Automatica 48 (2012) 25–35

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Connectivity preservation for multi-agent rendezvous with link failure☆

Feng Xiao [a,1], Long Wang [b], Tongwen Chen [c]

[a] School of Automation, Beijing Institute of Technology, Beijing 100081, China

[b] Intelligent Control Laboratory, Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China

[c] Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2V4

ARTICLE INFO

Article history:
Received 13 September 2010
Received in revised form
20 January 2011
Accepted 9 June 2011
Available online 5 October 2011

Keywords:
Multi-agent systems
Rendezvous
Directed networks
Connectivity preservation
Link failure

ABSTRACT

This paper proposes a new connectivity-preserving protocol in terms of rectangle-like regions. The protocol consists of a set of distributed control rules; their working together guarantees the network connectivity as well as rendezvous of a discrete-time multi-agent system. It is assumed that all agents share a common minimum sensing radius, but the information exchange may suffer from link failure and recovery. Consequently, the interaction topology is in fact directed and time-varying. By rigorous mathematical arguments, we show the effectiveness and robustness of the protocol in the presence of alignment errors in local coordinate orientations of agents and measurement errors in relative positions of neighbors. We also present simulations to demonstrate the effectiveness of the theoretical results.

1. Introduction

Connectivity preservation is an interesting and challenging topic in studying the stability and controllability of distributed multi-agent networks (Ji, Wang, Lin, & Wang, 2009; Lin, Broucke, & Francis, 2004; Tanner, 2004; Xie & Wang, 2009) and has attracted a great deal of attention from a number of researchers. This paper studies the connectivity preservation in the framework of rendezvous control of directed networks.

In recent studies of rendezvous problems, numerous protocols have been proposed for different practical situations (Dimarogonas & Kyriakopoulos, 2007; Fagnani, Johansson, Speranzon, & Zampieri, 2004; Lin, Francis, & Maggiore, 2005; Litus, Zebrowski, & Vaughan, 2009; Smith, Broucke, & Francis, 2007). As for the concern with connectivity preservation, the *circumcenter algorithm* is a well-known rendezvous protocol, proposed for a group of mobile robots with limited visibility in Ando, Oasa, Suzuki, and Yamashita (1999). The circumcenter algorithm is a distributed memoryless protocol and its validity of driving all robots to gather at a common location was proved under the assumptions that each robot is able to track its neighbors' positions instantaneously and every pair of robots are mutually visible. In the subsequent studies, the circumcenter algorithm has been generalized in various ways, for example, the synchronous and the asynchronous versions with continuous-time dynamics (Lin, Morse, & Anderson, 2007a,b), and the extended versions with milder discontinuous control laws (Conte & Pennesi, 2010) and with noisy measurements (Martínez, 2009). There are also some other representative connectivity-preserving rendezvous protocols, including the ones based on the distributed gradient method (Ji & Egerstedt, 2007) and the spectral analysis of interaction graphs (Yang et al., 2010; Zavlanos & Pappas, 2007). However, these algorithms are all built on the critical assumption of bidirectional interactions.

Different from ensuring connectivity directly by control laws, there exists some other related work, which considers the relationship between network connectivity and model parameters, and presents various sufficient conditions for connectivity preservation. In Sun and Huang (2009), the authors provided an initial-connectivity-based sufficient condition in terms of the degree of each node for dynamic connectivity under the linear feedback protocol, proposed in Olfati-Saber and Murray (2004). In Gustavi, Dimarogonas, Egerstedt, and Hu (2010), by studying the effect of the ratio of leaders-to-followers and the magnitude of goal attraction forces experienced by leaders on connectivity, the

☆ This work was supported by the Natural Sciences and Engineering Research Council of Canada, a University of Alberta Killam Postdoctoral Fellowship, 973 Program (2012CB821203) and NSFC (61020106005 and 10972002); in addition, Feng Xiao's work was supported in part by NSFC (60904062, 60925011). This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by the Associate Editor Masayuki Fujita under the direction of the Editor Ian R. Petersen.

E-mail addresses: fxiao@ece.ualberta.ca (F. Xiao), longwang@pku.edu.cn (L. Wang), tchen@ece.ualberta.ca (T. Chen).

[1] Corresponding author. He is currently a Killam Postdoctoral Fellow in the Department of Electrical and Computer Engineering, University of Alberta. Fax: +1 780 492 1811.

0005-1098/$ – see front matter
doi:10.1016/j.automatica.2011.09.027

authors presented connectivity-preserving conditions for a complete graph and a special case of incomplete graphs. We observe that so far these studied protocols have not been integrated with connectivity-preserving control rules and thus the given sufficient conditions are very conservative and depend on the network structures; this restricts their applications in practical distributed control tasks.

The main contribution of this paper is to present a new protocol for connectivity preservation in the rendezvous control of a discrete-time multi-agent system. The protocol is formulated by a collection of rectangle-like regions, and consists of several distributed control rules as well as a convex combination restriction, which guarantee that the distance between neighboring agents cannot exceed the minimum sensing radius, shared by all agents, and all agents will be eventually gathered at a common place. The novelty of this paper is summarized as follows.

First, the proposed protocol is a new direct connectivity-preserving control law, and it is effective in time-varying directed networks. This paper assumes that all agents share a common minimum sensing radius. Due to the limitation of on-board sensing instruments, directional sensing area, and the existence of obstacles, this paper further assumes that the information exchange may suffer from link failure and recovery, and thus the interaction topology is in fact time-varying, and it is directed at each time instant. These weak restrictions are natural and ubiquitous in many applications but they make the protocol design very challenging. Thus they are usually unseen in previous direct connectivity-preserving algorithms, such as various versions of the circumcenter algorithm (Ando et al., 1999; Conte & Pennesi, 2010; Lin et al., 2007a,b; Martínez, 2009), distributed gradient protocol (Ji & Egerstedt, 2007) and spectral analysis protocol (Yang et al., 2010; Zavlanos & Pappas, 2007).

Second, the proposed protocol is with memory, contrary to the memoryless circumcenter algorithm (Ando et al., 1999); but this property makes the usage of delayed information possible, which is natural in many practical situations. Furthermore, the protocol can be easily modified for the case with communication time delays; this feature distinguishes our protocol from others.

Third, by rigorous mathematical arguments, we show the effectiveness and robustness of the protocol in the presence of alignment errors in local coordinate orientations of agents and measurement errors in relative positions of neighbors. This paper assumes that each agent possesses a local Cartesian coordinate system. At each time, the subsequent position of each agent is selected from the intersection of a collection of rectangle regions, which are related to its local coordinate system. To ensure the proposed protocol to work efficiently, these local coordinate orientations should be aligned with each other by some consensus algorithms (Cortés, 2009; Ren & Beard, 2005), but here two kinds of alignment errors, namely, maximum static errors and maximum sum errors, are acceptable. Moreover, we also give an upper bound for allowable measurement errors without losing the network connectivity.

Finally, the theoretical work of this paper is a contribution to the development of network controllability theory and consensus theory; moreover, it has some potential applications in the formation of multi-agent systems. The consensus problem includes the rendezvous problem as a special case, and it emphasizes more abstract agreement quantities. Instances of them other than positions include anticipated attitude in multiple spacecraft alignment, velocity in flocking control, etc. (Lin, Francis, & Maggiore, 2007; Olfati-Saber, Fax, & Murray, 2007; Ren, Beard, & Atkins, 2007). This work is an extension to our previous result, presented in Xiao and Wang (2008), in which we proved the effectiveness of a class of consensus protocols with time-varying delays and switching interaction topologies. This paper develops additional connectivity-preserving control rules for the state-dependent graph case and shows their compatibility with the convex combination consensus protocols, the extended versions of the Vicsek model (Vicsek, Czirók, Ben-Jacob, Cohen, & Schochet, 1995), proposed in Xiao and Wang (2008). Similar control ideas can be applied in the formation control of directed networks.

