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t)\, dt, \quad n \geqslant 1. \tag{41}$$

$\omega = 2\pi f = 2\pi/T$.

The other well known representation of the Fourier series is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \tag{42}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega t}\, dt. \tag{43}$$

Note:

$$\int_{-T/2}^{T/2} e^{-in\omega t} e^{-im\omega t}\, dt = 0, n \neq m, \tag{44}$$

which is termed orthogonality. It can be shown that the basic elements (sinusoid waves etc) of a Fourier series satisfy this condition. This properties is sued to prove Fourier series relation. In addition, It is easy to see that the Fourier integral above can be regarded as the formal limit of the Fourier series as the period approaches infinity.

Example: $\Delta_T(t)$ denotes the sampling operator, which has the form of

$$\Delta_T(t) = T \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{45}$$

which can be represented by Fourier series

$$\Delta_T(t) = \sum_{n=-\infty}^{\infty} e^{in\omega t}. \tag{46}$$

In summary, a periodical signal is presented by Fourier series as a sum of simpler components (sinusoid and cosines) whose frequencies are related and whose amplitude are chosen to represent the signal. On the other hand, the sufficient condition to guarantee the existence and convergence of Fourier series is Dirichlet condition, which states that if a periodical signal $x(t)$ is piecewise continuous and has a left and a right hand derivatives in this interval then its Fourier series converges and the sum is $x(t)$. Here a signal is considered as piecewise continuous if it is continuous on all but at a finite number of points. More specifically, $x(t)$ should have only finite number of minima and maxima, finite number of discontinuities. The other important thing in Fourier analysis is Gibbs phenomenon, which describes a peculiar manner when $x(t)$ is approximated by a partial sum of Fourier series. A considerable error can be found in the vicinity of a discontinuity irrespective of how many terms are included.

E. Fourier Transform

For an integrable function $f(t)$ (i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$), the definition of the Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\ e^{-i\omega t}\, dt. \tag{47}$$

The definition of the inverse Fourier transform is

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\ e^{i\omega t}\, d\omega. \tag{48}$$

It is important to grasp the importance of the above equations. One tells us how the ’energy’ of $f(t)$ is continuously distributed in the frequency domain. On the other hand, it shows how the waveform may be synthesised from an infinite set of exponential functions, each weighted by the relevant value of $F(\omega)$.

One example: for a rectangular pulse $\mathrm{rect}(t/T)$, which is

$$\mathrm{rect}(t) = \sqcap(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2T} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2T} \\ 1 & \text{if } |t| < \frac{1}{2T}. \end{cases} \tag{49}$$

Then the Fourier transform pair is $\frac{\sin(\omega T)}{\omega T}$.

The other example: for Dirac delta function $\delta(t)$, its Fourier transform pair is $H(\omega) = \int_{-\infty}^{\infty} \delta(t)\ e^{-i\omega t}\, dt = 1$. While its inverse Fourier transform pair is $\delta(t) = \int_{-\infty}^{\infty} H(\omega)\ e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty}\ e^{i\omega t}\, d\omega$. Given $F(\omega) = \int_{-\infty}^{\infty} f(t)\ e^{-i\omega t}\, dt$, we can have $X(\xi)$ by multiply $e^{-j\xi t}$ and integrate

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\ e^{-i\xi t}\, dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right)\ e^{-i\xi t}\, dt \\ &= \int_{-\infty}^{\infty} F(\omega) \left(\int_{-\infty}^{\infty} e^{j(\omega-\xi)t} dt \right)\, d\omega \\ &= \int_{-\infty}^{\infty} F(\omega) \delta(\omega - \xi)\, d\omega \\ &= F(\xi). \end{aligned} \tag{50}$$

One important properties of Fourier transform is Parseval's theorem, which is

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{51}$$

F. *Discrete-Time Fourier Transform*

For a discrete set $f[n] = f(nT)$, the DTFT of $f[n]$ is:

$$F(\omega) = \sum_{n=-\infty}^{\infty} f[n]\, e^{-i\omega n}, \tag{52}$$

where $\omega = 2\pi f T \in [-\pi, \pi)$ denotes the continuous normalized radian frequency variable, and $1/T$ is sampling rate. The inverse DTFT one is:

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{i\omega n}\, d\omega. \tag{53}$$

G. *Discrete Fourier Transform*

The DFT for a sequence with N numbers is:

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-\frac{2\pi i}{N} kn} \qquad k = 0, \ldots, N-1. \tag{54}$$

The inverse DFT is given by:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} \qquad n = 0, \ldots, N-1. \tag{55}$$

H. Laplace Transform

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\, dt. \tag{56}$$

Make sure that you know the difference between Laplace transform and Fourier transform.

V. Important Elements

A. Causality

Given $h(t)$ is the impulse response of a system, we can have

$$h(t) = 0, \quad \forall\ t < 0. \tag{57}$$

B. Relationships

$$\begin{aligned} f(x) &\leftrightarrow\ F(\omega) \\ f(x-a) &\leftrightarrow\ e^{-ia\omega}\hat{f}(\omega) \\ f(ax) &\leftrightarrow\ \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right) \\ \frac{d^n f(x)}{dx^n} &\leftrightarrow\ (i\omega)^n \hat{f}(\omega) \end{aligned} \tag{58}$$

C. Convolution Theorem

The convolution theorem states:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}, \tag{59}$$

$$\mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}, \tag{60}$$

where $\mathcal{F}\{f\}$ denotes the Fourier transform of f.

