

Inertia in strategy switching transforms the strategy evolution

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A recent experimental study [Traulsen *et al.*, *Proc. Natl. Acad. Sci.* **107**, 2962 (2010)] shows that human strategy updating involves both direct payoff comparison and the cost of switching strategy, which is equivalent to inertia. However, it remains largely unclear how such a predisposed inertia affects 2×2 games in a well-mixed population of finite size. To address this issue, the “inertia bonus” (strategy switching cost) is added to the learner payoff in the Fermi process. We find how inertia quantitatively shapes the stationary distribution and that stochastic stability under inertia exhibits three regimes, with each covering seven regions in the plane spanned by two inertia parameters. We also obtain the extended “1/3” rule with inertia and the speed criterion with inertia; these two findings hold for a population above two. We illustrate the above results in the framework of the Prisoner’s Dilemma game. As inertia varies, two intriguing stationary distributions emerge: the probability of coexistence state is maximized, or those of two full states are simultaneously peaked. Our results may provide useful insights into how the inertia of changing status quo acts on the strategy evolution and, in particular, the evolution of cooperation.

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I. INTRODUCTION

Evolutionary game theory incorporates a frequency-dependent selection into the dynamics of genetic evolution [1]. For infinitely large well-mixed populations, the evolutionary dynamics is commonly described by the celebrated replicator equation [2]. For finite populations, many microscopic updating rules are brought forward, of which the most widely used are the Moran process [3,4] and the Fermi process [5,6]. These processes not only characterize genetic evolution, but also cultural evolution in human society. Payoff is not directly converted into offspring, but into the fitness of players, and their correlation is governed by a “selection intensity parameter.” Recent studies demonstrate that some evolutionary results pivot on the microscopic rules [7–9]. The problem naturally arises as to which updating rule best portrays imitation dynamics in the human world. Aiming at resolving the problem, Traulsen *et al.* [10] conducted a behavioral experiment and found that human strategy updating involves both direct payoff comparison and the cost of switching strategy.

In social settings, changing strategy involves real costs in many economic contexts, such as firms’ investment decisions (“setting up” or “shutting down” cost). But even in the absence of such tangible costs, if playing a new strategy is complex, then the switching strategy may impose mental pressure on individuals.

In this paper, we adopt the Fermi process with the “inertia bonus” (strategy switching cost) being added to learner payoff. Inertia is essentially equal to the cost of strategy switching

proposed by Szabó and Hauert [11] and the “stubbornness” in strategy switching advanced by Fu *et al.* [12], but their studies involve no or little effect of inertia. So far, a systematic theoretical analysis of inertia is still lacking. We would like to make a comprehensive investigation upon how inertia modifies the strategy evolution. As a particular example, we also study how inertia affects the cooperation evolution in the context of the Prisoner’s Dilemma game [13–18]. The game depicts the conflict of interest between what is best for the individual (defection) and what is best for the group (cooperation), and thus gives rise to the social dilemma. Spatially structured populations [19–32] are extensively adopted to escape from such a dilemma. To attain a clear-cut insight about how inertia alone guides strategy evolution, we focus on a well-mixed population, which offers a most disadvantageous environment for the evolution of cooperation.

The paper is organized as follows. Section II briefly introduces the Fermi process with inertia. Section III investigates what influence inertia exerts upon the stationary distribution. Section IV analyzes how inertia influences the stochastically stable states in the limit of strong selection. Section V shows the effect of inertia on the fixation probability and the conditional fixation time. We draw conclusions in Sec. VI.

II. MODEL

Consider a symmetric 2×2 game, in which the payoffs are given by the following matrix. An A individual playing the game with another A individual receives payoff a , and two interacting B individuals get d each. An A individual playing the game with a B individual gets b , whereas B obtains c :

$$\begin{array}{c} A \quad B \\ A \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ B \end{array} \quad (1)$$

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In a well-mixed population of finite size N , each individual can play with every other individual equally likely. The average payoffs of A individuals and B individuals are, respectively,

$$P_A(i) = \frac{a(i-1) + b(N-i)}{N-1}, \quad P_B(i) = \frac{ci + d(N-i-1)}{N-1},$$

where i is the number of A individuals in the population. Note that we exclude self-interaction in the above calculations.

We employ the Fermi process with inertia to describe the population dynamics in social and cultural evolution, and inertia substantially corresponds to the cost of switching strategy. The specific evolutionary process is stated as follows: At every time step, an individual, say the focal individual, is randomly singled out. He shifts his strategy with probability μ , otherwise he imitates the strategy of another randomly picked individual, say the model individual, with the probability denoted by the Fermi function with inertia. In the latter case, the composition of the population is likely to change only when the focal individual and the model individual stick to strategies of different types. If the focal individual takes strategy A , he switches to strategy B with probability $\frac{1}{1+e^{-\beta(P_B-P_A-\tau_1)}}$. If the focal individual employs strategy B , he imitates the A individual with likelihood $\frac{1}{1+e^{-\beta(P_A-P_B-\tau_2)}}$. The parameter $\beta \in [0, \infty)$ signifies the strength of selection. With $\tau_1 = 0$, $\tau_2 = 0$, the above update rule recovers the traditional Fermi process [6] which serves as the bench mark. Positive τ_1 and positive τ_2 represent that A individuals and B individuals are reluctant to switch strategies, respectively. So τ_1 and τ_2 are called A 's inertia and B 's inertia, respectively. For the sake of comprehensively analyzing, inertia parameters can be negative, suggesting that individuals are desirous of copying others' strategies.

The variation of the number of A individuals can be delineated by a Markov process. The state space is $V = \{0, \dots, N\}$.

When the mutation $\mu > 0$, the transition matrix is $(p_{ij})_{(N+1) \times (N+1)}$, where

$$p_{i,j} = 0 \quad \text{if } |i-j| > 1, \\ p_{0,1} = 1 - p_{0,0} = \mu = p_{N,N-1} = 1 - p_{N,N},$$

and, for $i = 1, \dots, N-1$,

$$p_{i,i+1} = \mu \frac{N-i}{N} + (1-\mu) \frac{i}{N} \frac{N-i}{N} \frac{1}{1+e^{-\beta[P_A(i)-P_B(i)-\tau_2]}}, \\ p_{i,i-1} = \mu \frac{i}{N} + (1-\mu) \frac{i}{N} \frac{N-i}{N} \frac{1}{1+e^{-\beta[P_B(i)-P_A(i)-\tau_1]}}, \\ p_{i,i} = 1 - p_{i,i-1} - p_{i,i+1}. \quad (2)$$

As the process is ergodic, there exists a unique invariant distribution expressed by $\pi(\mu) = [\pi(\mu, 0), \dots, \pi(\mu, N)]$. The distribution can be derived explicitly [33] for $k = 0, \dots, N$:

$$\pi(\mu, k) = \frac{\lambda(\mu, k)}{\lambda(\mu, 0) + \dots + \lambda(\mu, N)} \quad \text{where} \quad (3) \\ \lambda(\mu, k) = \prod_{i=0}^{k-1} \frac{p_{i,i+1}}{p_{i+1,i}} \quad \text{and} \quad \lambda(\mu, 0) = 1.$$

Here the quantities appealing to us are the stationary distribution and the stochastically stable states.

When the mutation $\mu = 0$, the transition matrix is $(\widehat{p}_{i,j})_{(N+1) \times (N+1)}$, where

$$\widehat{p}_{i,j} = 0 \quad \text{if } |i-j| > 1, \\ 1 = \widehat{p}_{0,0} = 1 - \widehat{p}_{0,1} = \widehat{p}_{N,N} = 1 - \widehat{p}_{N,N-1},$$

and, for $i = 1, \dots, N-1$,

$$\widehat{p}_{i,i+1} = \frac{i}{N} \frac{N-i}{N} \frac{1}{1+e^{-\beta[P_A(i)-P_B(i)-\tau_2]}}, \\ \widehat{p}_{i,i-1} = \frac{i}{N} \frac{N-i}{N} \frac{1}{1+e^{-\beta[P_B(i)-P_A(i)-\tau_1]}}, \\ \widehat{p}_{i,i} = 1 - \widehat{p}_{i,i-1} - \widehat{p}_{i,i+1}. \quad (4)$$

The states $v = 0$ and N are absorbing states, and the quantities of interest are the fixation probability and the conditional fixation time.

We separately investigate how inertia acts upon the above four quantities in detail.

III. THE FERMI PROCESS WITH SMALL MUTATION: STATIONARY DISTRIBUTION

For arbitrary mutation, how inertia affects the shape of the stationary distribution is difficult to quantitatively analyze. However, for small mutation, the quantitative result can be obtained on the basis of the following method. Given a small mutation rate, the first-order approximation of $\lambda(\mu, k)$ is in fact a discrete function of k and is expanded as a continuous function in the interval of $[1, 99]$. By analyzing the effect of inertia on the monotonicity of the continuous function, we determine the threshold values of inertia, which are crucial to analyze how inertia transforms the profile of the distribution.