This paper is organized as follows. The preliminary notion is assembled in Section 2. The model is set up in Section 3; and the rendezvous and connectivity-preserving results are presented in Sections 4 and 5, respectively. Simulations are given in Section 6. Finally, concluding remarks are stated in Section 7. In the Appendix, the proof of the convergence result with measurement errors is attached.

2. Preliminaries

In this section, we introduce several notions in graph theory and list some useful notation for later reference; for further details, see Xiao and Wang (2008) and Godsil and Royal (2001).

A *directed graph* $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and an edge set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$. Let $\mathcal{V}(\mathcal{G})$ be $\{v_1, v_2, \ldots, v_n\}$. A *path* in a directed graph $\mathcal{G}$ from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of finite vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}(\mathcal{G})$ for $j = 1, 2, \ldots, k-1$. A directed graph $\mathcal{G}$ is said to *have a spanning tree* if there exists a vertex, called the *root*, such that it can be connected to all other vertices through paths. If for any i, j, $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ implies that $(v_j, v_i) \in \mathcal{E}(\mathcal{G})$, then $\mathcal{G}$ is *undirected*. In this case, $\mathcal{G}$ is *connected* if $\mathcal{G}$ has a spanning tree.

Notation. Let $\mathcal{I}_n = \{1, 2, \ldots, n\}$, and let $\|\cdot\|_\infty$ and $\|\cdot\|_2$ represent the maximum norm and Euclidean norm respectively. For any set of vectors $\mathcal{S}$ and any matrix A with compatible dimensions, $A\mathcal{S} = \{A\xi : \xi \in \mathcal{S}\}$. For any point $\xi \in \mathbb{R}^2$, let $\xi^{(1)}$ and $\xi^{(2)}$ denote its first and second coordinates respectively; i.e., $\xi = [\xi^{(1)}, \xi^{(2)}]^T$. Given a point $p \in \mathbb{R}^2$ in the global coordinate system and angle $\theta \in \mathbb{R}$, let $\mathcal{O}(p, \theta)$ denote the local coordinate system with p as its origin and θ as the axis angle difference from the global coordinate axes.

3. Problem formulation

The objective of this paper is to present a distributed control protocol for the rendezvous and connectivity preservation of a group of agents, moving in a two-dimensional plane. As we will see, the main results can be extended easily to high-dimensional cases. This section will formulate the problem to be studied.

Suppose that the multi-agent system consists of n agents, labeled 1 through n. Let $p_i(t) \in \mathbb{R}^2$ denote the position of agent i at time t in a given global Cartesian coordinate system and suppose that each agent i possesses a moving (time-varying) local Cartesian coordinate system, denoted by $\mathcal{O}_i(t)$, with agent i located at the origin. For convenience, denote the angle difference between the coordinate axes of $\mathcal{O}_i(t)$ and the global coordinate axes by $\theta_i(t)$. Then we have $\mathcal{O}_i(t) = \mathcal{O}(p_i(t), \theta_i(t))$, as shown in Fig. 1. In the sequel, we use $p_j|_{\mathcal{O}_i(t)} = [p_j^{(1)}|_{\mathcal{O}_i(t)}, p_j^{(2)}|_{\mathcal{O}_i(t)}]^T$ to denote the coordinates of agent j in the local coordinate system $\mathcal{O}_i(t)$. Clearly, $p_j(t)|_{\mathcal{O}_i(t)}$ represents the relative position of agent j, related to agent i, and $p_i(t)|_{\mathcal{O}_i(t)} = 0$.

The main results of this paper will be developed based on the assumption that the local coordinate orientations of all agents have already been aligned with each other, but alignment errors are allowable. For convenience, the global coordinate system is assumed to be with the closest orientation in some metric to all of the local coordinate systems. Therefore, $\theta_i(t)$ represents the orientation alignment error of the local coordinate system $\mathcal{O}_i(t)$. This coordinate orientation alignment can be achieved by performing the traditional asymptotical consensus algorithms (Olfati-Saber & Murray, 2004; Ren & Beard, 2005) or finite-time consensus algorithms (Wang & Xiao, 2010). Interested readers may also refer to Cortés (2009) for possible alignment strategies.

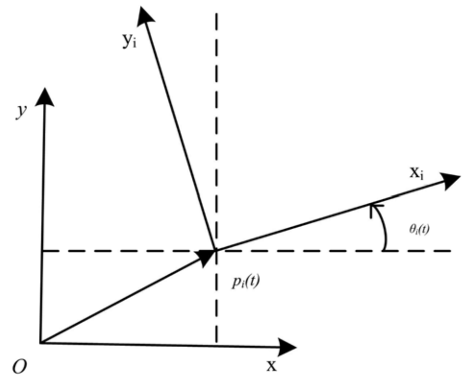


Fig. 1. The local coordinate system $\mathcal{O}_i(t)$, where O, $\mathbf{x}$, $\mathbf{y}$ represent the global origin and the unit vectors for the global horizontal and vertical axes, respectively, and $\mathbf{x}_i$ and $\mathbf{y}_i$ represent the unit vectors for the local horizontal and vertical axes of $\mathcal{O}_i(t)$, respectively.

With the above preparations, assume that agent i in the network is governed by the following discrete-time equation:

$$p_i(t+1)|_{\mathcal{O}_i(t)} = u_i(t), \quad t \in \mathbb{N}, \tag{1}$$

where $u_i(t)$ is a distributed state feedback, called *protocol* or *algorithm*, to be designed based on the relative positions of agents within the local sensing range of agent i. In the global coordinate system, Eq. (1) can be equivalently represented by

$$p_i(t+1) = p_i(t) + \Theta(\theta_i(t))u_i(t), \quad t \in \mathbb{N}, \tag{2}$$

where

$$\Theta(\theta_i(t)) = \begin{bmatrix} \cos(\theta_i(t)) & -\sin(\theta_i(t)) \\ \sin(\theta_i(t)) & \cos(\theta_i(t)) \end{bmatrix}$$

is a rotation matrix.

To characterize the interaction topology among agents, assume that each agent has a limited sensing range and can measure the relative positions of agents within the range. The sensing areas of different agents may be in different shapes, depending on the on-board sensing instruments and outside environments. Furthermore, assume that there exists a common minimum sensing radius R, that is, if $\|p_i - p_j\|_2 \le R$, then by rotating detectors, agent i has the possibility of getting the relative position of agent j, and vice versa. Then the adjacency relations between agents, determined by R, can be modeled by an undirected proximity graph. However, because of the irregular detection areas of agents, existence of obstacles, or external interference, it is in fact true that agent i cannot measure the position information of all agents within the distance of R. In other words, the information exchange between agents may suffer from link failure.

It is assumed the failed links are recoverable. Specifically, there exists a common *recovery time* $T \in \mathbb{N}$, such that if $\|p_j(k) - p_i(k)\|_2 \le R$ for all k, $t \le k \le t+T$, then agent i should obtain the relative position of agent j at some time in the time interval $[t, t+T]$, and it is also true for agent j, but its successful measurement may not occur at the same time as agent i. Therefore, if $T = 0$, then no link failure occurs, and if $T \ge 1$, then the interaction between agents is *not bidirectional.*

4. Rendezvous protocol in the general form

This section will present the rendezvous protocol based on the above assumption and perform its convergence analysis. The form of the protocol is quite general, and it leaves open the possibility of being combined with additional connectivity-preserving rules, given in the next section.