D. Aliasing

Provided the sampling rate is $f_s = 1/T_s$, the sampling signal $x[n]$ of $x(t)$ is

$$x[n] = x(nT_s) = x(t) \cdot \Delta_{Ts}(t) = x(t) \cdot T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{61}$$

which can be further represented using Fourier series

$$x[n] = x(t) \cdot \sum_{n=-\infty}^{\infty} e^{in\omega_s t}. \tag{62}$$

where $\omega_s = 2\pi f_s$.

Applying continuous Fourier transform on Eq. (63),

$$
\begin{aligned}
\mathcal{F}\{x[n]\} &= \int_{-\infty}^{\infty} x[n]\ e^{-i\omega t}\, dt \\
&= \sum_{n=-\infty}^{\infty} [\int_{-\infty}^{\infty} x(t) e^{in\omega_s t}\ e^{-i\omega t}\, dt] \\
&= \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)
\end{aligned} \tag{63}
$$

E. Application of Fourier Transform-Spectral Analysis

F. Application of Fourier Transform-FFT

G. Application of Fourier Transform-Pseudospectral

VI. Fundamentals of Active Control Theory

This section introduces the fundamentals of active control and beyond.

A. Introduction

Active control involves driving a number of actuators to create a sound or vibration signal out of phase with that generated by the vehicle, thus attenuating it by destructive interference. Here the term of vehicle can denote a car, an aircraft or any machine. The successful application of active control requires that there is both a good spatial and a good temporal matching between the field due to the actuators, or secondary sources, and that due to the vehicle. In other words, active control needs a clear physical insight of the original noise, and imposes high requirement for a linear actuator and a suitable active control algorithm.

The requirement for spatial matching gives rise to clear limits on the upper frequency of active noise control, due to the physical requirement that the acoustic wavelength must be small compared with the zone of control. The requirement for temporal matching requires a signal processing system that can adapt to changes in the vehicle speed and load. Both the physical limitations and the signal processing control strategies will be described in this report, together with a description of some of the practical systems that have found their way into production at the time of writing. Active noise and vibration control can provide a useful alternative to passive noise and vibration control, particularly at low frequencies and on vehicles with particular problems. Although active control has been experimentally demonstrated in vehicles for over 20 years, it is only recently that the levels of integration within the vehicles electronic systems have allowed the cost to become affordable, particularly due to the rapid development of digital processor techniques. Active control may now allow a reduction in the weight of conventional passive methods of low frequency noise control, helping the push towards lighter, more fuel efficient, vehicles.

B. Preparation

1) Method of Lagrangian Multipliers:* In mathematical optimization, the method of Lagrange multipliers provides a strategy for finding the maxima and minima of a function subject to constraints.

For instance, consider the optimization problem:

$$\text{maximise } f(x,y) \text{ subject to } g(x,y)=c. \tag{64}$$

We introduce a new variable (λ) called a Lagrange multiplier, and study the Lagrange function defined by

$$\Lambda(x,y,\lambda)=f(x,y)+\lambda\cdot\Big(g(x,y)-c\Big). \tag{65}$$

* Referred wikipedia.

2) Operations on Matrices:

$$\frac{\partial \mathbf{b}^T\theta}{\partial\theta} = \mathbf{b}, \tag{66}$$

$$\frac{\partial \theta^T\mathbf{A}\theta}{\partial\theta} = 2\mathbf{A}\theta. \tag{67}$$

where A is a symmetrical matrix.

3) The Matrix Inversion Formula: Provided

$$\mathbf{A} = \mathbf{B}^{-1} + \mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T, \tag{68}$$

we can have

$$\mathbf{A}^{-1} = \mathbf{B} + \mathbf{B}\mathbf{C}(\mathbf{D} + \mathbf{C}^T\mathbf{B}\mathbf{C})^{-1}\mathbf{C}^T\mathbf{B}. \tag{69}$$

4) Independent and Uncorrelated:

C. Wiener Filters

1) Introduction: A Wiener filer can restore a corrupted signal based on a statistical approach. Specifically, given a signal $s(t)$, which is sampled by an ADC and is unavoidable corrupted by a noise $n(t)$, a Wiener filter, $a(t)$, can be constructed to filter out the noise and obtain an estimated signal $x(t)$ using the following convolution:

$$y(t) = a(t) * (x(t) + n(t)), \tag{70}$$

where $x(t) + n(t)$ can be denoted by $w(t)$. Define $e(t) = (x(t+\alpha) - y(t))$, where α is the delay between measurements, we have

$$e^2(t) = x^2(t+\alpha) - 2x(t+\alpha)y(t) + y^2(t). \tag{71}$$

Remind that

$$y(t) = \int_{-\infty}^{\infty} a(\tau)\left[x(t-\tau) + n(t-\tau)\right] d\tau, \tag{72}$$

and

$$R_{ff}(\tau) = \mathrm{E}\left[f(t)\overline{f}(t-\tau)\right], \tag{73}$$

we can have

$$\begin{aligned} E(e^2) &= R_{xx}(0) - 2\int_{-\infty}^{\infty} a(\tau)x(t+\alpha)w(t-\tau)\,d\tau + E(\int_{-\infty}^{\infty} a(\tau)w(t-\tau)\,d\tau)^2 \\ &= R_{xx}(0) - 2\int_{-\infty}^{\infty} a(\tau)R_{wx}(\tau+\alpha)\,d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\tau)a(\theta)R_{ww}(\tau-\theta)\,d\tau\,d\theta. \end{aligned} \tag{74}$$

A Wiener filter can be designed by finding $a(t) = \arg\min(E(e^2))$, where a_i is a value at which $E(e^2)$ is minimised, in other words, $\arg\min f(x) := \{x \mid \forall y : f(y) \geqslant f(x)\}$.