From Eqs. (3) and (4), in combination with $p_{N,N-1} = p_{0,1} = \mu$, we have, for small mutation rates,

$$\forall k \in \{1, \dots, N-1\}, \\ \lambda(\mu, k) = \frac{\mu}{p_{k,k-1}} \prod_{i=1}^{k-1} \frac{p_{i,i+1}}{p_{i,i-1}} \approx \frac{\mu}{\widehat{p}_{k,k-1}} \prod_{i=1}^{k-1} \frac{\widehat{p}_{i,i+1}}{\widehat{p}_{i,i-1}} = O(\mu), \quad (5)$$

$$\lambda(\mu, 0) = 1, \quad \lambda(\mu, N) = \prod_{i=1}^{N-1} \frac{p_{i,i+1}}{p_{i,i-1}} \\ = \prod_{i=1}^{N-1} \frac{\widehat{p}_{i,i+1} + \mu \left(\frac{N-i}{N} - \widehat{p}_{i,i+1} \right)}{\widehat{p}_{i,i-1} + \mu \left(\frac{i}{N} - \widehat{p}_{i,i-1} \right)} \\ \approx \prod_{i=1}^{N-1} \frac{\widehat{p}_{i,i+1}}{\widehat{p}_{i,i-1}} + O(\mu), \quad (6)$$

where $O(\mu)$ stands for the infinitesimal of the same order of μ . Intuitively, the process spends $1 - O(\mu)$ of the time at full states and $O(\mu)$ of the time at coexistence states.

Since $\pi(\mu, k) = \frac{\lambda(\mu, k)}{\lambda(\mu, 0) + \dots + \lambda(\mu, N)}$, it is straightforward to see that $\pi(\mu) = [\pi(\mu, 0), \dots, \pi(\mu, N)]$ is the normalized vector of $\lambda(\mu) = [\lambda(\mu, 0), \dots, \lambda(\mu, N)]$. In fact, given the mutation rate, $\pi(\mu, k)$ and $\lambda(\mu, k)$ are discrete functions of

k , whose corresponding continuous functions share the same monotonicity. Let

$$F(k, c_1, c_2) = \frac{1}{\widehat{p_{k,k-1}}} \prod_{i=1}^{k-1} \frac{\widehat{p_{i,i+1}}}{\widehat{p_{i,i-1}}} \\ = \frac{N}{k} \frac{N}{N-k} (1 + c_1 e^{m \times k + n}) \prod_{i=1}^{k-1} \frac{1 + c_1 e^{m \times i + n}}{1 + c_2 e^{-m \times i - n}}, \quad (7)$$

where $c_1 = e^{\beta \tau_1}$, $c_2 = e^{\beta \tau_2}$, $m = \beta \frac{a+d-c-b}{N-1}$, and $n = \beta \frac{-a-(N-1)d+Nb}{N-1}$. Then, $\lambda(\mu, k) = \mu F(k, c_1, c_2)$ for $k = \{1, \dots, N-1\}$.

Let $f(x, c_1, c_2)$ be a continuous and differentiable function of x , and, for $k \in \{1, \dots, N-1\}$, $f(k, c_1, c_2) = F(k, c_1, c_2)$. The explicit formula of $f(x, c_1, c_2)$ is not obvious, but that of $\ln f(x, c_1, c_2)$ can be effortlessly approximated as

$$\ln f(x, c_1, c_2) \approx 2 \ln N - \ln x - \ln(N-x) \\ + \int_0^x \ln(1 + c_1 e^{ms+n}) ds \\ - \int_1^x \ln(1 + c_2 e^{-ms-n}) ds.$$

It is easy to verify that the derivative of $f(x, c_1, c_2)$ with respect to x has the same sign as that of $\ln f(x, c_1, c_2)$. Therefore, to achieve the variety of the shape of the stationary distribution under inertia, we can analyze how c_1 and c_2 affect the derivative of $\ln f(x, c_1, c_2)$.

Denoting the derivative of $\ln f(x, c_1, c_2)$ by $h(x, c_1, c_2)$, we have

$$h(x, c_1, c_2) = \frac{\partial \ln f}{\partial x} = -\frac{1}{x} + \frac{1}{N-x} \\ + \ln(1 + c_1 e^{mx+n}) - \ln(1 + c_2 e^{-mx-n}). \quad (8)$$

Let $h(x, c_1, 1) = 0$, the threshold values of A 's inertia are $\{c_1^*(1), \dots, c_1^*(N-1)\}$, where

$$c_1^*(x) = \frac{e^{[\ln(1+e^{-mx-n}) + \frac{1}{x} - \frac{1}{N-x}] - 1}}{e^{mx+n}}. \quad (9)$$

Let $h(x, 1, c_2) = 0$, the threshold values of B 's inertia are $\{c_2^*(1), \dots, c_2^*(N-1)\}$, where

$$c_2^*(x) = \frac{e^{[\ln(1+e^{mx+n}) - \frac{1}{x} + \frac{1}{N-x}] - 1}}{e^{-mx-n}}. \quad (10)$$

Especially, inertia is illustrated to alleviate and even remove the social dilemma whenever the interaction mode is characterized as the Prisoner's Dilemma game ($c > a > d > b$). Strategy A and B represent cooperation and defection, respectively. In this case, we find that $n = \beta \frac{-a-(N-1)d+Nb}{N-1}$ is always negative, while $m = \beta \frac{a+d-c-b}{N-1}$ can be positive or negative. Fixating b and c , we pick two arrays of a and d to ensure that m of two games enjoys different signs. Amazingly, some inertia makes these two games exhibit profoundly different dynamics.

In the following two examples, the mutation rate $\mu = 0.0001$, the selection intensity $\beta = 1$, and the population size $N = 100$.

A sketch of the analysis upon the effect of A 's inertia is as follows. In the first step, we determine the threshold values of A 's inertia $\{c_1^*(1), \dots, c_1^*(99)\}$ by Eq. (9). The maximum and minimum of $\{c_1^*(1), \dots, c_1^*(99)\}$ are denoted by $c_1^*(M)$ and $c_1^*(m)$. In the second step, since $h(k, c_1, 1)$ is an increasing function of c_1 , we obtain the following conclusions:

(1) For $c_1 = e^{\tau_1} \leq c_1^*(m)$, we have $h(k, e^{\tau_1}, 1) \leq 0$ for any $k \in \{1, \dots, 99\}$, then the stationary distribution has a unique maximum at $v = 0$.

(2) For $c_1^*(m) < c_1 = e^{\tau_1} < c_1^*(M)$, the shape of the stationary distribution depends on the profile of $c_1^*(x), x \in [1, 99]$. If $c_1^*(x)$ decreases monotonously, the stationary distribution simultaneously peaks at $v = 0$ and 100. If $c_1^*(x)$ increases monotonously, the stationary distribution peaks at the coexistence state.

(3) For $c_1 = e^{\tau_1} \geq c_1^*(M)$, we have $h(k, e^{\tau_1}, 1) \geq 0$ for any $k \in \{1, \dots, 99\}$, then the stationary distribution has a unique maximum at $v = 100$.

Similarly, a sketch of the analysis upon the effect of B 's inertia is as follows. In the first step, we determine the threshold values of B 's inertia $\{c_2^*(1), \dots, c_2^*(99)\}$ by Eq. (10). The maximum of $\{c_2^*(1), \dots, c_2^*(99)\}$ is denoted by $c_2^*(M)$. Meanwhile, we find that $c_2^*(1) < c_2^*(2) < \dots < c_2^*(Z-1) < 0 \leq c_2^*(Z)$. In the second step, since $h(k, 1, c_2)$ is a decreasing function of c_2 , we obtain the following conclusions:

(1) For $c_2 = e^{\tau_2} \geq c_2^*(M)$, we have $h(k, 1, e^{\tau_2}) \leq 0$ for any $k \in \{1, \dots, 99\}$, then the stationary distribution has a unique maximum at $v = 0$.

(2) For $c_2 = e^{\tau_2} < c_2^*(M)$, the shape of the stationary distribution depends on the profile of $c_2^*(x), x \in [1, 99]$. If $c_2^*(x)$ increases monotonously, the stationary distribution simultaneously peaks at $v = 0$ and 100.

(3) Since $c_2^*(1) < c_2^*(2) < \dots < c_2^*(Z-1) < 0 \leq c_2^*(Z)$, we have $h(k, 1, e^{\tau_2}) \leq 0$ for any τ_2 and $k \in \{1, \dots, Z-1\}$, then the peak point $v = 0$ of the stationary distribution always exists no matter how small τ_2 is. This is because cooperators always imitate defectors with a nonzero probability $\frac{1}{1+e^{-\beta(p_B-p_A)}}$ in the process only with τ_2 .