4.1. Registered interaction topology

With link failure, the information of neighbors, determined by the minimum sensing radius R, may be lost. To ensure the failed links recover in time, we should consider the usage of delayed information in the design of protocol u_i, and thus introduce an important concept, "*registered interaction topology*".

Let D with $0 < D < \frac{\sqrt{2}R}{2}$ be the *registering radius*, which is a protocol parameter. For any i, if agent i gets the instantaneous relative position vector,[2] $p_j(t)|_{\mathcal{O}_i(t)}$, of agent j at time t, and $\|p_j(t)|_{\mathcal{O}_i(t)}\|_\infty \le D$, then agent i may register agent j as one of its neighbors and preserve its role for the next T time steps. Denote all registered neighbors of agent i at time t by $\mathcal{N}_i(t)$. At any time t, if $j \in \mathcal{N}_i(t-1)$ and the relative position of agent j, $p_j(t)|_{\mathcal{O}_i(t)}$, with $\|p_j(t)|_{\mathcal{O}_i(t)}\|_2 \le R$, is obtained again by agent i, then the "registered neighbor" role is renewed and preserved for the next T time steps.

Definition 1 (*Registered Interaction Topology* $\mathcal{G}(t)$)**.** The registered interaction topology $\mathcal{G}(t)$ is a directed graph with vertex set $\mathcal{V}(\mathcal{G}(t)) = \{v_1, v_2, \ldots, v_n\}$ modeling the agents, and $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(t))$ if and only if $j \in \mathcal{N}_i(t)$.

Remarks.

(1) Note that the registered interaction topology $\mathcal{G}(t)$ is *not* the real topology of information channels; but it in fact reflects the union of information channels over the latest $T+1$ time steps; as we will see, $\mathcal{G}(t)$ is strongly related to the proposed protocol and its edges indicate the inter-usage of information between agents.
(2) Recall the assumption of link failure and recovery. If agent i registers agent j as one of its neighbors at time t_0 and $\|p_j(t) - p_i(t)\|_2 \le R$ for all t with $t_0 \le t \le t_1$, then it follows from the renewal strategy of "registered neighbor" role that $j \in \mathcal{N}_i(t)$ for all t, $t_0 \le t \le t_1$.

4.2. Rendezvous protocol and convergence analysis

To present the rendezvous protocol in a compact form, we need to introduce two point sets. Let W_L and W_U be real numbers such that $0 < W_L \le W_U$. For each agent i, the first point set is a rectangle, defined by

$$\begin{aligned}\mathcal{D}_i^{\text{rect}}(t, \Delta\theta) = \Bigg\{ \xi \in \mathbb{R}^2 : \xi^{(1)} &= \frac{1}{\sum\limits_{j \in \mathcal{N}_i(t) \cup \{i\}} W'_{ij}} \\ &\quad \times \sum_{j \in \mathcal{N}_i(t)} W'_{ij} p_j^{(1)}(t - \tau_{ij}(t))|_{\mathcal{O}(p_i(t), \theta_i(t)+\Delta\theta)}, \\ \xi^{(2)} &= \frac{1}{\sum\limits_{j \in \mathcal{N}_i(t) \cup \{i\}} W''_{ij}} \\ &\quad \times \sum_{j \in \mathcal{N}_i(t)} W''_{ij} p_j^{(2)}(t - \tau_{ij}(t))|_{\mathcal{O}(p_i(t), \theta_i(t)+\Delta\theta)}, \\ &\quad W_L \le W'_{ij} \le W_U,\, W_L \le W''_{ij} \le W_U \Bigg\}.\end{aligned}$$

In the above equation, $t - \tau_{ij}(t)$ denotes the latest time, at which the instantaneous position of agent j, with $\|p_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))}\|_2 \le R$, is obtained by agent i. Then by the definition of registered neighbors, $0 \le \tau_{ij}(t) \le T$. Here, it should be assumed that by local information, agent i has the capability to calculate $p_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)}$. We give an example to illustrate the iterative computing process. Clearly, if $\tau_{ij}(t) = 0$, it has been assumed that the instantaneous information $p_j(t)|_{\mathcal{O}_i(t)}$ is obtained by agent i at time t. If $\tau_{ij}(t) \ge 1$, suppose that $p_j(t - 1 - \tau_{ij}(t-1))|_{\mathcal{O}_i(t-1)}$ is obtained by agent i.

[2] If this assumption is relaxed by the one of getting time delayed relative positions, the main results are still obtainable as long as the maximum time delay can be estimated.

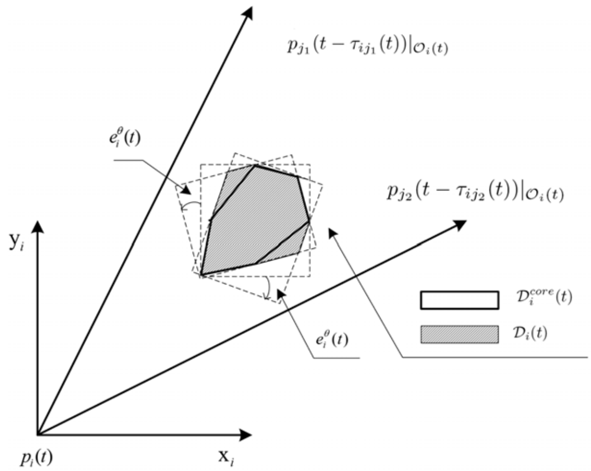


Fig. 2. Illustration of $\mathcal{D}_i(t)$ for agent i with two registered neighbors, where $\mathcal{D}_i^{\mathrm{core}}(t)$ is defined by Eq. (6).

Then $p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}$ can be calculated by

$$p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} = \Theta(\theta_i(t-1)-\theta_i(t)) \times \left(p_j(t-1-\tau_{ij}(t-1))|_{\mathcal{O}_i(t-1)} - u_i(t-1)\right).$$

Assume that each agent has an estimation of the upper bound of its axis error $\theta_i(t)$, denote by $e_i^\theta(t)$, such that for any i, t, $|\theta_i(t)| \le e_i^\theta(t)$. Then the second point set is defined by

$$\mathcal{D}_i(t) = \bigcap_{|\Delta\theta|\le e_i^\theta(t)} \Theta(\Delta\theta)\mathcal{D}_i^{\mathrm{rect}}(t,\Delta\theta).$$

The illustration of $\mathcal{D}_i(t)$ is shown in Fig. 2. Clearly, $\mathcal{D}_i(t)$ is a rectangle-like polytope if $e_i^\theta(t)$ is sufficiently small. Then the rendezvous protocol is given in the following form:

$$u_i(t) \in \mathcal{D}_i(t), \quad i \in \mathcal{I}_n, \ t \in \mathbb{N}. \tag{3}$$

Note that

$$u_i(t) \in \Theta(-\theta_i(t))\mathcal{D}_i^{\mathrm{rect}}(t,-\theta_i(t)),$$

and the coordinate axes of $\mathcal{O}(p_i(t),0)$ are parallel to those of the global coordinate system. Thus, it follows from Eq. (2) that in the global coordinate system, protocol (3) has the following form:

$$p_i^{(k)}(t) = \frac{1}{\sum_{j\in\mathcal{N}_i(t)\cup\{i\}} \alpha_{ij}^k(t)} \times \left(\sum_{j\in\mathcal{N}_i(t)} \alpha_{ij}^k(t)p_j^{(k)}(t-\tau_{ij}(t)) + \alpha_{ii}^k(t)p_i^{(k)}(t)\right), \quad k=1,2, \tag{4}$$

where $W_L \le \alpha_{ij}^k(t) \le W_U$.