2) Implementations: Due to causality, $w(t+\tau)(\tau > 0)$ is unknown, $a(\tau) = 0$ when $\tau < 0$. In addition, only a discrete filter with finite length can be implemented. A discrete Wiener filter with N th length is discussed below.

$$y[n] = \sum_{i=0}^{N} a_i w[n-i], \tag{75}$$

where the values of a_i can be obtained by

$$a_i = \arg\min \; E\{e^2[n]\}. \tag{76}$$

$$\begin{aligned} E\{e^2[n]\} &= E\{(y[n]-x[n])^2\} \\ &= E\{y^2[n]\} + E\{x^2[n]\} - 2E\{x[n]y[n]\} \\ &= E\{\big(\textstyle\sum_{i=0}^{N} a_i w[n-i]\big)^2\} + E\{x^2[n]\} - 2E\{\textstyle\sum_{i=0}^{N} a_i w[n-i]x[n]\}. \end{aligned} \tag{77}$$

Calculate its derivatives we have

$$\begin{aligned} \tfrac{\partial}{\partial a_i} E\{e^2[n]\} &= 2E\{\big(\textstyle\sum_{j=0}^{N} a_j w[n-j]\big) w[n-i]\} - 2E\{x[n]w[n-i]\} \quad i = 0,\ldots,N \\ &= 2\textstyle\sum_{j=0}^{N} E\{w[n-j]w[n-i]\}a_j - 2E\{w[n-i]x[n]\}. \end{aligned} \tag{78}$$

Assuming that both $w[n]$ and $x[n]$ are stationary, the autocorrelation of $w[n]$ and the cross-correlation between $w[n]$ and $x[n]$ can be defined as below:

$$R_{ww}[m] = E\{w[n]w[n+m]\} \qquad R_{wx}[m] = E\{w[n]x[n+m]\}. \tag{79}$$

The derivatives of $E\{e^2[n]\}$ therefore is:

$$\frac{\partial}{\partial a_i} E\{e^2[n]\} = 2\sum_{j=0}^{N} R_{ww}[j-i]a_j - 2R_{xw}[i] \quad i = 0,\ldots,N \tag{80}$$

Letting the derivatives to zero for the minimisation of $E\{e^2[n]\}$, we have

$$\sum_{j=0}^{N} R_{ww}[j-i]a_j = R_{xw}[i] \quad i = 0,\ldots,N \tag{81}$$

which can be written in a complete form (matrix form):

$$\underbrace{\begin{bmatrix} R_{ww}[0] & R_{ww}[1] & \cdots & R_{ww}[N] \\ R_{ww}[1] & R_{ww}[0] & \cdots & R_{ww}[N-1] \\ \vdots & \vdots & \ddots & \vdots \\ R_{ww}[N] & R_{ww}[N-1] & \cdots & R_{ww}[0] \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} R_{xw}[0] \\ R_{xw}[1] \\ \vdots \\ R_{xw}[N] \end{bmatrix}}_{V} \tag{82}$$

where T is a symmetric Toeplitz matrix (that is, if the ij th element of the matrix M is m_{ij}, then we have $m_{ij} = m_{i-1,j-1}$), it must be positive definite and invertible. Hence $A = T^{-1}V$. Eq. (82) is the Wiener-Hopf equation.

3) Practical issues: Firstly, the expectations of $R_{ww}[m]$ and $R_{ws}[m]$ are actually computed by

$$R_{ww}[m] = \frac{1}{N}\sum_{j=0}^{N}(w[n]w[n+m]) \qquad R_{ws}[m] = \frac{1}{N}\sum_{j=0}^{N}(w[n]s[n+m]). \tag{83}$$

It also works for slowly varying R_{ww} and R_{ws}, but an appropriate of N should be chosen, based on the sampling rate and the varying procedure.

The computational complexity of the Wiener filter is $O(N^2)$.

D. Least Mean Squares Filter

1) Introduction: The method of steepest descent is used in designing an LMS filter. The core problem is to find a coefficient vector $\mathbf{h}(n)$ which minimizes a cost function:

$$C(n) = E\left\{|e(n)|^2\right\}, \tag{84}$$

where $e(n) = [\mathbf{h}(n) - \hat{\mathbf{h}}(n)] * x(n) + v(n)$, $\mathbf{h}(n)$ denotes the unknown system, $v(n)$ is the interference particularly in measurements, and $\hat{\mathbf{h}}(n)$ are the coefficients of the LMS filter.

$$\nabla C(n) = \nabla E\left\{e(n)\, e^*(n)\right\} = 2E\left\{\nabla(e(n))\, e^*(n)\right\} \tag{85}$$

where $\nabla e(n) = \partial e(n)/\partial \hat{\mathbf{h}}(n) = -\mathbf{x}(n)$. Note the Einstein notation (or Einstein summation convention) is used here. Hence,

$$\nabla C(n) = -2E\left\{\mathbf{x}(n)\, e^*(n)\right\} \tag{86}$$

Keep in mind that $\nabla C(n)$ points towards the steepest ascent of the cost function. The optimal coefficients of $\hat{\mathbf{h}}$ can therefore be found by

$$\hat{\mathbf{h}}(n+1) = \hat{\mathbf{h}}(n) - \frac{\mu}{2}\nabla C(n) = \hat{\mathbf{h}}(n) + \mu\, E\left\{\mathbf{x}(n)\, e^*(n)\right\}. \tag{87}$$

By far LMS is the most commonly used adaptive filering algorithm, due to the following reasons:

(1) the algorithm is fast;

(2) the algorithm is the first one of its kind;

(3) its computational cost is small (to be discussed in next section);

(4) recursive algorithm, can work adaptively, that is, real-time;

(5) track slow changes in the signal statistics.