Example 1. Consider the Prisoner's Dilemma game ($a = 5, b = 0, c = 6, d = 2$). Figure 1 indicates the monotonic decrement of $c_1^*(x)$ and the monotonic increment of $c_2^*(x)$, respectively:

(1) For $e^{\tau_1} \leq c_1^*(99)$ [$c_1^*(99)$ is the minimum point in Fig. 1(a)], or for $e^{\tau_2} \geq c_2^*(99)$ [$c_2^*(99)$ is the maximum point in Fig. 1(b)], the stationary distribution takes on the only maximal value at the full defection state $v = 0$ and the dilemma is not alleviated but worsened [e.g., Figs. 2(a) and 2(d)].

(2) For $c_1^*(99) < e^{\tau_1} < c_1^*(1)$ [$c_1^*(1)$ is the maximum point in Fig. 1(a)], or for $e^{\tau_2} < c_2^*(99)$, the stationary distribution exhibits a U shape with two local maxima at the full states $v = 0$ and 100 and the dilemma is lightened [e.g., Figs. 2(b) and 2(e)].

(3) For $e^{\tau_1} \geq c_1^*(1)$, the stationary distribution assumes the single maximal value at the full cooperation state $v = 100$ and the dilemma is thoroughly eliminated [e.g., Fig. 2(c)]. Note that we show that the distribution under very small τ_2 possesses the peak point $v = 0$ in Fig. 2(f).

Example 2. Consider the Prisoner's Dilemma game ($a = 4, b = 0, c = 6, d = 1$). From Fig. 3(a), $c_1^*(x)$ can be approximately seen as an increasing function. From Fig. 3(b), $c_2^*(x)$

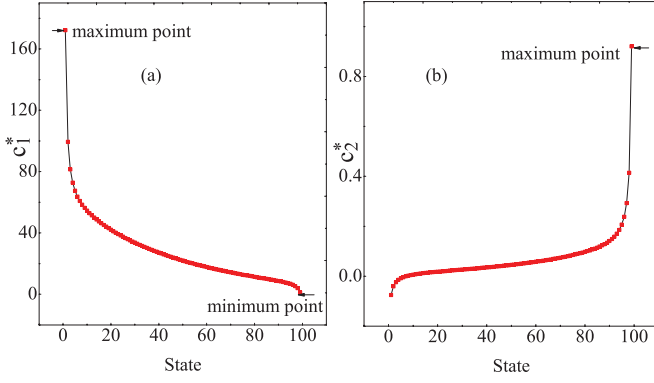


FIG. 1. (Color online) For the game ($a=5, b=0, c=6, d=2$), stationary points $c^*(x)$ [representing $c_1^*(x)$ or $c_2^*(x)$] for the equilibrium probability of the state x given by Eqs. (9) and (10). (a) In the absence of B 's inertia, i.e., $c_2 = 1$, $c_1^*(x)$ decreases monotonously with x . (b) In the absence of A 's inertia, i.e., $c_1 = 1$, $c_2^*(x)$ increases monotonously with x . Relevant parameters: $\beta = 1, N = 100$.

is not a monotone function, which results in quite interesting phenomena:

(1) For $e^{\tau_1} \leq c_1^*(8)$ [$c_1^*(8)$ is the minimum point in Fig. 3(a)], or for $e^{\tau_2} \geq c_2^*(99)$ [$c_2^*(99)$ is the maximum point in Fig. 3(b)], a single maximum of the stationary distribution appears at the full defection state $v = 0$ and cooperation is completely inhibited [e.g., Figs. 4(a) and 4(d)].

(2) For $c_1^*(8) < e^{\tau_1} < c_1^*(93)$ [$c_1^*(93)$ is the maximum point in Fig. 3(a)], a unique maximum value of the stationary distribution arises at a coexistence state and cooperation is supplied with an opportunity to survive [e.g., Fig. 4(b)].

(3) For $c_2^*(16) \leq e^{\tau_2} < c_2^*(99)$ [$c_2^*(16)$ is threshold $_1$ in Fig. 3(b)], the local maxima of the stationary distribution turn up at both the full defection state $v = 0$ and the full cooperation state $v = 100$ but the latter is much smaller, so the dilemma is slightly attenuated.

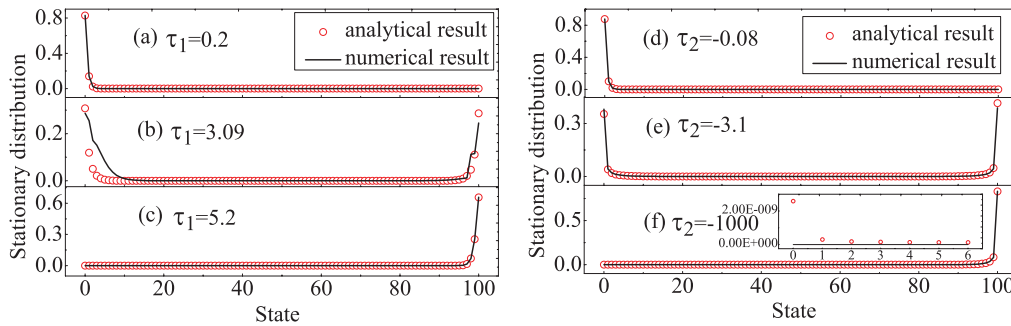


FIG. 2. (Color online) Stationary distribution of the game ($a = 5, b = 0, c = 6, d = 2$) under the influence of A 's inertia (τ_1) or B 's inertia (τ_2). The distribution undergoes a qualitative change with increasing τ_1 or decreasing τ_2 . Three different scenarios occur for the stationary distribution in the absence of τ_2 (i.e., $\tau_2 = 0$): (a) For $\tau_1 \leq \ln c_1^*(99)$, the stationary distribution reaches maximization at the absorbing state $v = 0$; (b) for $\ln c_1^*(99) < \tau_1 < \ln c_1^*(1)$, the local peak $v = 0$ is withheld and another local peak $v = 100$ comes into being; and (c) for $\tau_1 \geq \ln c_1^*(1)$, the stationary probability distribution no longer has the relative maximum $v = 0$ and exhibits a single maximal value at the boundary $v = 100$. Two different scenarios occur for the stationary distribution with $\tau_1 = 0$: (d) For $\tau_2 \geq \ln c_2^*(99)$, the stationary distribution demonstrates its maximality at the Nash equilibrium $v = 0$; and (e) for $\tau_2 < \ln c_2^*(99)$, the system spends most of the time near two absorbing states, leading to local maxima of the stationary distribution at $v = 0$ and 100. In addition, (f) the local peak $v = 0$ in the distribution exists under very small τ_2 (see inset for detail). In all six cases, the numerical result (obtained by the Monte Carlo simulation average over 10^8 time steps) depicted by lines accords well with the analytical result [obtained by Eq. (3)] given by symbols. Relevant parameters: $\mu = 0.0001, \beta = 1, N = 100$.

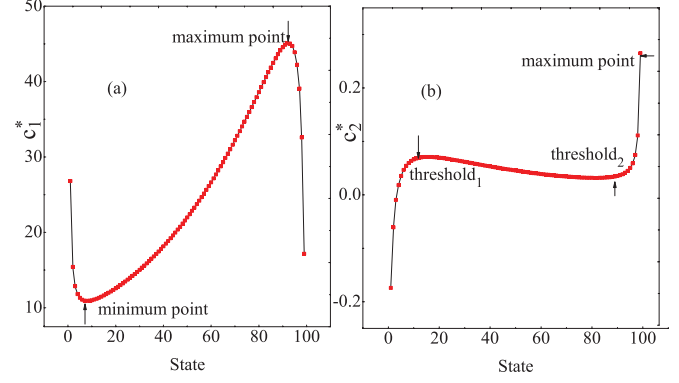


FIG. 3. (Color online) For the game ($a=4, b=0, c=6, d=1$), stationary points $c^*(x)$ [representing $c_1^*(x)$ or $c_2^*(x)$] for the equilibrium probabilities of the state x depicted by Eqs. (9) and (10). (a) In the absence of B 's inertia, i.e., $c_2 = 1$, on the whole, $c_1^*(x)$ increases monotonously with x . (b) In the absence of A 's inertia, i.e., $c_1 = 1$, $c_2^*(x)$ exhibits a varied monotonicity with x . Relevant parameters: $\beta = 1, N = 100$.

(4) For $c_2^*(81) \leq e^{\tau_2} < c_2^*(16)$ [$c_2^*(81)$ is threshold $_2$ in Fig. 3(b)], the relative maxima of the stationary distribution at the full states $v = 0$ and 100 don't disappear and another peak appears at a coexistence state [e.g., Fig. 4(e)]; correspondingly the relaxation of the dilemma is conspicuous.

(5) For $e^{\tau_2} < c_2^*(81)$, the distribution again just carries two local peaks at the full states $v = 0$ and 100, but the latter is much larger, so the dilemma is significantly relieved [e.g., Fig. 4(f)].