Clearly, the above protocol is in fact a revised version of the consensus protocol, presented in Xiao and Wang (2008), but protocol (3) is more appropriate for the high-dimensional case ($\mathbb{R}^2$ in this paper) in the absence of an agreement of a global coordinate system; and it is also compatible with the connectivity-preserving rules, proposed in the subsequent subsection.

As a direct subsequence of Theorem 2 in Xiao and Wang (2008), we have the following convergence result.

Theorem 1 (*Convergence*). *If the registered interaction topology $\mathcal{G}(t)$ with recovery time T and registering radius D always contains a spanning tree, then protocol (3) solves the rendezvous problem; i.e., there exists a common value $p^* \in \mathbb{R}^2$, such that $\lim_{t\to\infty} p_i(t) = p^*$ for any $i \in \mathcal{I}_n$.*

Remark. The above conclusion still holds if the union of registered interaction topology $\mathcal{G}(t)$ contains a spanning tree periodically.

5. Connectivity-preserving control

This section will present the connectivity-preserving results of this paper. The first two subsequent add two additional connectivity-preserving rules into the protocol (3) and provide sufficient conditions for their compatibility respectively. The last two subsections show the effectiveness in connectivity preservation and the robustness against measurement errors respectively.

5.1. Additional control rules for connectivity preservation

This subsection provides the following Rules (R1, R2) in the selection of local feedback $u_i(t)$ from $\mathcal{D}_i(t)$ for the network connectivity preservation:

(R1): for any i and t, $\|u_i(t)\|_\infty \le S$, where S is maximum step length such that $D+(T+1)S \le \frac{\sqrt{2}R}{2}$;

(R2): for any $i \in \mathcal{I}_n$, $t \in \mathbb{N}$, $j \in \mathcal{N}_i(t)$, and any $k \in \{1,2\}$, if $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \ge D$, then $u_i^{(k)}(t) \ge 0$, and if $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \le -D$, then $u_i^{(k)}(t) \le 0$.

Rule (R2) says that registering radius D is a critical distance in the sense that if the relative distance $\|p_j(t-\tau_{ij(t)})|_{\mathcal{O}_i(t)}\|_\infty$ between agent i and its registered neighbor j is larger than D, then agent i will try to avoid further increasing of the distance between them. However, because of link failure and alignment errors in local coordinate orientations, the increase of the distance cannot be avoided. So it is required that registering radius D and maximum step length S should be small enough so that $D+(T+1)S$ is much less than sensing radius R. Furthermore, another upper bound on the maximum step length S can be added into Rule (R1) in some practical applications, especially in the case with input saturation restriction. In Section 5.3, explicit upper bounds on S and D will be given to ensure the connectivity. On the other hand, since the movement restriction of agents in Rule (R2) is described in each local coordinate direction, it is convenient to choose maximum norm $\|\cdot\|_\infty$, instead of Euclidean norm $\|\cdot\|_2$, to represent the distance; and in this sense, the sensing radius becomes $\frac{\sqrt{2}R}{2}$.

5.2. Compatibility with the rendezvous protocol

This subsection gives sufficient conditions, in terms of parameters W_U and W_L, so that we can always find proper local feedback $u_i(t)$ in $\mathcal{D}_i(t)$, satisfying Rules (R1, R2); i.e., Rules (R1, R2) are compatible with the rendezvous protocol in the form of Eq. (3).

Theorem 2 (*Compatibility*). *Assume that for all t with $t' < t$, the control law $u_i(t')$ is selected according to Rule* (R1); *i.e., $\left\|u_i(t')\right\|_\infty \le S, t' < t$; the maximum alignment error $e_{\max}^\theta = \sup\{e_i^\theta(t) : t \in \mathbb{N}, i \in \mathcal{I}_n\}$ is small enough so that $e_{\max}^\theta < \arctan\left(\min\left\{\frac{D}{2R^0}, \frac{S}{4R^0}\right\}\right)$, and W_U and W_L meet the following condition:*

(C1):

$$\frac{W_U}{W_L} \ge \max\left\{N\left(\frac{R^0}{S-2R^0\tan(e_{\max}^\theta)} - 1\right), \frac{(N-1)R^0 + 2R^0 N\tan(e_{\max}^\theta)}{D-2R^0\tan(e_{\max}^\theta)}\right\},$$

where $R^0 = R + \sqrt{2}TS\cos\left(\frac{\pi}{4} - 2e_{\max}^\theta\right)$, $N = \max_{i,t}|\mathcal{N}_i(t)|$, and $|\mathcal{N}_i(t)|$ denotes the cardinality of set $\mathcal{N}_i(t)$. Then we can always find $u_i(t) \in \mathcal{D}_i(t)$, satisfying Rules (R1, R2).

Remarks.

(1) It can be observed in Condition (C1) that the choice of parameters W_U and W_L depends solely on the values of $R, D, T,$

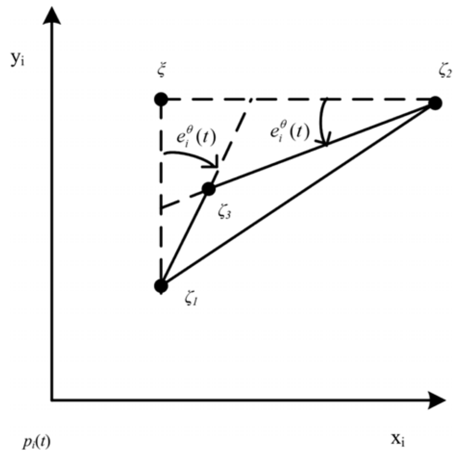


Fig. 3. Proof of Lemma 2.

S, $e^{\theta}_{\max}$ and N; and the larger the maximum alignment error $e^{\theta}_{\max}$ and recovery time T are, the larger $\frac{W_U}{W_L}$ is required to be. N is the only parameter depending on the evolution of the system. However, it can be roughly estimated that $N \leq n-1$. Moreover, it will be shown that the initial network connectivity will be preserved under Rules (R1, R2). Thus if we assume that no additional edge is added into the registered interaction topology, then $N = \max_i |\mathcal{N}_i(0)|$, which can be estimated more accurately.

(2) The condition that $e^{\theta}_{\max} < \arctan\left(\min\left\{\frac{D}{2R^0}, \frac{S}{4R^0}\right\}\right)$ and (C1), provided in Theorem 2, are just sufficient for compatibility; they could be somewhat conservative.

(3) Following the proof of Theorem 2, we will provide a feasible distributed numerical algorithm for the choice of u_i from $\mathcal{D}_i(t)$ for each agent i without knowing the specific values of W_L and W_U.

To show the above theorem, we need to give a further discussion on the properties of point sets $\mathcal{D}^{\text{rect}}_i(t,0)$ and $\mathcal{D}_i(t)$.

Lemma 1 gives an upper bound of the side length of $\mathcal{D}^{\text{rect}}_i(t,0)$.