2) Practical issues: Firstly, how to obtain $E\left\{\mathbf{x}(n)\, e^*(n)\right\}$?

Secondly, what is the computational cost of the LMS algorithm?

TABLE I: The computational cost of LMS

Operations	*	$\hat{\mathbf{h}}(n+1)$
multiplication	M	M

(Note: addition cost is omitted here. Why?)

3) Variant of LMS: NLMS: Normalised LMS filter converges fast as its input range is limited to a normalised range.

E. Recursive Least Squares Filter

1) Introduction: Given the input samples $u(N)$ and the desired response $d(N)$, design a linear filter

$$y(n) = \sum_{k=0}^{M} w_k u(n-k). \tag{88}$$

Search w_k recursively to minimise the sum of error squares

$$E(n) = \sum_{i=0}^{n} \beta(n,i) e(i)^2. \tag{89}$$

where $E(n)$ is the cost function, $e(i)$ the error signal that is

$$e(i) = d(i) - y(i) = d(i) - \sum_{k=0}^{M} w_k u(i-k). \tag{90}$$

$\beta(n,i)$ is the forgetting factor that reducing the influence of old data, usually takes the form of

$$\beta(n,i) = \lambda^{n-i} \qquad 0 < \lambda < 1. \tag{91}$$

The cost function $E(n)$ is minimised when all its partial derivatives is equal to zero

$$\frac{\partial E(\mathbf{w})}{\partial w_m} = \sum_{i=0}^{n} \lambda^{n-i} e(i) \frac{\partial e(i)}{\partial w_m} = \sum_{i=0}^{n} \lambda^{n-i} e(i)\, u(i-m) = 0. \tag{92}$$

Replace $e(n)$ with the definition of the error signal

$$\sum_{i=0}^{n} \lambda^{n-i} \left[d(i) - \sum_{k=0}^{M} w_k u(i-k) \right] u(i-m) = 0. \tag{93}$$

Rearranging the previous equation yields

$$\sum_{k=0}^{M} w_k \left[\sum_{i=0}^{n} \lambda^{n-i}\, u(i-k) u(i-m) \right] = \sum_{i=0}^{n} \lambda^{n-i} d(i) u(i-m), \tag{94}$$

which can be expressed in a matrix form

$$\mathbf{R}_x(n)\, \mathbf{w}_n = \mathbf{r}_{dx}(n). \tag{95}$$

2) Algorithm:

$$\mathbf{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-p) \end{bmatrix} \tag{96}$$

$$\alpha(n) = d(n) - \mathbf{w}(n-1)^T \mathbf{x}(n) \tag{97}$$

$$\mathbf{g}(n) = \mathbf{P}(n-1)\mathbf{x}^*(n) \left\{\lambda + \mathbf{x}^T(n)\mathbf{P}(n-1)\mathbf{x}^*(n)\right\}^{-1} \tag{98}$$

$$\mathbf{P}(n) = \lambda^{-1}\mathbf{P}(n-1) - \mathbf{g}(n)\mathbf{x}^T(n)\lambda^{-1}\mathbf{P}(n-1) \tag{99}$$

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \alpha(n)\mathbf{g}(n) \tag{100}$$

F. Kalman Filter

1) Lemma: best linear unbiased estimator: Suppose unknown θ is linearly related to a noisy input $\mathbf{b}$ by

$$\mathbf{b} = \mathbf{A}\theta + e \qquad e \sim N(0, V). \tag{101}$$

The BLUE of θ is

$$\mathbf{b} = (\mathbf{A}^T V^{-1} \mathbf{A})^{-1} \mathbf{A}^T V^{-1} \mathbf{b}. \tag{102}$$

2) Derivation: In Kalman filter, we have two inputs at each time instance t, one being the prediction from previous estimations

$$\hat{x}_t^- = x_t + e_t^-, e_t^- \sim N(0, Q), \tag{103}$$

and the other from the present measurement (innovation?)

$$y_t = Cx_t + v_t, v_t \sim N(0, R). \tag{104}$$

Combine these two together we have

$$\begin{pmatrix} \hat{x}_t^- \\ y_t \end{pmatrix} = \begin{pmatrix} I \\ C \end{pmatrix} x_t + \begin{pmatrix} e_t^- \\ v_t \end{pmatrix} \qquad \begin{pmatrix} e_t^- \\ v_t \end{pmatrix} \sim N(0, \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}) \tag{105}$$