(6) For $e^{\tau_1} \geq c_1^*(93)$, the unique maximum point of the distribution lies at the full cooperation state $v = 100$, so cooperators totally win the victory against defectors [e.g., Fig. 4(c)].

For any given game, the quantitative effects of A 's inertia and B 's inertia on the shape of the stationary distribution are

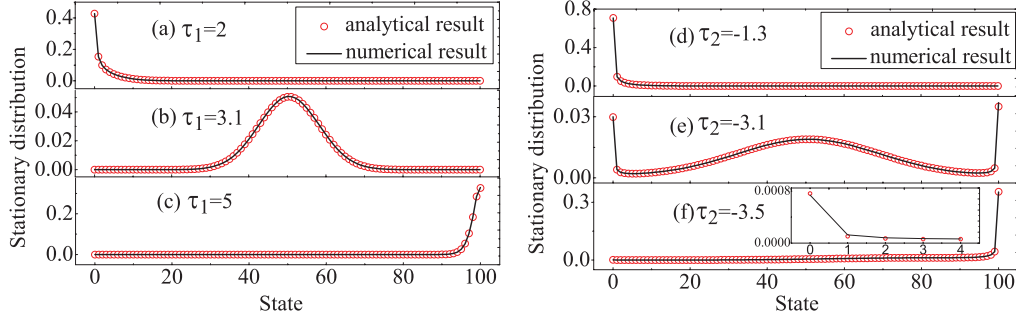


FIG. 4. (Color online) Stationary distribution of the game ($a = 4, b = 0, c = 6, d = 1$) under the influence of A 's inertia (τ_1) or B 's inertia (τ_2). The distribution undergoes a qualitative change with increasing τ_1 or decreasing τ_2 . Three different scenarios occur for the stationary distribution in the absence of B 's inertia (i.e., $\tau_2 = 0$): (a) For $\tau_1 \leq \ln c_1^*(8)$, the system spends most of the time near the state of all B individuals; (b) for $\ln c_1^*(8) < \tau_1 < \ln c_1^*(93)$, τ_1 leads the system away from the state $v = 0$ and the distribution is centered at a coexistence state; and (c) for $\tau_1 \geq \ln c_1^*(97)$, the stationary probability distribution has a single absolute maximum at the boundary $v = 100$. Four different scenarios occur for the stationary distribution with $\tau_1 = 0$: (d) For $\tau_2 \geq \ln c_2^*(99)$, the stationary distribution has a single maximum value at the endpoint $v = 0$ [note that we don't portray the case for $\ln c_2^*(16) \leq \tau_2 < \ln c_2^*(99)$, where the stationary distribution keeps a maximum at $v = 0$, and a second maximum (smaller) appears at the other absorbing state $v = 100$]; (e) for $\ln c_2^*(81) < \tau_2 < \ln c_2^*(16)$, the system not only keeps the relative maxima $v = 0$ and 100 , but also another local peak appears for intermediate values; and (f) for $\tau_2 \leq \ln c_2^*(81)$, the peak at a coexistence state disappears, while the maximum at $v = 100$ and a second (smaller) maximum at $v = 0$ are held (see inset for detail of the distribution near $v = 0$). In all six cases, the numerical result (obtained by the Monte Carlo simulation average over 10^8 time steps) depicted by lines accords well with the analytical result [obtained by Eq. (3)] given by symbols. Relevant parameters: $\mu = 0.0001$, $\beta = 1$, $N = 100$.

considered separately. In particular, the influences of both on the same Prisoner's Dilemma game are slightly different. The dilemma under A 's inertia can be alleviated or even thoroughly eliminated. Whereas B 's inertia always provides defection with an opportunity of survival and the dilemma can't be entirely removed. The effects of inertia on two Prisoner's Dilemma games with $a + d - b - c$ of different signs are strikingly different. For $a + d - b - c > 0$, inertia can lead the stationary distribution to simultaneously peak at two full states. For $a + d - b - c < 0$, inertia can lead the stationary distribution to peak at a coexistence state.

Remark 1. As an alternative of the above discussion, the approximation formula for the probability density function can be obtained by the Kolmogorov-Fokker-Planck forward equation [33]. The probability density, $p(x, t)$, which gives us that the fraction of individuals playing strategy A is x at time t , satisfies

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial [b(x)p(x, t)]}{\partial x} + \frac{1}{2} \frac{\partial^2 [a(x)p(x, t)]}{\partial x^2}, \quad (11)$$

where $b(x) = p_{xN, xN+1} - p_{xN, xN-1}$ and $a(x) = \frac{1}{N}(p_{xN, xN+1} + p_{xN, xN-1})$.

The probability density function is easily found to be

$$p^{(s)}(x) = C e^{2 \int_0^x \frac{b(s)}{a(s)} ds} / a(x). \quad (12)$$

The constant C follows from the constraint $\int_0^1 p^{(s)} ds = 1$.

By the Itô calculus, Eq. (11) is equivalent to a Langevin equation [34]:

$$\dot{x} = a(x) + b(x)\xi, \quad (13)$$

where ξ is uncorrelated Gaussian noise and $b(x) = 0$ for both $x = 0$ and 1 .

The stochastic dynamics of a system described by Eq. (13) can be easily analyzed by the Gibbs form of the stationary probability function [35]:

$$p^{(s)}(x) \propto e^{V_{\text{eff}}}. \quad (14)$$

The extremes of the effective potential V_{eff} correspond to the stationary fixed points of the noise-sustained dynamics.

IV. THE FERMI PROCESS WITH $\mu \rightarrow 0$: STOCHASTICALLY STABLE STATE

The stochastically stable equilibrium was originally put forward by Foster and Young [36], whereafter the concept has been extensively researched by Kandori *et al.* [37] and Mikisz *et al.* [38–41]. The former proposed a similar model where the risk-dominant strategy is stochastically stable. The latter required that individuals should only play one game at every time step, and found that the population undergoes several equilibrium transitions with the transformed population size.

A state $v \in V$ is stochastically stable if $\pi^*(v) = \lim_{\mu \rightarrow 0} \pi(\mu, v) > 0$.

From Eq. (5), we see $\lambda^*(k) = \lim_{\mu \rightarrow 0} \lambda(\mu, k) = 0$ for $k \in \{1, \dots, N-1\}$. Since $\lambda(\mu, 0) = 1$ and by Eq. (3), $\pi(\mu, k) \leq \lambda(\mu, k)$. It follows that $\pi^*(k) = \lim_{\mu \rightarrow 0} \pi(\mu, k) = 0$ for $k \in \{1, \dots, N-1\}$.

By Eq. (6), we have $\lambda^*(0) = \lim_{\mu \rightarrow 0} \lambda(\mu, 0) = 1$ and $\lambda^*(N) = \lim_{\mu \rightarrow 0} \lambda(\mu, N) = \prod_{i=1}^{N-1} \frac{p_{i,i+1}}{p_{i,i-1}}$. It follows that

$$\pi^*(0) = \frac{1}{1 + \lambda^*(N)}, \quad \pi^*(N) = \frac{\lambda^*(N)}{1 + \lambda^*(N)}. \quad (15)$$

Let $\pi^* = [\pi^*(0), \pi^*(1), \dots, \pi^*(N-1), \pi^*(N)]$. As discussed above, the stationary distribution $\pi(\mu)$ approaches $\pi^* = [\pi^*(0), 0, \dots, 0, \pi^*(N)]$ as the mutation rate μ goes to zero (for the Moran process, Fudenberg *et al.* [42] have

obtained the same result). Consequently, the possible stochastically stable states are $v = 0$ and N , and $\lambda^*(N)$ is critical for the detailed analysis of how inertia alters the stochastically stable states.

We are interested in the stochastic stability under strong selection and the expression of $\lambda^*(N)$ is annoyingly complicated; therefore the limit of the approximation of $\ln \lambda^*(N)$ as β tends to $+\infty$ is our focus.

We see $\lambda^*(N) = \lim_{\mu \rightarrow 0} \lambda(\mu, N) = \prod_{i=1}^{N-1} \frac{\widehat{p_{i,i+1}}}{p_{i,i-1}}$. Inserting Eq. (4) into $\lambda^*(N)$ leads to

$$\lambda^*(N) = \prod_{i=1}^{N-1} \frac{1 + e^{-\beta[p_B(i) - p_A(i) - \tau_1]}}{1 + e^{-\beta[p_A(i) - p_B(i) - \tau_2]}} = \prod_{i=1}^{N-1} \frac{1 + e^{\beta(\rho i + \zeta)}}{1 + e^{\beta(-\rho i + \eta)}}, \quad (16)$$

where $\rho = \frac{a+d-c-b}{N-1}$, $\zeta = \Omega + \tau_1$, $\eta = -\Omega + \tau_2$, and $\Omega = \frac{-a-(N-1)d+Nb}{N-1}$. Taking the natural logarithm of Eq. (16) and replacing the sums by integrals induces

$$\ln \lambda^*(N) \approx \int_0^{N-1} \ln[1 + e^{\beta(\rho x + \zeta)}] dx - \int_0^{N-1} \ln[1 + e^{\beta(-\rho x + \eta)}] dx. \quad (17)$$

For $a + d - b - c \neq 0$, the integral of transcendental functions in Eq. (17) can't be written down explicitly. To calculate the limit of $\ln \lambda^*(N)$, we exchange the order of the limit and the integral. The following two theorems, whose proofs are introduced in Appendix A, guarantee the rationality of the swap.