Lemma 1. *Under Rule (R1), for any $i \in \mathcal{I}_n$, $t \in \mathbb{N}$, $j \in \mathcal{N}_i(t)$, $\tau_{ij}(t) \leq T$, if $e^{\theta}_{\max} \leq \frac{\pi}{8}$, then $\|p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}\|_\infty \leq R^0$; and thus the maximum side length of $\mathcal{D}^{\text{rect}}_i(t,0)$ is not greater than $2R^0$.*

Proof. If $T=0$, the result is obvious. Assume that $T \geq 1$. Then

$$\begin{aligned}
p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} &= p_i(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} + \Theta(\theta_i(t-\tau_{ij}(t)) \\
&\quad - \theta_i(t))p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))} \\
&= \Theta(-\theta_i(t))(p_i(t-\tau_{ij}(t)) - p_i(t)) + \Theta(\theta_i(t \\
&\quad - \tau_{ij}(t)) - \theta_i(t))p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))} \\
&= \Theta(-\theta_i(t))\Big(-\Theta(\theta_i(t-\tau_{ij}(t)))u_i(t-\tau_{ij}(t)) \\
&\quad - \Theta(\theta_i(t-\tau_{ij}(t)+1))u_i(t-\tau_{ij}(t)+1) \\
&\quad - \cdots - \Theta(\theta_i(t-1))u_i(t-1)\Big) \\
&\quad + \Theta(\theta_i(t-\tau_{ij}(t)) - \theta_i(t))p_j(t \\
&\quad - \tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))},
\end{aligned}$$

where the last equation follows from Eq. (2). Note that $\|p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))}\|_2 \leq R$. We get

$$\begin{aligned}
&\|p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}\|_\infty \\
&\quad \leq R + \sqrt{2}TS \max_{0 \leq e' \leq e^{\theta}_{\max}} \cos\left(\frac{\pi}{4} - 2e'\right) = R^0. \quad \square
\end{aligned}$$

The next lemma gives a connection between sets $\mathcal{D}_i(t)$ and $\mathcal{D}^{\text{rect}}_i(t,0)$.

Lemma 2. *For any i, t, and any $\xi \in \mathcal{D}^{\text{rect}}_i(t,0)$, if $e^{\theta}_{\max} < \frac{\pi}{4}$, then there exists $\zeta \in \mathcal{D}_i(t)$ such that*

$$\|\xi - \zeta\|_\infty \leq 2R^0 \tan(e^{\theta}_{\max}). \tag{5}$$

Proof. If $\xi \in \mathcal{D}_i(t)$, then we choose $\zeta = \xi$, which satisfies inequality (5).

Suppose that $\xi \notin \mathcal{D}_i(t)$. Note that $\mathcal{D}^{\text{rect}}_i(t,0)$ is the minimum rectangle covering $\mathcal{D}_i(t)$, with sides parallel to the coordinate axes of $\mathcal{O}_i(t)$. Therefore, there exist at most four maximum continuous regions in $\mathcal{D}^{\text{rect}}_i(t,0)$, not intersected with $\mathcal{D}_i(t)$, and they should be located at the four corners of the rectangle $\mathcal{D}^{\text{rect}}_i(t,0)$.

Without loss of generality, suppose that ξ belongs to the region at the upper left corner. Then there exist $\zeta_1, \zeta_2 \in \mathcal{D}_i(t)$, such that $\zeta^{(1)}_1 = \xi^{(1)}$, $\zeta^{(2)}_1 < \xi^{(2)}$, $\zeta^{(1)}_2 > \xi^{(1)}$, and $\zeta^{(2)}_2 = \xi^{(2)}$. If $|\zeta^{(2)}_1 - \xi^{(2)}| \leq 2R^0 \tan(e^{\theta}_i(t))$ or $|\zeta^{(1)}_2 - \xi^{(1)}| \leq 2R^0 \tan(e^{\theta}_i(t))$, then we can choose $\zeta = \zeta_1$ or $\zeta = \zeta_2$.

If none of the above cases holds, then by the definition of $\mathcal{D}_i(t)$, it can be shown that $\zeta_3 \in \mathcal{D}_i(t)$, as depicted in Fig. 3. Since the maximum side length of $\mathcal{D}^{\text{rect}}_i(t,0)$ is not greater than $2R^0$, $|\zeta^{(1)}_3 - \xi^{(1)}|$ and $|\zeta^{(2)}_3 - \xi^{(2)}|$ are no greater than $2R^0 \tan(e^{\theta}_{\max})$. Thus $\|\zeta_3 - \xi\|_\infty \leq 2R^0 \tan(e^{\theta}_i(t)) \leq 2R^0 \tan(e^{\theta}_{\max})$ and $\zeta = \zeta_3$ is a suitable choice. $\square$

Proof of Theorem 2. Define the set

$$\mathcal{D}^{\text{core}}_i(t) = \Bigg\{\xi = \frac{1}{\sum\limits_{j \in \mathcal{N}_i(t) \cup \{i\}} W_{ij}} \sum_{j \in \mathcal{N}_i(t)} W_{ij} p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} : W_L \leq W_{ij} \leq W_U \Bigg\}. \tag{6}$$

Clearly, $\mathcal{D}^{\text{core}}_i(t) \subset \mathcal{D}_i(t)$, see Fig. 2 for its illustration.

Step 1: We first prove that there exists $\xi \in \mathcal{D}^{\text{core}}_i(t)$, such that

$$\|\xi\|_\infty \leq S - 2R^0 \tan(e^{\theta}_{\max}).$$

It follows from the definition of $\mathcal{D}^{\text{core}}_i(t)$ that

$$\xi = \frac{\sum\limits_{j \in \mathcal{N}_i(t)} W_L p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}}{W_U + |\mathcal{N}_i(t)| W_L} \in \mathcal{D}^{\text{core}}_i(t). \tag{7}$$

Note that $S < R \leq R^0$. Thus we have

$$e^{\theta}_{\max} < \arctan\frac{1}{4} < \frac{\pi}{8};$$

and by Lemma 1,

$$\begin{aligned}
\|\xi\|_\infty &\leq \frac{|\mathcal{N}_i(t)| W_L R^0}{W_U + |\mathcal{N}_i(t)| W_L} = \frac{R^0}{\frac{W_U}{|\mathcal{N}_i(t)| W_L} + 1} \\
&\leq \frac{R^0}{\frac{W_U}{N W_L} + 1} \leq S - 2R^0 \tan(e^{\theta}_{\max}),
\end{aligned}$$

where the last inequality follows from Condition (C1).

Step 2: In this step, we aim to get the conclusion that for any $k \in \{1,2\}$, if $p^{(k)}_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \geq D$, $j \in \mathcal{N}_i(t)$, then there exists $\xi \in \mathcal{D}^{\text{core}}_i(t)$ such that

$$2R^0 \tan(e^{\theta}_{\max}) \leq \xi^{(k)} \leq S - 2R^0 \tan(e^{\theta}_{\max});$$

and if $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \le -D, j \in \mathcal{N}_i(t)$, then there exists $\xi \in \mathcal{D}_i^{\text{core}}(t)$ such that

$$-S + 2R^0 \tan(e_{\max}^{\theta}) \le \xi^{(k)} \le -2R^0 \tan(e_{\max}^{\theta}).$$

We only prove the first case, and second case can be proved similarly.