Applying BLUE we have

$$\hat{x}_t = P_t \begin{pmatrix} I \\ C \end{pmatrix}^T \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}^{-1} \begin{pmatrix} \hat{x}_t^- \\ y_t \end{pmatrix}, \tag{106}$$

where

$$P_t = \left[\begin{pmatrix} I \\ C \end{pmatrix}^T \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}^{-1} \begin{pmatrix} I \\ C \end{pmatrix} \right]^{-1}. \tag{107}$$

$$\begin{aligned} \hat{x}_t &= P_t(Q^{-1}\hat{x}_t^- + C^T R^{-1} y_t) \\ P_t &= (Q^{-1} + C^T R^{-1} C)^{-1}. \end{aligned}$$

$$\begin{aligned} K_t &= [P_t^- - P_t^- H_t'(H_t P_t^- H_t' + R_t)^{-1} H_t P_t^-] H_t' R ...... - H_t P_t^- H_t'] R_t^{-1} \\ &= P_t^- C'(C P_t^- C' + R)^{-1}, \end{aligned}$$

$$\begin{aligned} P_t &= P_t^- - P_t^- H_t'(H_t P_t^- H_t' + R_t)^{-1} H_t P_t^- \\ &= P_t^- - K_t H_t P_t^- \\ &= (I - K_t H_t) P_t^-. \end{aligned}$$

G. *Counterparts in Control*

Adaptive control involves modifying the control law used by a controller to cope with the fact that the parameters of the system being controlled are slowly time-varying or uncertain. For example, as an aircraft flies, its mass will slowly decrease as a result of fuel consumption; we need a control law that adapts itself to such changing conditions. Adaptive control is different from robust control in the sense that it does not need a priori information about the bounds on these uncertain or time-varying parameters; robust control guarantees that if the changes are within given bounds the control law need not be changed, while adaptive control is precisely concerned with control law changes. Adaptive control includes gain scheduling and Model Reference Adaptive Controllers (MRAC). The latter is the focus here.

MRAC method is firstly proposed in MIT at 1950s for agile aircraft control design. The MIT method is explained briefly below. Given a process $y = kG(s)u$, where k is a constant and unknown, u is the control input, the desired response is $y_m = k_0 G(s) u_c$, a controller should be designed to minimise the difference between y and y_m through adjusting u. Suppose $u = \theta u_c$, a cost function can be constructed, it is $J(\theta) = 0.5e^2$, where $e = y - y_m = kG(s)\theta u_c - k_0 G(s) u_c$. As aforementioned, it is reasonable to adjust the parameter θ in the direction of the negative gradient of J:

$$\frac{d\theta}{dt} = -\gamma \frac{\partial J}{\partial \theta} = -\gamma e \frac{\partial e}{\partial \theta}. \tag{108}$$

We also have

$$\frac{\partial e}{\partial \theta} = kG(s) u_c = \frac{k}{k_0} y_m. \tag{109}$$

Hene the MIT rule of adaptive control is

$$\frac{d\theta}{dt} = -\gamma y_m e. \tag{110}$$

VII. BEAMFORMING

$$p = \frac{1}{4\pi r} e^{-j\omega r/c} p_0. \tag{111}$$

$$p_1 = \frac{1}{4\pi r_1} e^{-j\omega r_1/c} p_0. \tag{112}$$

$$p_2 = \frac{1}{4\pi r_2} e^{-j\omega r_2/c} p_0. \tag{113}$$

$$p_0 = \frac{1}{2}(\underbrace{4\pi r_1 e^{j\omega r_1/c}}_{w_1} p_1 + \underbrace{4\pi r_2 e^{j\omega r_2/c}}_{w_2} p_2). \tag{114}$$

A. Sound field model

The notations generally appeared in literature [1], [2] are adopted below. Given a microphone array with M microphones, the output $\mathbf{x}(t)$ denotes time domain measurements of microphones, $\mathbf{x} \in \Re^{M\times 1}$ and t denotes time. For a single signal of interest $s(t) \in \Re^1$ in a free sound propagation space, using Green's function for the wave equation, we can have

$$\mathbf{x}(t) = \frac{1}{4\pi \mathbf{r}} s(t-\tau), \tau = \frac{\mathbf{r}}{C}. \tag{115}$$

where C is the speed of sound, $\mathbf{r} \in \Re^{m\times 1}$ are the distances between the signal of interest s and microphones, and τ is the related sound propagation time between s and microphones. For most aeroacoustic applications beamforming is generally conducted in frequency domain. [3] The frequency domain version of Eq. (115) is:

$$\mathbf{X}(j\omega) = \frac{1}{4\pi \mathbf{r}} S(j\omega) e^{-j\omega\tau} = \mathbf{a}_0(\mathbf{r}, j\omega) S(j\omega), \tag{116}$$

where $j = \sqrt{-1}$, $\mathbf{a}$ is the steering vector, ω is angular frequency, $(j\omega)$ and $(\mathbf{r}, j\omega)$ can be omitted for brevity, $\mathbf{X}$ and S are counterparts in frequency domain, and we can simply write Eq. (116) as $\mathbf{X} = \mathbf{a}_0 S$.

The situation becomes more complex for a practical case, for which the array output vector can be given by

$$\mathbf{X} = \mathbf{a}_0 S + \mathbf{I} + \mathbf{N}, \tag{117}$$

where $\mathbf{I}$ is the interference from coherent signals and/or reflections, $\mathbf{N}$ denotes the collective error from facility background noise and sensor noise. It is worthwhile to note that the signal of interest (S), interference ($\mathbf{I}$) and noise ($\mathbf{N}$) are of zero-mean signal waveforms and generally assumed statistically independent for the simplicity.