Theorem 1. Let $h(x, \beta) = \ln[1 + e^{\beta\omega(x)}]$ where $\omega(x) \geq 0$ for all x in a measurable set χ , then $\lim_{\beta \rightarrow +\infty} \int_{\chi} h(x, \beta) dx = \int_{\chi} \lim_{\beta \rightarrow +\infty} h(x, \beta) dx$.

Theorem 2. Let $h(x, \beta) = \ln[1 + e^{\beta\omega(x)}]$ where $\omega(x) \leq 0$ for all x in a measurable set χ , then $\lim_{\beta \rightarrow +\infty} \int_{\chi} h(x, \beta) dx = \int_{\chi} \lim_{\beta \rightarrow +\infty} h(x, \beta) dx$.

We rearrange Eq. (17):

$$\begin{aligned} \ln \lambda^*(N) \approx & \int_{\chi_1} \ln[1 + e^{\beta(\rho x + \zeta)}] dx + \int_{\chi_2} \ln[1 + e^{\beta(\rho x + \zeta)}] dx \\ & - \int_{\chi_3} \ln[1 + e^{\beta(-\rho x + \eta)}] dx \\ & - \int_{\chi_4} \ln[1 + e^{\beta(-\rho x + \eta)}] dx. \end{aligned}$$

where χ_1, χ_2, χ_3 , and χ_4 are measurable intervals which satisfy $\forall x \in \chi_1, \rho x + \zeta \geq 0$, $\forall x \in \chi_2, \rho x + \zeta \leq 0$, $\forall x \in \chi_3, -\rho x + \eta \geq 0$, and $\forall x \in \chi_4, -\rho x + \eta \leq 0$.

By Theorem 1, together with the equivalent relation of limit, we have

$$\begin{aligned} \lim_{\beta \rightarrow +\infty} \int_{\chi_1} \ln[1 + e^{\beta(\rho x + \zeta)}] dx &= \int_{\chi_1} \lim_{\beta \rightarrow +\infty} \ln[1 + e^{\beta(\rho x + \zeta)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_1} \beta(\rho x + \zeta) dx, \\ \lim_{\beta \rightarrow +\infty} \int_{\chi_3} \ln[1 + e^{\beta(-\rho x + \eta)}] dx &= \int_{\chi_3} \lim_{\beta \rightarrow +\infty} \ln[1 + e^{\beta(-\rho x + \eta)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_3} \beta(-\rho x + \eta) dx. \end{aligned}$$

Due to Theorem 2, along with the equivalent relation of limit, we get

$$\begin{aligned} \lim_{\beta \rightarrow +\infty} \int_{\chi_2} \ln[1 + e^{\beta(\rho x + \zeta)}] dx &= \int_{\chi_2} \lim_{\beta \rightarrow +\infty} \ln[1 + e^{\beta(\rho x + \zeta)}] dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_2} e^{\beta(\rho x + \zeta)} dx, \\ \lim_{\beta \rightarrow +\infty} \int_{\chi_4} \ln[1 + e^{\beta(-\rho x + \eta)}] dx &= \int_{\chi_4} \lim_{\beta \rightarrow +\infty} \ln[1 + e^{\beta(-\rho x + \eta)}] dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_4} e^{\beta(-\rho x + \eta)} dx. \end{aligned}$$

We only investigate the case for $a + d - b - c > 0$. For the game fulfilling $a + d - b - c < 0$, we can explore a transformed game where A and B are interchanged. In the limit of strong selection ($\beta \rightarrow +\infty$), the possible values of $\ln \lambda^*(N)$ are $+\infty$, 0 , and $-\infty$. Therefore, the resulting stochastic stability falls into three regimes: “only A is stochastically stable,” “only B is stochastically stable,” and “both A and B are stochastically stable with the same equilibrium probability.” The detailed analysis (see Appendix B) shows that the plane spanned by τ_1 and τ_2 is subdivided into 21 domains, and each regime covers seven domains.

For $a + d - b - c = 0$, Eq. (17) reduces to $\ln \lambda^*(N) = (N-1) \ln(1 + e^{\beta\zeta}) - (N-1) \ln(1 + e^{\beta\eta})$. Then,

(1) If $\zeta > \eta$ and $\zeta > 0$ (i.e., $\tau_1 - \tau_2 > -2\Omega$ and $\tau_1 > -\Omega$), then $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = +\infty$. We say that only A is stochastically stable. In addition, if $\zeta = 0$ and $\eta < 0$ (i.e., $\tau_1 = -\Omega$ and $\tau_2 < \Omega$), then $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = (N-1) \ln 2$. For a large population, the system spends much more time at the state $v = N$ and it is approximate that only A is stochastically stable.

(2) If $\eta > \zeta$ and $\eta > 0$ (i.e., $\tau_1 - \tau_2 < -2\Omega$ and $\tau_2 > \Omega$), then $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = -\infty$. We say that only B is stochastically stable. In addition, if $\eta = 0$ and $\zeta < 0$ (i.e., $\tau_2 = \Omega$ and $\tau_1 < -\Omega$), then $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = -(N-1) \ln 2$. For a large population, the system spends much more time at the state $v = 0$ and it is approximate that only B is stochastically stable.

(3) If $\zeta = \eta \geq 0$ (i.e., $\tau_1 - \tau_2 = -2\Omega$ and $\tau_1 \geq -\Omega$), or if $\zeta < 0$ and $\eta < 0$ (i.e., $\tau_1 < -\Omega$ and $\tau_2 < \Omega$), then $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = 0$. We say that both A and B are stochastically stable, and the population spends equal time at both states $v = 0$ and N in the long run.

It is well known that the simplified Prisoner's Dilemma game ($a = 1, b = 0, c = 1 + u, d = u$) satisfies the condition $a + d = b + c$. When there exists no inertia in the process, only defection is stochastically stable, while, according to the above conclusions, approximate inertia parameters τ_1 and τ_2 make cooperation stochastically stable.

Remark 2. Fudenberg and Imhof [43] demonstrated that $\pi^* = [\pi^*(0), \dots, \pi^*(N)]$ is a good approximation of $\pi(\mu) = [\pi(\mu, 0), \dots, \pi(\mu, N)]$ for sufficiently small mutation rates. In view of how small the mutation rate for the approximation is, Wu *et al.* determined the critical mutation rate under which the total variation distance between $\pi(\mu)$ and π^* [44] is smaller than the given tolerance of the error.

V. THE FERMI PROCESS WITHOUT MUTATION: THE FIXATION PROBABILITY AND THE CONDITIONAL FIXATION TIME

In the no-mutation process, $v = 0$ and N are absorbing states. A mutant either replaces the other type or is eliminated. We are keen on how inertia influences fixation probabilities and conditional fixation times.

A. Fixation probabilities

The probability of a single A mutant taking over a population of B individuals is called A 's fixation probability and denoted by ρ_A . Similarly, the probability of a single B mutant taking over a population of A individuals is called B 's fixation probability and expressed as ρ_B . Since the Fermi process with inertia is a birth-death process, the fixation probabilities have simple explicit expressions [45]:

$$\rho_A = \frac{1}{1 + \sum_{k=1}^{N-1} \prod_{j=1}^k \frac{\widehat{p_{j,j-1}}}{p_{j,j+1}}}, \quad \rho_B = \frac{\prod_{j=1}^{N-1} \frac{\widehat{p_{j,j-1}}}{p_{j,j+1}}}{1 + \sum_{k=1}^{N-1} \prod_{j=1}^k \frac{\widehat{p_{j,j-1}}}{p_{j,j+1}}}. \quad (18)$$

The most widely used updating rules in a finite population are the Moran process and the Fermi process.

For the Moran process, Nowak *et al.* [4] found the “1/3” and “1/2” rules for large populations under weak selection. The 1/2 rule means that, if at an A frequency of 1/2 the fitness of A is greater (or smaller) than that of B , then $\rho_A > \rho_B$ (or $\rho_A < \rho_B$). The 1/3 rule reads that, if at an A frequency of 1/3 the fitness of A is greater (or smaller) than that of B , then $\rho_A > 1/N$ (or $\rho_A < 1/N$). The case where the fitness of A is identical with that of B at an A frequency of 1/3 has been investigated by Bomze and Pawlowitsch [46].