Suppose that $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \ge D$. Since $\mathcal{D}_i^{\text{core}}(t)$ is a convex polytope, by the conclusion of Step 1, it suffices to prove that there exists some $\xi \in \mathcal{D}_i^{\text{core}}(t)$ such that $\xi^{(k)} \ge 2R^0 \tan(e_{\max}^{\theta})$. Choose

$$\xi = \frac{1}{W_U + |\mathcal{N}_i(t)|W_L}\left(\sum_{l\in\mathcal{N}_i(t), l\neq j} W_L p_l(t-\tau_{il}(t))|_{\mathcal{O}_i(t)} + W_U p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}\right) \in \mathcal{D}_i^{\text{core}}(t). \quad (8)$$

Then it follows from (C1) that

$$\xi^{(k)} \ge \frac{-(N-1)R^0 W_L + DW_U}{W_U + NW_L} \ge 2R^0 \tan(e_{\max}^{\theta}).$$

Step 3: Finally, we prove that the solution, satisfying Rules (R1, R2), exists. Define the following four subcases:

(S1): there exists j, such that $p_j^{(1)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \ge D$;

(S2): there exists j, such that $p_j^{(1)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \le -D$;

(S3): there exists j, such that $p_j^{(2)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \ge D$;

(S4): there exists j, such that $p_j^{(2)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \le -D$.

Step 3.1: Consider the case when none of the above subcases holds.

By the result of Step 1, there exists $\xi \in \mathcal{D}_i^{\text{core}}(t)$, such that

$$\|\xi\|_\infty \le S - 2R^0 \tan(e_{\max}^{\theta}).$$

Since $\mathcal{D}_i^{\text{core}}(t) \subset \mathcal{D}_i(t)$ and hence $u_i(t) = \zeta$ is a proper choice.

Step 3.2: Consider the case when only one of the above subcases holds.

We only give the possible choice of $u_i(t)$ when only (S1) holds, but (S2, S3, S4) do not hold; the other cases can be proved similarly.

By the result of Step 2, there exists $\xi_1 \in \mathcal{D}_i^{\text{core}}(t)$, such that

$$2R^0 \tan(e_{\max}^{\theta}) \le \xi_1^{(1)} \le S - 2R^0 \tan(e_{\max}^{\theta}). \quad (9)$$

By the result of Step 1, there exists $\xi_2 \in \mathcal{D}_i^{\text{core}}(t)$, such that

$$\|\xi_2\|_\infty \le S - 2R^0 \tan(e_{\max}^{\theta}). \quad (10)$$

Because $\mathcal{D}_i^{\text{core}}(t) \subset \mathcal{D}_i^{\text{rect}}(t,0)$ and $\mathcal{D}_i^{\text{rect}}(t,0)$ is a rectangle,

$$\zeta = \begin{bmatrix}\xi_1^{(1)} \\ \xi_2^{(2)}\end{bmatrix} \in \mathcal{D}_i^{\text{rect}}(t,0).$$

By Lemma 2, there exists $\eta \in \mathcal{D}_i(t)$ such that

$$\|\eta - \zeta\|_\infty \le 2R^0 \tan(e_{\max}^{\theta}).$$

It follows from the above inequality and inequalities (9) and (10), $\|\eta\|_\infty \le S$ and $\eta^{(1)} \ge 0$. Choose $u_i(t) = \eta$, which satisfies Rules (R1, R2).

Step 3.3: Consider the case then only (S1, S2) hold or when only (S3, S4) hold.

We only consider the case when only (S1, S2) hold, but (S3, S4) do not hold; the other case can be proved similarly.

With the same arguments as in Step 3.2, there exist $\xi_1, \xi_2 \in \mathcal{D}_i(t)$, such that $\xi_1^{(1)} \ge 0, \xi_2^{(1)} \le 0$, $\|\xi_1\|_\infty \le S$, and $\|\xi_2\|_\infty \le S$. Since $\mathcal{D}_i(t)$ is a convex set, there exists some α such that $0 \le \alpha \le 1$, $\eta = \alpha\xi_1 + (1-\alpha)\xi_2$ with $\eta^{(1)} = 0$. Obviously, $\|\eta\|_\infty \le S$. Choose $u_i(t) = \eta$, which satisfies Rules (R1, R2).

Step 3.4: Consider the case when only (S1, S3) hold or when only (S1, S4) hold, or when only (S2, S3) hold or when only (S2, S4) hold.

We only consider the case when only (S1, S3) hold, but (S2, S4) do not hold; the other cases can be proved similarly.

By the result of Step 2, there exist $\xi_1, \xi_2 \in \mathcal{D}_i^{\text{core}}(t)$, such that $2R^0 \tan(e_{\max}^{\theta}) \le \xi_1^{(1)} \le S - 2R^0 \tan(e_{\max}^{\theta})$ and $2R^0 \tan(e_{\max}^{\theta}) \le \xi_2^{(2)} \le S - 2R^0 \tan(e_{\max}^{\theta})$. Because $\mathcal{D}_i^{\text{rect}}(t,0)$ is a rectangle, we have

$$\zeta = \begin{bmatrix}\xi_1^{(1)} \\ \xi_2^{(2)}\end{bmatrix} \in \mathcal{D}_i^{\text{rect}}(t,0).$$

By Lemma 2, there exists $\eta \in \mathcal{D}_i(t)$ such that $\|\eta - \zeta\|_\infty \le 2R^0 \tan(e_{\max}^{\theta})$, and thus $0 \le \eta^{(1)} \le S, 0 \le \eta^{(2)} \le S$. Hence $u_i(t) = \eta$ is a proper choice.

Step 3.5: Consider the case when only one of (S1, S2, S3, S4) does not hold.

We only consider the case when only (S4) does not hold; the other cases can be proved similarly.

With the same arguments as in Step 3.4, there exists $\xi_1 \in \mathcal{D}_i(t)$, with $0 \le \xi_1^{(1)}, \xi_1^{(2)} \le S$, and there exists $\xi_2 \in \mathcal{D}_i(t)$, with $-S \le \xi_2^{(1)} \le 0$ and $0 \le \xi_2^{(2)} \le S$. Since $\mathcal{D}_i(t)$ is a convex set, there exists some α such that $0 \le \alpha \le 1$, $\eta = \alpha\xi_1 + (1-\alpha)\xi_2$ with $\eta^{(1)} = 0$. Obviously, $\|\eta\| \le S$ and $\eta^{(2)} \ge 0$. Hence $u_i(t) = \eta$ is a proper choice.

Step 3.6: Consider the case when all (S1, S2, S3, S4) hold.

By the result of Step 2, we get that rectangle $\{\xi : \|\xi\|_\infty \le 2R^0 \tan(e_{\max}^{\theta})\} \subset \mathcal{D}_i^{\text{rect}}(t,0)$. With the same argument as in the proof of Lemma 2, it can be shown that $0 \in \mathcal{D}_i(t)$. Then $u_i(t) = 0$ is the proper choice. $\square$

The above proof as well as the proof of Lemma 2 provides us in fact a feasible distributed numerical algorithm for each agent i for choosing feedback u_i from $\mathcal{D}_i(t)$, with restrictions (R1, R2), $i \in \mathcal{I}_n$. The numerical algorithm is a combination of the following three basic components.

Sub-algorithm 1 (*corresponding to Step* 1):

Goal: To find $\xi \in \mathcal{D}_i^{\text{core}}(t)$ such that $\|\xi\|_\infty \le S - 2R^0 \tan(e_{\max}^{\theta})$.