Let $\mathbf{R}_X$, $\mathbf{R}_{IN}$ and $\mathbf{R}_S$ denote the $M \times M$ theoretical covariance matrix of the array output vector, interference-plus-noise covariance and signal of interest covariance matrices, respectively, we can have

$$\mathbf{R}_X = \mathrm{E}\left\{\mathbf{X}\mathbf{X}^*\right\}, \tag{118}$$

$$\mathbf{R}_{IN} = \mathrm{E}\left\{(\mathbf{I}+\mathbf{N})(\mathbf{I}+\mathbf{N}))^*\right\}, \tag{119}$$

$$\mathbf{R}_S = \mathrm{E}\left\{\sigma^2\mathbf{a}_0\mathbf{a}_0^*\right\}, \tag{120}$$

where $(.)^*$ stands for conjugate transpose, $\mathrm{E}\{.\}$ denotes the statistical expectation, and $\sigma^2 = \mathrm{E}\left\{|S|^2\right\}$ is the variance of S. In practical aeroacoustic measurements, **N** denotes background noise and can be measured without the presence of any test model. The measurements of **X** can be conducted thereafter with the placement of a test model within the test section. The statistics of the signal of interest, σ^2, can be estimated by beamformers, and a suitable beamforming method with narrow main lobe and small side lobes can reduce interference from unknown **I**.

The covariance matrix **R** is unavailable in practical applications and normally is approximated by the sample covariance matrices, which are

$$\hat{\mathbf{R}}_X \approx \frac{1}{K}\sum_{k=1}^{K}\mathbf{X}\mathbf{X}^*, \tag{121}$$

$$\hat{\mathbf{R}}_{IN} \approx \frac{1}{K}\sum_{k=1}^{K}\mathbf{N}\mathbf{N}^*, \tag{122}$$

$$\hat{\mathbf{R}}_S \approx \hat{\mathbf{R}}_X - \hat{\mathbf{R}}_{IN}, \tag{123}$$

where $\hat{(.)}$ denotes approximations, and K is the number of sampling blocks that is preferably larger than the number of microphones M to maintain full rank of $\hat{\mathbf{R}}$.

B. Conventional beamforming

A narrowband beamformer output for each frequency bin of interest can be written by

$$\mathbf{Y} = \mathbf{W}^*\mathbf{X}, \tag{124}$$

where **Y** is beamformer output and $\mathbf{W} \in \mathfrak{C}^{M\times 1}$ is beamformer weight vector. For the conventional beamformer of delay-and-sum type, the beamformer weight vector is achieved by minimizing the approximation error

$$\min_{\mathbf{W}} \|\mathbf{W}^*\mathbf{X} - S\|, \tag{125}$$

whose solution is $\mathbf{W}_{opt} = (\mathbf{a}_0\mathbf{a}_0^*)^{-1}\mathbf{a}_0$ that yields the following estimation of σ^2:

$$\hat{\sigma}^2 = \mathbf{W}_{opt}^*\hat{\mathbf{R}}_S\mathbf{W}_{opt} = \mathbf{a}_0^*(\mathbf{a}_0\mathbf{a}_0^*)^{-1}\hat{\mathbf{R}}_S(\mathbf{a}_0\mathbf{a}_0^*)^{-1}\mathbf{a}_0. \tag{126}$$

C. Adaptive beamforming

The fundamental idea behind Capon beamforming is to obtain $\mathbf{W}_{opt}$ through maximizing the signal-to-interference-plus ratio (SINR),

$$\mathrm{SINR} = \frac{\mathbf{W}^*\mathbf{R}_S\mathbf{W}}{\mathbf{W}^*\mathbf{R}_{IN}\mathbf{W}}, \tag{127}$$

and maintaining distortionless response toward the direction of signal of interest. [1], [4] In other words, the expected effect of the noise and interferences should be minimized, thus leading to the following linearly constrained quadratic problem: [5]

$$\min_{\mathbf{W}} \mathbf{W}^*\mathbf{R}_{IN}\mathbf{W} \quad \text{subject to } \mathbf{W}^*\mathbf{a}_0 = 1. \tag{128}$$

The solution can be easily derived with Lagrange multiplier [6] , that is $\mathbf{W}_{opt} = \alpha\mathbf{R}_{IN}^{-1}\mathbf{a}_0$, where α is a constant that equals $(\mathbf{a}_0^*\mathbf{R}_{IN}^{-1}\mathbf{a}_0)^{-1}$. In practical aeroacoustic applications the covariance matrix is replaced by the sampling covariance matrix. We can have

$$\hat{\sigma}^2 = \hat{\mathbf{W}}_{opt}^*\hat{\mathbf{R}}_S\hat{\mathbf{W}}_{opt}, \text{where } \hat{\mathbf{W}}_{opt} = \frac{\hat{\mathbf{R}}_X^{-1}\mathbf{a}_0}{\mathbf{a}_0^*\hat{\mathbf{R}}_X^{-1}\mathbf{a}_0}, \tag{129}$$