For the Fermi process, the 1/2 rule is extended to any selection intensity and the 1/3 rule still holds [5,6,47]. However, to our best knowledge, no study has worked on the fixation probability for the case where the fitness of A is equal to that of B at an A frequency of 1/3.

For weak selection, from Eq. (4), the ratio of the transition probabilities can be approximated as

$$\frac{\widehat{p_{i,i-1}}}{\widehat{p_{i,i+1}}} \sim 1 + \beta(P_B - P_A + \tau_2/2 - \tau_1/2). \quad (19)$$

Then, the payoff matrix Eq. (1) can be rescaled as

$$\begin{array}{cc} & \begin{array}{c} A \\ B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a + \tau_1/2 & b + \tau_1/2 \\ c + \tau_2/2 & d + \tau_2/2 \end{pmatrix}. \end{array} \quad (20)$$

Corresponding to Eq. (20), the respective payoffs are $P'_A(i) = P_A(i) + \frac{\tau_1}{2}$ and $P'_B(i) = P_B(i) + \frac{\tau_2}{2}$, and the respective transition probabilities are

$$\begin{aligned} p'_{i,i-1} &= \frac{i}{N} \frac{N-i}{N} \frac{1}{1 + e^{-\beta[P'_A(i) - P'_B(i)]}}, \\ p'_{i,i+1} &= \frac{i}{N} \frac{N-i}{N} \frac{1}{1 + e^{-\beta[P'_B(i) - P'_A(i)]}}. \end{aligned} \quad (21)$$

By Eq. (21), we again get, under weak selection ($\beta \ll 1$),

$$\frac{p'_{i,i-1}}{p'_{i,i+1}} \sim 1 + \beta(P_B - P_A + \tau_2/2 - \tau_1/2). \quad (22)$$

This is why the payoff matrix Eq. (1) is rescaled as the payoff matrix Eq. (20) when ρ_A and $\frac{\rho_A}{\rho_B}$ are considered.

The 1/2 rule is applied to the rescaled matrix Eq. (20), which generates the 1/2 rule with inertia for weak selection (the rescaled matrix only holds for weak selection) and sufficiently large populations:

If $\frac{a}{2} + \frac{b}{2} - \frac{c}{2} - \frac{d}{2} > \frac{(\tau_2 - \tau_1)}{2}$, then $\rho_A > \rho_B$, i.e., A is more likely to replace B than vice versa.

If $\frac{a}{2} + \frac{b}{2} - \frac{c}{2} - \frac{d}{2} \leq \frac{(\tau_2 - \tau_1)}{2}$, then $\rho_A \leq \rho_B$, i.e., B is more likely to replace A than vice versa.

A strategy is risk-dominant if it has a higher payoff against a player equiprobably choosing feasible strategies. The original 1/2 rule tells us that the risk-dominant strategy takes over the whole population with a higher fixation probability. Nevertheless, the above outcome says that we can modify inertia parameters τ_1 and τ_2 so that the risk-dominant strategy no longer has the advantage.

Homoplastically, the 1/3 rule is applied to the rescaled matrix Eq. (20), which creates the 1/3 rule with inertia for weak selection and sufficiently large populations:

If $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} > \frac{1}{2}(\tau_2 - \tau_1)$, then $\rho_A > 1/N$, i.e., selection favors strategy A .

If $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} < \frac{1}{2}(\tau_2 - \tau_1)$, then $\rho_A < 1/N$, i.e., selection doesn't favor strategy A .

A problem is naturally posed as to what will happen in the case of $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1)$. We consider the inverse of $\rho_A(\beta)$:

$$\begin{aligned} \frac{1}{\rho_A(\beta)} &= 1 + \sum_{k=1}^{N-1} \prod_{j=1}^k \frac{p'_{j,j-1}}{p'_{j,j+1}} \\ &= 1 + \sum_{k=1}^{N-1} \exp \left\{ -\beta \left[\frac{k^2}{N-1} s + \frac{k}{N-1} (s+t) \right] \right\}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} s &= \frac{a - b - c + d}{2}, \\ t &= -a + bN - d(N-1) + \frac{N-1}{2}(\tau_1 - \tau_2), \end{aligned} \quad (24)$$

and perform a Taylor series expansion of $\frac{1}{\rho_A}$ at $\beta = 0$:

$$\frac{1}{\rho_A(\beta)} = \frac{1}{\rho_A(0)} + \left[\frac{1}{\rho_A(0)} \right]' \beta + \frac{\left[\frac{1}{\rho_A(0)} \right]''}{2!} \beta^2 + O(\beta^2).$$

After some easy calculations,

$$\begin{aligned} \frac{1}{\rho_A(0)} &= N, \\ \left[\frac{1}{\rho_A(0)} \right]' &= -\frac{N}{6} [(N-2)a + (2N-1)b - (N+1)c \\ &\quad - (2N-4)d - \frac{3}{2}(N-1)(\tau_2 - \tau_1)]. \end{aligned} \quad (25)$$

For the case $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1)$, the first-order derivative Eq. (25) reduces to $-\frac{N}{6}[-2a - b - c + 4d + \frac{3}{2}(\tau_2 - \tau_1)]$. If $-2a - b - c + 4d + \frac{3}{2}(\tau_2 - \tau_1) > 0$, then $\rho_A > 1/N$. When the formula $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1)$ is taken into consideration, this inequality can be simplified as $b - c - \frac{1}{2}(\tau_2 - \tau_1) > 0$. Likewise, if $b - c - \frac{1}{2}(\tau_2 - \tau_1) < 0$, then $\rho_A < 1/N$.

When $\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1)$ and $b - c - \frac{1}{2}(\tau_2 - \tau_1) = 0$, i.e., $[\frac{1}{\rho_A(0)}]' = 0$, the first-order approximation does not allow us to assess ρ_A relative to $1/N$. From Eq. (26), we get that the second-order approximation is positive for $N > 2$, namely, $\rho_A < 1/N$ in such a case:

$$\begin{aligned} \left[\frac{1}{\rho_A(0)} \right]'' &= \frac{N(6N^3 - 9N^2 + N + 1)}{30(N-1)} s^2 + \frac{N^2}{2} s(s+t) \\ &+ \frac{2N^2 - N}{6(N-1)} (s+t)^2 \\ &= \frac{N(2N^3 - 3N^2 - 3N + 2)}{135(N-1)} s^2. \end{aligned} \quad (26)$$

To sum up the above arguments, we acquire the extended 1/3 rule with inertia for weak selection ($\beta \ll 1$) and a population whose size exceeds two:

If

$$\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} > \frac{1}{2}(\tau_2 - \tau_1),$$

or if

$$\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1), \quad b - c - \frac{1}{2}(\tau_2 - \tau_1) > 0,$$

then $\rho_A > 1/N$, i.e., selection favors cooperation.

If

$$\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} < \frac{1}{2}(\tau_2 - \tau_1),$$

or if

$$\frac{a}{3} + \frac{2b}{3} - \frac{c}{3} - \frac{2d}{3} = \frac{1}{2}(\tau_2 - \tau_1), \quad b - c - \frac{1}{2}(\tau_2 - \tau_1) \leq 0,$$

then $\rho_A < 1/N$, i.e., selection doesn't favor cooperation.

B. The conditional fixation time

Given that the Markov process reaches the absorbing state $v = N$, the time that it takes when starting in the state $v = 1$ is called A 's conditional fixation time and denoted by T^A . Similarly, given that the Markov process reaches the absorbing state $v = 0$, the time that it takes when starting in the state $v = N - 1$ is called B 's conditional fixation time and expressed as T^B . Antal and Scheuring confirmed a surprising and interesting result $T^A = T^B$ [48], hence the conditional fixation time for the birth-death process has a unified explicit form T_{fix} :

$$T_{\text{fix}}(N) = \sum_{n=1}^{N-1} \frac{y_{0,n-1} y_{n,N-1}}{\widehat{p_{n,n+1}} q_n y_{0,N-1}}, \quad (27)$$

where

$$\begin{aligned} q_n &= \prod_{i=1}^n \frac{\widehat{p_{i,i-1}}}{\widehat{p_{i,i+1}}}, \quad y_{n,m} = \sum_{k=n}^m \prod_{i=1}^k \frac{\widehat{p_{i,i-1}}}{\widehat{p_{i,i+1}}}, \\ \forall n, m &\in \{0, \dots, N\}, \\ \frac{1}{\widehat{p_{n,n+1}}} &= \frac{N}{n} \frac{N}{N-n} (1 + e^{-\beta(P_A - P_B - \tau_2)}), \end{aligned}$$

where the empty sum is defined to be zero and the empty product is defined to be one.