Feasible solution:

(1) Set $\xi := \frac{\sum_{j\in\mathcal{N}_i(t)} p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}}{1+|\mathcal{N}_i(t)|}$;

(2) If $\|\xi\|_\infty > S - 2R^0 \tan(e_{\max}^{\theta})$, set $\xi := (S - 2R^0 \tan(e_{\max}^{\theta}))\frac{\xi}{\|\xi\|_\infty}$.

Sub-algorithm 2 (*corresponding to Step* 2):

Goal: For any $k \in \{1, 2\}, j \in \mathcal{N}_i(t)$,

(1) If $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \ge D$, to find $\xi \in \mathcal{D}_i^{\text{core}}(t)$, such that $2R^0 \tan(e_{\max}^{\theta}) \le \xi^{(k)} \le S - 2R^0 \tan(e_{\max}^{\theta})$;

(2) If $p_j^{(k)}(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)} \le -D$, to find $\xi \in \mathcal{D}_i^{\text{core}}(t)$, such that $-S + 2R^0 \tan(e_{\max}^{\theta}) \le \xi^{(k)} \le -2R^0 \tan(e_{\max}^{\theta})$.

Feasible solution to the first case (similar to the second case):

(1) Set $\xi := \frac{\sum_{j\in\mathcal{N}_i(t)} W_L p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}}{1+|\mathcal{N}_i(t)|}$;

(2) If $\xi^{(k)} < 2R^0 \tan(e_{\max}^{\theta})$, set $\xi := \alpha\xi + (1-\alpha)p_j(t-\tau_{ij}(t))|_{\mathcal{O}_i(t)}$, such that $\xi^{(k)} = 2R^0 \tan(e_{\max}^{\theta})$;

(3) Else if $\xi^{(k)} > S - 2R^0 \tan(e_{\max}^{\theta})$, let ξ' be the solution in Sub-algorithm 1, and set $\xi := \alpha\xi + (1-\alpha)\xi'$, such that $\xi^{(k)} = S - 2R^0 \tan(e_{\max}^{\theta})$.

Sub-algorithm 3 (*corresponding to* Lemma 2):

Goal: Given $\xi_1, \xi_2 \in \mathcal{D}_i^{\text{core}}(t)$, and $\zeta = [\xi_1^{(1)}, \xi_2^{(2)}]$, to find $\eta \in \mathcal{D}_i(t)$ such that $\|\eta - \zeta\|_\infty \le 2R^0 \tan(e_{\max}^{\theta})$.

Feasible solution:

(1) If $\|\xi_1 - \zeta\|_\infty \le 2R^0 \tan(e^\theta_{\max}(t))$ or $\|\xi_2 - \zeta\|_\infty \le 2R^0 \tan(e^\theta_{\max}(t))$, then $\eta := \xi_1$ or $\eta := \xi_2$ is the feasible choice;
(2) Else let $\alpha = \frac{|\xi_1^{(1)}-\xi_2^{(1)}|-\tan(e^\theta_{\max}(t))|\xi_1^{(2)}-\xi_2^{(2)}|}{1-\tan^2(e^\theta_{\max}(t))}$ and let $\beta = \frac{|\xi_1^{(2)}-\xi_2^{(2)}|-\tan(e^\theta_{\max}(t))|\xi_1^{(1)}-\xi_2^{(1)}|}{1-\tan^2(e^\theta_{\max}(t))}$; and set $\eta^{(1)} := \xi_2^{(1)} + \frac{\alpha}{|\xi_1^{(1)}-\xi_2^{(1)}|} \times (\xi_1^{(1)} - \xi_2^{(1)})$ and $\eta^{(2)} := \xi_1^{(2)} + \frac{\beta}{|\xi_2^{(2)}-\xi_1^{(2)}|}(\xi_2^{(2)} - \xi_1^{(2)})$.

Note that parameters W_L and W_U do not appear in the above three sub-algorithms. However, from the fact that ξ in Step 1, determined by Eq. (7), is a convex combination of agent i and the centroid of agent i and its registered neighbors, and ξ in Step 2, determined by Eq. (8), is a convex combination of agent j and the centroid, we have that if compatible condition (C1) is satisfied, then the above three sub-algorithms are surely solvable. Simulations based on these sub-algorithms will be provided in Section 6.

5.3. Effectiveness in connectivity preservation

This subsection studies the effectiveness of the proposed protocol in the presence of two kinds of alignment errors of coordinate orientations. The first is maximum static error $e^\theta_{\max}$ for the case with time-invariant local coordinate orientations; the second is maximum sum error e^θ_Σ, defined by

$$e^\theta_\Sigma = \sup\left\{\lim_{l\to\infty}\sum_{t=0}^{l}|\theta_i(t)|, i \in \mathcal{I}_n\right\},$$

for the case with time-varying local coordinate orientations.

Theorem 3 (*Connectivity Preservation*). *Assume that $e^\theta_{\max} < \arctan\left(\min\left\{\frac{D}{2R^0}, \frac{S}{4R^0}\right\}\right)$; for the case with time-invariant local coordinate orientations, assume that*

$$D^0 + (T+1)S^0 + 2d_{\max}\tan(e^\theta_{\max}) \le \frac{\sqrt{2}R}{2},$$

where $D^0 = \frac{D}{\cos(e^\theta_{\max})} + (R+TS^0)\tan(e^\theta_{\max})$, $S^0 = \sqrt{2}S\cos(\frac{\pi}{4} - e^\theta_{\max})$, and $d_{\max} = \max_{i,j,k}\{|p_i^{(k)}(0) - p_j^{(k)}(0)|\}$; and for the case with time-varying local coordinate orientations, assume that

$$D^0 + (T+1)S^0 + 2Se^\theta_\Sigma \le \frac{\sqrt{2}R}{2}. \tag{11}$$

Then if the system evolves under Rules (R1, R2), and $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t_0))$ for some time t_0, then $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t))$ for all $t \ge t_0$. Thus, under the above conditions as well as the compatible Condition (C1), protocol (3) under Rules (R1, R2) will drive all agents to reach a common place asymptotically, as long as the registered interaction topology $\mathcal{G}(t)$ is connected for some time t.

Remarks.

(1) If $e^\theta_{\max} = 0$, i.e., all local coordinate orientations have been already aligned with each other, then the above two inequalities are reduced to $D + (T+1)S \le \frac{\sqrt{2}R}{2}$, which is the least requirement for S, as described in (R1).
(2) If the alignment algorithm for local coordinate orientations is performed in parallel with the rendezvous control, and $\theta_i(t)$, $i \in \mathcal{I}_n$, converge asymptotically to zero at a vanishing speed μ, $0 < \mu < 1$, i.e., $|\theta_i(t)| \le \mu^t|\theta_i(0)|$, then we can get an estimation that $\lim_{l\to\infty}\sum_{t=0}^{l}|\theta_i(t)| \le \frac{1}{1-\mu}|\theta_i(0)|$.
(3) By Eq. (4), if the maximum distance between agents is not larger than R, then the network connectivity cannot be broken any more and thus the connectivity-preserving restrictions (R1, R2, C1) can be relaxed in this case.
(4) It should be emphasized that the connectivity of registered interaction topology $\mathcal{G}(t)$ does not means that the information channels between mutually registered neighbors are bidirectional and exist all time; and it only means that the distances between them do never exceed the minimum sensing radius R.

We first list the following facts about the system under Rules (R1, R2) before proving the above theorem.

In the global coordinate system, the maximum step length, described by Rule (R1), is estimated by the following lemma.