Eq. (129) is different from the classical form of adaptive beamforming,

$$\hat{\sigma}^2 = \hat{\mathbf{W}}_{opt}^*\hat{\mathbf{R}}_X\hat{\mathbf{W}}_{opt}, \text{where } \hat{\mathbf{W}}_{opt} = \frac{\hat{\mathbf{R}}_X^{-1}\mathbf{a}_0}{\mathbf{a}_0^*\hat{\mathbf{R}}_X^{-1}\mathbf{a}_0}, \tag{130}$$

whose solutions contain both background noise and the desired signal. Hence, Eq. (130) is not adopted in this work. In addition, the covariance matrice of the desired signal $\hat{\mathbf{R}}_X$ and the noise $\hat{\mathbf{R}}_{IN}$ can be approximated using Eqs. (122)-(123), respectively. One could also propose an adaptive beamforming algorithm as below

$$\hat{\sigma}^2 = \hat{\mathbf{W}}_{opt}^*\hat{\mathbf{R}}_S\hat{\mathbf{W}}_{opt}, \text{where } \hat{\mathbf{W}}_{opt} = \frac{\hat{\mathbf{R}}_{IN}^{-1}\mathbf{a}_0}{\mathbf{a}_0^*\hat{\mathbf{R}}_{IN}^{-1}\mathbf{a}_0}. \tag{131}$$

In this work we found this algorithm fails to generate satisfactory results for practical data. The potential reason could be the mismatches in background noise covariance matrices. That is, $\hat{\mathbf{R}}_{IN}$ in Eq. (131) is achieved without the presence of any test model. The installation of a test model, however, could alter the background noise covariance matrix. As a result, the following of this work adopts Eq. (129) to implement the adaptive beamforming method.

It is worthwhile to emphasize that the present problem [Eq. (128)] is originally proposed for rank-one signal (point source) cases. However, practical aeroacoustics normally consist of distributed coherent noise sources and the $\hat{\mathbf{R}}_X$ has a full rank. Direct using the Eq. (129) [the solution of Eq. (128)] for acoustic imaging could be arguable. Hence, specific modifications of $\hat{\mathbf{R}}_X$ have been conducted for the practical data and details can be found in Sec. **??**.

D. Robust adaptive beamforming

The main objective of this paper is to investigate the performance of the adaptive beamforming for practical aeroacoustic applications. In previous works we have already demonstrated that conventional beamforming with delay-and-sum approach is independent of sample data and has been applied for various aeroacoustic cases. [7]–[9] The adaptive beamforming method, however, is well known for its great sensitivity to any mismatches, perturbations and data errors. A systematic solution has been proposed to improve the robustness of adaptive beamforming with respect to any mismatches in steering vectors. [1], [10] In this work the array is placed outside of free stream and should have little mismatch in steering vector (detailed setup is given in the next section). It is therefore assumed that main computational errors of an adaptive beamformer should come from the ill-condition of sample matrices. Regularization method can be used to mitigate the ill-conditioning by adding a constant to diagonal elements of sample matrices. This is the so-called diagonal loading technique, which is one of the most popular approaches for robust adaptive beamforming. The regularized problem can be described by

$$\min_{\mathbf{W}} \mathbf{W}^* \mathbf{R}_{IN} \mathbf{W} + \epsilon \mathbf{W}^* \mathbf{W} \quad \text{subject to } \mathbf{W}^* \mathbf{a}_0 = 1, \tag{132}$$

where the diagonal loading factor ϵ imposes a penalty to avoid inappropriately large array vector $\mathbf{W}$. The loaded version of Eq. (129) is

$$\hat{\sigma}^2 = \hat{\mathbf{W}}_{opt}^* \hat{\mathbf{R}}_S \hat{\mathbf{W}}_{opt}, \text{where } \hat{\mathbf{W}}_{opt} = \frac{(\hat{\mathbf{R}}_X + \epsilon \mathbf{I}_M)^{-1} \mathbf{a}_0}{\mathbf{a}_0^* (\hat{\mathbf{R}}_X + \epsilon \mathbf{I}_M)^{-1} \mathbf{a}_0}, \tag{133}$$

where $\mathbf{I}_M$ is $M \times M$ identity matrix. It is easy to see that the diagonal loading ensures the invertibility of the loaded matrix $\hat{\mathbf{R}}_X + \epsilon \mathbf{I}_M$ regardless of ill-condition of $\hat{\mathbf{R}}_X$.

The choose of the diagonal loading factor ϵ follows a somewhat ad hoc way. A couple of empirical criteria for ϵ with respect to the so-called white noise gain parameter have been given previously. [11] The latter parameter, however, still lacks explicit relationship or clear physical meaning. An iterative procedure is used in this work to tune ϵ, which is set to $\lambda \times$max [eig($\hat{\mathbf{R}}_X$)], eig (.) denotes the eigenvalues of a matrix. The diagonal loading parameter λ can be iteratively chosen between 0.01 and 0.5. A smaller λ normally produces an image with better resolution, but the computation is also easy to blow. For the following practical case we found the value of λ can be quickly determined within a couple round of iterations. The diagonal loading approach is hence used in the rest of this paper for its simple implementation.

In summary the beamforming algorithm for a narrowband frequency range is conduced as follows.

Step 1: Compute sample covariance matrices $\hat{\mathbf{R}}_X$ and $\hat{\mathbf{R}}_S$, and compute eigenvalues of $\hat{\mathbf{R}}_X$.

Step 2: Given an observed plane, which has N gridpoints, construct steering vector $\mathbf{a}_0$ for each gridpoint.

Step 3: Calculate the diagonal loading factor ϵ with an initial guess of $\lambda = 0.01$. $\epsilon = \lambda \times$ max [eig($\hat{\mathbf{R}}_X$)].

Step 4: Repeat conducting N times of adaptive beamforming equation [Eq. (133)] for each gridpoint, and an acoustic image can be produced.