The conditional fixation time for the neutral game is an important reference. Neutrality results from vanishing selection intensity (i.e., $\beta = 0$), which means (1) no game is being played, (2) all individuals have the same fitness, or (3) inertia is not introduced. In this case, it is easy to see

$$\begin{aligned} q_n &= 1, \quad y_{n,m} = m - n + 1, \quad \forall n \leq m \in \{0, \dots, N\}, \\ \frac{1}{\widehat{p_{n,n+1}}} &= \frac{2N \times N}{n \times (N-n)}. \end{aligned}$$

Substituting these into Eq. (27), the conditional fixation time for the neutral game is $2N(N-1)$, which measures the average "speed" of fixation and is different from the corresponding quantity $N(N-1)$ for the Moran process.

The conditional fixation time without inertia has been studied by Altrock and Traulsen [49]. Here, we investigate how inertia transforms the conditional fixation time. Note that we can not employ the rescaled matrix Eq. (20) in this case.

For weak selection ($\beta \ll 1$), let $x(n) = \frac{n(n+1)}{N-1} s + \frac{n}{N-1} t$ and $z(n) = \frac{2n}{N-1} s + \frac{1}{N-1} t - \frac{\tau_1 - \tau_2}{2}$ where s and t have been given in Eq. (24); the first-order approximations of all factors in Eq. (27) are expounded below:

$$q_n \approx 1 - \beta \times x(n),$$

$$y_{n,m} \approx m - n + 1 - \beta \times \sum_{k=n}^m x(k), \quad (28)$$

$$\frac{1}{\widehat{p_{n,n+1}}} \approx \frac{N \times N}{(N-n) \times n} \{2 - \beta \times [z(n) - \tau_2]\}.$$

Inserting Eq. (28) into Eq. (27), we have

$$\begin{aligned} T_{\text{fix}} &\approx 2N(N-1) - \beta \frac{N(N-2)}{18} \\ &\times \left[(N+3)(a-b-c+d) - 9 \frac{N-1}{N-2} (\tau_1 + \tau_2) \right]. \end{aligned} \quad (29)$$

As a consequence, the speed criterion with inertia for weak selection is

If

$$(N+3)(a-b-c+d) > 9 \frac{N-1}{N-2} (\tau_1 + \tau_2),$$

then $t_{\text{fix}} < 2N(N-1)$, i.e., fixation is fast.

If

$$(N+3)(a-b-c+d) < 9 \frac{N-1}{N-2} (\tau_1 + \tau_2),$$

then $t_{\text{fix}} > 2N(N-1)$, i.e., fixation is slow.

When inertia is not introduced into the Fermi process (i.e., $\tau_1 = \tau_2 = 0$), A 's fixation probability and the conditional

fixation time are seen as benches and denoted by ρ_A^B and T_{fix}^B . In the case of $\tau_1 = \tau_2$ under weak selection, we get $\rho_A = \rho_A^B$, and so A 's fixation probability is unaffected if two strategies have the same inertia. In the case of $\tau_1 = -\tau_2$, we get $T_{\text{fix}} = T_{\text{fix}}^B$ under weak selection, and so the conditional fixation time is unaffected if two strategies have the opposite inertia.

In particular, for the Prisoner's Dilemma games satisfying $a + d < b + c$, we have $\rho_A^B < 1/N$ and $T_{\text{fix}}^B > 2N(N - 1)$, hence cooperation is not favored by selection and fixation is slow. In consideration of the extended 1/3 rule with inertia and the speed criterion with inertia, we can regulate inertia so that cooperation is favored by selection and fixation is fast.

VI. DISCUSSIONS AND CONCLUSIONS

The costs of switching strategy were originally proposed by Szabó and Hauert [11], but their roles were not addressed at length. It has been experimentally substantiated that inertia does exist in the human imitation behavior and possesses the property of asymmetry; i.e., the cost for A switching to B may differ from that for the reverse process [10]. The costs actually correspond to A 's inertia and B 's inertia in our model, which are governed by tunable quantities τ_1 and τ_2 , respectively. We make a comprehensive analysis upon the effect of inertia on the quantities of interest: the stationary distribution, the stochastically stable state, the fixation probability, and the conditional fixation time.

We analyzed how inertia quantitatively varies the profile of the stationary distribution where the threshold values of inertia are crucial.

We found that inertia leads the stochastic stability in the limit of strong selection to demonstrate three possible regimes, "only A is stochastically stable," "both A and B are stochastically stable with an equal equilibrium probability," and "only B is stochastically stable." Correspondingly, the plane spanned by τ_1 and τ_2 is divided into 21 regions, seven of which correspond to one regime. Substantially, a methodical analysis upon the impact of inertia on $\pi^*(0)/\pi^*(N)$ is constructed. It is easy to verify that ρ_A/ρ_B is equal to $\pi^*(0)/\pi^*(N)$. Consequently, a systematic analysis of how inertia works on ρ_A/ρ_B can correspondingly be furnished.

We obtained the extended 1/3 rule with inertia and the speed criterion with inertia for weak selection. In our model, strategy A 's imitation probability $\frac{1}{1+e^{-\beta(\Delta\pi-\tau_1)}}$ varies from that of strategy B $\frac{1}{1+e^{-\beta(\Delta\pi-\tau_2)}}$, where $\Delta\pi$ is the payoff difference between the focal individual and the model individual. As for the fixation probability under weak selection, the rescaled matrix is applied and the switching probability has a unified form $g(\beta\Delta\pi) = \frac{1}{1+e^{-\beta\Delta\pi}}$. Consequently, the extended 1/3 rule with inertia holds for such pairwise comparison processes satisfying $g'(0) > 0$ and $g(0) > 0$ according to the conclusion in [50]. For the conditional fixation time under weak selection, the rescaled matrix no longer applies and the unified form of imitation probability can't characterize the pairwise comparison process integrating inertia. Nonetheless, adopting the discussion similar to [50], the speed criterion with inertia can also be extended for a special class of pairwise comparison processes.

We illustrated the above conclusions by taking the Prisoner's Dilemma game. The effects of two strategies' inertia are slightly different. The dilemma under cooperation's inertia can be alleviated or even entirely removed, while that under defection's inertia can't be thoroughly eliminated. The impacts of inertia upon the game with $a + d - b - c$ of different signs are significantly different. For $a + d - b - c > 0$, the stationary distribution under inertia can simultaneously maximize at two full states, while, for $a + d - b - c < 0$, its largest value can be achieved at a coexistence state. Appropriate inertia makes it possible that the state of full cooperation is stochastically stable. In the absence of mutation, under the help of appropriate inertia, selection favors cooperation and fixation is fast.

In conclusion, we made a thorough analysis on what influence inertia executes upon the strategy evolution and, particularly, the evolution of cooperation.

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APPENDIX A

We are now in a position to prove formally Theorems 1 and 2, and we first introduce four necessary lemmas for them.

Lemma 1 [51]. If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \leq f_{n+1}(x)$, then $\lim_{n \rightarrow +\infty} \int f_n(x) dx = \int \lim_{n \rightarrow +\infty} f_n(x) dx$.

Lemma 2 [52]. Let $f(x)$ increase monotonically, suppose that $\lim_{n \rightarrow +\infty} x_n = +\infty$, then $\lim_{x \rightarrow +\infty} f(x) = \lim_{n \rightarrow +\infty} f(x_n)$.

Lemma 3 [51]. If $|f_n(x)| \leq g(x)$ for all n , and $g(x)$ is integrable, then $\lim_{n \rightarrow +\infty} \int f_n(x) dx = \int \lim_{n \rightarrow +\infty} f_n(x) dx$.

Lemma 4 [52]. If arbitrarily choosing a sequence $\{x_n\}$ and $\lim_{n \rightarrow +\infty} x_n = +\infty$, we have that $\lim_{n \rightarrow +\infty} f(x_n) = A$. Then we say that $f(x)$ approaches A as x approaches $+\infty$; or $f(x)$ has limit A at $+\infty$ and write $f(x) \rightarrow A$ as $x \rightarrow +\infty$. And the reverse is also true.

Proof of Theorem 1

Proof. Let $(\beta_n)_{n=1}^{+\infty}$ be an increasing sequence and $\beta_n \rightarrow +\infty$. Then, $\{h(x, \beta_n)\}$ is a sequence of non-negative measurable functions, and $h(x, \beta_n) \leq h(x, \beta_{n+1})$ since $\omega(x) \geq 0$ for all $x \in \chi$. So Lemma 1 implies $\lim_{n \rightarrow +\infty} \int_{\chi} h(x, \beta_n) dx = \int_{\chi} \lim_{n \rightarrow +\infty} h(x, \beta_n) dx$.

According to $\omega(x) \geq 0$ for all $x \in \chi$, we also get that $\int_{\chi} h(x, \beta) dx$ is an increasing function with respect to β . Then Lemma 2 suggests $\lim_{\beta \rightarrow +\infty} \int_{\chi} h(x, \beta) dx = \lim_{n \rightarrow +\infty} \int_{\chi} h(x, \beta_n) dx$.