Step 5: Check the image quality. The whole computation is done if the quality is satisfactory. Otherwise double the value of λ and repeat steps 3-5.

VIII. DIGITAL SIGNAL PROCESSOR*

A. Introduction

According Wikipedia, a digital signal processor (DSP) is a specialized microprocessor with an optimized architecture for the fast operational needs of digital signal processing. It has the following characteristics: a) Real-time digital signal processing capabilities. DSPs typically have to process data in real time, i.e., the correctness of the operation depends heavily on the time when the data processing is completed. (Think of the definition of real time)
b) High throughput. DSPs can sustain processing of high-speed streaming data, such as audio and multimedia data processing. (Think of MP3)
c) Deterministic operation. The execution time of DSP programs can be foreseen accurately, thus guaranteeing a repeatable, desired performance. (Real time again, and software dependent)
d) Re-programmability by software. Different system behaviour might be obtained by re-coding the algorithm executed by the DSP instead of by hardware modifications.

DSPs appeared on the market in the early 1980s. Over the last 15 years they have been the key enabling technology for many electronics products in fields such as communication systems, multimedia, automotive, instrumentation and military.

B. History

DSPs appeared on the market in the early 1980s. Since then, they have undergone an intense evolution in terms of hardware features, integration, and software development tools. DSPs are now a mature technology. This section gives an overview of the evolution of the DSP over their 25-year life span; specialized terms such as Harvard architecture, pipelining, instruction set or JTAG are used. The reader is referred to the following paragraphs for explanations of their meaning.

In the late 1970s there were many chips aimed at digital signal processing; however, they are not considered to be digital signal processing owing to either their limited programmability or their lack of hardware features such as hardware multipliers. The first marketed chip to qualify as a programmable DSP was NECs MPD7720, in 1981: it had a hardware multiplier and adopted the Harvard architecture (more information on this architecture is given in Section 3.1). Another early DSP was the TMS320C10, marketed by TI in 1982. Figure 3 shows a selective chronological list of DSPs that have been marketed from the early 1980s until now.

During the market development phase, DSPs were typically based upon the Harvard architecture. The first generation of DSPs included multiply, add, and accumulator units. Examples are TIs TMS320C10 and Analog Devices (ADI) ADSP-2101. The second generation of DSPs retained the architectural structure of the first generation but added features such as pipelining, multiple arithmetic units, special address generator units, and Direct Memory Access (DMA). Examples include TIs TMS320C20 and Motorolas DSP56002. While the first DSPs were capable of fixed- point operations only, towards the end of the 1980s DSPs with floating point capabilities started to appear. Examples are Motorolas DSP96001 and TIs TMS320C30. It should be noted that the floating-point format was not always IEEE-compatible. For instance, the TMS320C30 internal calculations were carried out in a proprietary format; a hardware chip converter [6] was available to convert to the standard

IEEE format. DSPs belonging to the development phase were characterized by fixed-width instruction sets, where one of each instruction was executed per clock cycle. These instructions could be complex, and encompassing several operations. The width of the instruction was typically quite short and did not overcome the DSP native word width. As for DSP producers, the market was nearly equally shared between many manufacturers such as Fujitsu, Hitachi, IBM, NEC, Toshiba, Texas Instruments and, towards the end of the 1980s, Motorola, Analog Devices and Zoran.

During the market consolidation phase, enhanced DSP architectures such as Very Long Instruction Word (VLIW) and Single Instruction Multiple Data (SIMD) emerged. These architectures increase the DSP performance through parallelism. Examples of DSPs with enhanced architectures are TIs TMS320C6xxx DSPs, which was the first DSP to implement the VLIW architecture, and ADIs TigerSHARC, that includes both VLIW and SIMD features. The number of on-chip peripherals increased greatly during this phase, as well as the hardware features that allow many processors to work together. Technologies that allow real-time data exchange between host processor and DSP started to appear towards the end of the 1990s. This constituted a real sea change in DSP system debugging and helped the developers enormously. Another phenomenon observed during this phase was the reduction of the number of DSP manufacturers. The number of DSP families was also greatly reduced, in favour of wider families that granted increased code compatibility between DSPs of different generations belonging to the same family. Additionally, many DSP families are not general- purpose but are focused on specific digital signal processing applications, such as audio equipment or control loops.

C. Architecture Features

DSP is different from general purpose processors. It was designed and optimised specifically for digital signal processing. Architecture features include:

- Hardware modulo addressing, allowing circular buffers to be implemented;
- Driving multiple arithmetic units may require memory architectures to support several accesses per instruction cycle;
- Separate program and data memories (Harvard architecture), and sometimes concurrent access on multiple data busses;
- Special SIMD (single instruction, multiple data) operations;
- Use VLIW techniques so each instruction drives multiple arithmetic units in parallel;
- Special arithmetic operations, such as fast multiply-accumulates (MACs). Many fundamental DSP algorithms, such as FIR filters or the Fast Fourier transform (FFT) depend heavily on multiply and accumulate performance;
- Bit-reversed addressing, a special addressing mode useful for calculating FFTs;
- Deliberate exclusion of a memory management unit. DSPs frequently use multi-tasking operating systems, but have no support for virtual memory or memory protection. Operating systems that use virtual memory require more time for context switching among processes, which increases latency.

[*] The material in ”Digital signal processor fundamentals and system design” by M. E. Angoletta is extensively referred in this section. The related contents on wikipedia is also used here. The former document will be provided in class.

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