Again applying Lemma 2 to $h(x, \beta_n)$, we have $\lim_{\beta \rightarrow +\infty} h(x, \beta) = \lim_{n \rightarrow +\infty} h(x, \beta_n)$. Then $\lim_{\beta \rightarrow +\infty} \int_{\chi} h(x, \beta) dx = \int_{\chi} \lim_{\beta \rightarrow +\infty} h(x, \beta) dx$.

Proof of Theorem 2

Proof. For every sequence $(\beta_n)_{n=1}^{+\infty}$ such that $\beta_n \rightarrow +\infty$. $|h(x, \beta_n)| \leq \ln 2$ for all n since $\omega(x) \leq 0$, $\forall x \in \chi$.

Apparently $\ln 2$ is integrable in χ . By Lemma 3, we have $\lim_{n \rightarrow +\infty} \int_{\chi} h(x, \beta_n) dx = \int_{\chi} \lim_{n \rightarrow +\infty} h(x, \beta_n) dx$.

Due to the arbitrariness of $\{\beta_n\}$, Lemma 4 suggests $\lim_{\beta \rightarrow +\infty} \int_{\chi} h(x, \beta) dx = \int_{\chi} \lim_{\beta \rightarrow +\infty} h(x, \beta) dx$.

APPENDIX B

Let $\phi(x) = \rho x + \xi$ and $\varphi(x) = -\rho x + \eta$. The sketch of the analysis process is as follows.

In the first step, we determine the interval χ_1 where $\phi(x) \geq 0$, the interval χ_2 where $\phi(x) \leq 0$, the interval χ_3 where $\varphi(x) \geq 0$, and the interval χ_4 where $\varphi(x) \leq 0$. Since $\phi(x)$ is an increasingly continuous function, the intervals χ_1 and χ_2 depend on the endpoint values $\phi(0)$ and $\phi(N - 1)$.

If $\phi(0) \geq 0$, then $\chi_1 = [0, N - 1]$.

If $\phi(0) < 0$ and $\phi(N - 1) > 0$, then $\chi_1 = [M, N - 1]$ and $\chi_2 = [0, M]$ where $\phi(M) = 0$.

If $\phi(N - 1) \leq 0$, then $\chi_2 = [0, N - 1]$.

Similarly, since $\varphi(x)$ is a decreasingly continuous function, the intervals χ_3 and χ_4 depend on the endpoint values $\varphi(0)$ and $\varphi(N - 1)$.

If $\varphi(N - 1) \geq 0$, then $\chi_3 = [0, N - 1]$.

If $\varphi(0) > 0$ and $\varphi(N - 1) < 0$, then $\chi_3 = [0, G]$ and $\chi_4 = [G, N - 1]$ where $\varphi(G) = 0$.

If $\varphi(0) \leq 0$, then $\chi_4 = [0, N - 1]$.

In the second step, we exchange the order of the limit and the integral and calculate $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N)$.

$$\begin{aligned} & \lim_{\beta \rightarrow +\infty} \int_{\chi_1} \ln[1 + e^{\beta\phi(x)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_1} \beta\phi(x) dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_1} \beta(\rho x + \xi) dx \\ &= \lim_{\beta \rightarrow +\infty} \beta \left(\frac{1}{2} \rho x^2 + \xi x \right) \Big|_{\chi_1} \end{aligned}$$

$$\begin{aligned} & \lim_{\beta \rightarrow +\infty} \int_{\chi_2} \ln[1 + e^{\beta\phi(x)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_2} e^{\beta\phi(x)} dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_2} e^{\beta(\rho x + \xi)} dx \\ &= \lim_{\beta \rightarrow +\infty} \frac{e^{\beta(\rho x + \xi)}}{\beta\rho} \Big|_{\chi_2} \end{aligned}$$

$$\begin{aligned} & \lim_{\beta \rightarrow +\infty} \int_{\chi_3} \ln[1 + e^{\beta\varphi(x)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_3} \beta\varphi(x) dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_3} \beta(-\rho x + \eta) dx \\ &= \lim_{\beta \rightarrow +\infty} \beta \left(-\frac{1}{2} \rho x^2 + \eta x \right) \Big|_{\chi_3} \end{aligned}$$

$$\begin{aligned} & \lim_{\beta \rightarrow +\infty} \int_{\chi_4} \ln[1 + e^{\beta\varphi(x)}] dx \\ &= \lim_{\beta \rightarrow +\infty} \int_{\chi_4} e^{\beta\varphi(x)} dx = \lim_{\beta \rightarrow +\infty} \int_{\chi_4} e^{\beta(-\rho x + \eta)} dx \\ &= \lim_{\beta \rightarrow +\infty} \frac{e^{\beta(-\rho x + \eta)}}{-\beta\rho} \Big|_{\chi_4} \end{aligned}$$

In the third step, we obtain the stochastic stability.

If $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = +\infty$, then only A is stochastically stable.

If $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = 0$, then both A and B are stochastically stable with the same equilibrium probability.

If $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N) = -\infty$, then only B is stochastically stable.

Following the above analysis process, the plane spanned by τ_1 and τ_2 is divided into 21 regions in accordance with $\lim_{\beta \rightarrow +\infty} \ln \lambda^*(N)$.

In the following seven regions, only A is stochastically stable:

(1) $\tau_1 \geq -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $\tau_2 - \tau_1 < 2\Omega + \rho(N - 1)$.

(2) $\tau_1 \geq -\Omega$, $\Omega < \tau_2 < \rho(N - 1) + \Omega$ and $(N - 1)\tau_1 - G\tau_2 > -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 - (N - 1)\Omega - G\Omega$.

(3) $\tau_1 \geq -\Omega$ and $\tau_2 \leq \Omega$.

(4) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - (N - 1)\tau_2 > \frac{1}{2}\rho M^2 - \rho(N - 1)^2 - (2N - 2 - M)\Omega$.

(5) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\Omega < \tau_2 < \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - G\tau_2 > -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 + \frac{1}{2}\rho M^2 - (N - 1 - M + G)\Omega$.

(6) $-\frac{1}{2}\rho(M + N - 1) - \Omega < \tau_1 < -\Omega$ and $\tau_2 \leq \Omega$.

(7) $\tau_1 \leq -\Omega - \rho(N - 1)$ and $\Omega < \tau_2 < \Omega + \frac{1}{2}\rho G$.

In the following seven regions, both A and B are stochastically stable:

(1) $\tau_1 \geq -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $\tau_2 - \tau_1 = 2\Omega + \rho(N - 1)$.

(2) $\tau_1 \geq -\Omega$, $\Omega < \tau_2 < \rho(N - 1) + \Omega$ and $(N - 1)\tau_1 - G\tau_2 = -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 - (N - 1)\Omega - G\Omega$.

(3) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - (N - 1)\tau_2 = \frac{1}{2}\rho M^2 - \rho(N - 1)^2 - (2N - 2 - M)\Omega$.

(4) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\Omega < \tau_2 < \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - G\tau_2 = -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 + \frac{1}{2}\rho M^2 - (N - 1 - M + G)\Omega$.

(5) $\tau_1 = -\frac{1}{2}\rho(M + N - 1) - \Omega$ and $\tau_2 \leq \Omega$.

(6) $\tau_1 \leq -\Omega - \rho(N - 1)$ and $\tau_2 = \Omega + \frac{1}{2}\rho G$.

(7) $\tau_1 < -\rho(N - 1) - \Omega$, $\tau_2 < \Omega$.

In the following seven regions, only B is stochastically stable

(1) $\tau_1 \geq -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $\tau_2 - \tau_1 > 2\Omega + \rho(N - 1)$.

(2) $\tau_1 \geq -\Omega$, $\Omega < \tau_2 < \rho(N - 1) + \Omega$ and $(N - 1)\tau_1 - G\tau_2 < -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 - (N - 1)\Omega - G\Omega$.

(3) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\tau_2 \geq \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - (N - 1)\tau_2 < \frac{1}{2}\rho M^2 - \rho(N - 1)^2 - (2N - 2 - M)\Omega$.

(4) $-\Omega - \rho(N - 1) < \tau_1 < -\Omega$, $\Omega < \tau_2 < \Omega + \rho(N - 1)$ and $(N - 1 - M)\tau_1 - G\tau_2 < -\frac{1}{2}\rho(N - 1)^2 - \frac{1}{2}\rho G^2 + \frac{1}{2}\rho M^2 - (N - 1 - M + G)\Omega$.

(5) $-\Omega - \rho(N - 1) < \tau_1 < -\frac{1}{2}\rho(M + N - 1) - \Omega$ and $\tau_2 \leq \Omega$.

(6) $\tau_1 \leq -\Omega - \rho(N - 1)$ and $\tau_2 \geq \Omega + \rho(N - 1)$.

(7) $\tau_1 \leq -\Omega - \rho(N - 1)$ and $\Omega + \frac{1}{2}\rho G < \tau_2 < \Omega + \rho(N - 1)$.

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