Lemma 3. *For any i and t, if $\|u_i(t)\|_\infty \le S$ and $e^\theta_{\max} < \frac{\pi}{4}$, then $\|p_i(t+1) - p_i(t)\|_\infty \le S^0$.*

Proof. It follows from Eq. (2) that

$$\begin{aligned}\|p_i(t+1) - p_i(t)\|_\infty &\le S\big(|\cos(\theta_i(t))| + |\sin(\theta_i(t))|\big)\\ &= \sqrt{2}S\cos\left(\frac{\pi}{4} - |\theta_i(t)|\right)\\ &\le \sqrt{2}S\cos\left(\frac{\pi}{4} - e^\theta_{\max}\right) = S^0. \quad \square\end{aligned}$$

Lemma 4. *For any $\xi \in \mathbb{R}^2$, $k \in \{1, 2\}$, and $|\theta| \le e^\theta_{\max} < \frac{\pi}{4}$, if $\|\xi\|_\infty \le R + TS^0$, then $\xi^{(k)} \ge D^0$ implies that $(\Theta(\theta)\xi)^{(k)} \ge D$, and $\xi^{(k)} \le -D^0$ implies that $(\Theta(\theta)\xi)^{(k)} \le -D$.*

Proof. If $\xi^{(1)} \ge D^0$, then

$$\begin{aligned}(\Theta(\theta)\xi)^{(1)} &= \cos(\theta)\xi^{(1)} - \sin(\theta)\xi^{(2)}\\ &\ge \cos(\theta)\left(\frac{D}{\cos(e^\theta_{\max})} + (R+TS^0)\tan(e^\theta_{\max})\right)\\ &\quad - \sin(\theta)\xi^{(2)}\\ &\ge D + (R+TS^0)\sin(e^\theta_{\max}) - \sin(\theta)\xi^{(2)}\\ &\ge D,\end{aligned}$$

where the last inequality follows from that $\|\xi\|_\infty \le R + TS^0$.

With the same arguments, we can prove the other cases. $\square$

Lemma 5. *Suppose that $\|u_i(t)\|_\infty \le S$, and $|\theta_i(t)| < \frac{\pi}{4}$, $i \in \mathcal{I}_n$. For any k, if $u_i^{(k)}(t) \le 0$, then $(\Theta(\theta_i(t))u_i(t))^{(k)} \le S|\sin(\theta_i(t))|$; if $u_i^{(k)}(t) \ge 0$, then $(\Theta(\theta_i(t))u_i(t))^{(k)} \ge -S|\sin(\theta_i(t))|$.*

Proof. If $k = 1$ and $u_i^{(1)}(t) \le 0$, then

$$\begin{aligned}(\Theta(\theta_i(t))u_i(t))^{(1)} &= \cos(\theta_i(t))u_i(t)^{(1)} - \sin(\theta_i(t))u_i(t)^{(2)}\\ &\le -\sin(\theta_i(t))u_i(t)^{(2)} \le S|\sin(\theta_i(t))|.\end{aligned}$$

The other cases can be proved similarly. $\square$

Proof of Theorem 3. Since $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t_0))$, by the definition of registered neighbors and their renewal strategy, there exist a time t_0^-, $t_0^- \le t_0$, and $k \in \{i, j\}$, such that $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t_0^-))$, and $\|p_i(t_0^-)|_{\mathcal{O}_k(t_0^-)} - p_j(t_0^-)|_{\mathcal{O}_k(t_0^-)}\|_\infty \le D$, which implies that $\|p_i(t_0^-) - p_j(t_0^-)\|_\infty \le \sqrt{2}D\cos\left(\frac{\pi}{4} - e^\theta_{\max}\right) < D^0$. Furthermore, by the fact that $D^0 + (T+1)S^0 \le \frac{\sqrt{2}}{2}R$ and Lemma 3, we can get that $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t))$, $t_0^- \le t \le t_0^- + T + 1$. Next, we assume that $t_0 = t_0^-$ without loss of generality; and prove this theorem by contradiction.

We first give the follow assumption and then show that it will lead to a contradiction:

(AC): Assume that there exists a time t_1 such that $\|p_i(t) - p_j(t)\|_2 \le R$ for all $t_0 \le t \le t_1$ and $\|p_j(t_1+1) - p_i(t_1+1)\|_2 > R$.

Obvious, $t_1 \geq t_0 + T + 1$, and by the assumption of link failure and recovery, Assumption (AC) implies that $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t)), t_0 \leq t \leq t_1$ and if the above assumption is false, then $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t))$ for all $t \geq t_0$.

Without loss of generality, suppose that $p_j^{(1)}(t_1+1) - p_i^{(1)}(t_1+1) > \frac{\sqrt{2}R}{2}$. Since $\|p_i(t_0) - p_j(t_0)\|_\infty < D^0$, there exists a time t_0', $t_0 < t_0' \leq t_1$, such that

$$D^0 + TS^0 \leq p_j^{(1)}(t_0') - p_i^{(1)}(t_0') < D^0 + (T+1)S^0,$$

and

$$p_j^{(1)}(t) - p_i^{(1)}(t) \geq D^0 + (T+1)S^0, \quad t = t_0'+1, \ldots, t_1+1.$$

By Lemma 3, the maximum step length along each global coordinate axis is not larger than S^0, and thus the above inequalities imply that for $t = t_0', \ldots, t_1, p_j^{(1)}(t-\tau_{ij}(t)) - p_i^{(1)}(t) \geq D^0$ and $p_i^{(1)}(t - \tau_{ji}(t)) - p_j^{(1)}(t) \leq -D^0$. It follows from Lemma 4 and Rule (R2) that

$$\begin{cases} u_i^{(1)}(t) \geq 0 \\ u_j^{(1)}(t) \leq 0, \end{cases} \quad t = t_0', \ldots, t_1. \tag{12}$$

Case 1: Consider the case when the coordinate orientation of each agent is fixed. For simplicity, we drop the time parameter in the expression $\theta_i(t)$.

The above inequality (12) implies that the two agents are located between the two lines for $t = t_0', t_0'+1, \ldots, t_1+1$,

$$\begin{cases} \text{Line 1: } \xi(s) = p_i(t_0') - s\sin(\theta_i)\mathbf{x} + s\cos(\theta_i)\mathbf{y}, & s \in (+\infty, \infty) \\ \text{Line 2: } \xi(s) = p_j(t_0') - s\sin(\theta_j)\mathbf{x} + s\cos(\theta_j)\mathbf{y}, & s \in (+\infty, \infty), \end{cases}$$

where $\mathbf{x}, \mathbf{y}$ are the unit vectors codirectional with the global horizontal and vertical axes respectively.

On the other hand, by Eq. (4), there exists a square region with side length $d_{\max}$ and the movement of all agents is restricted to this region all the time. Therefore, $p_j^{(1)}(t) - p_i^{(1)}(t) < D^0 + (T+1)S^0 + 2d_{\max}\tan(e_{\max}^{\theta}) \leq \frac{\sqrt{2}R}{2}$ for all $t, t_0' \leq t \leq t_1+1$, which contradicts Assumption (AC).

Case 2: Consider the case when the coordinate orientation of each agent is time-varying. By Lemma 5,

$$\begin{cases} (\Theta(\theta_i(t))u_i(t))^{(1)} \geq -S|\sin(\theta_i(t))| \\ (\Theta(\theta_j(t))u_j(t))^{(1)} \leq S|\sin(\theta_j(t))|, \end{cases} \quad t = t_0', t_0'+1, \ldots, t_1+1.$$

Therefore, by inequality (11),

$$\begin{aligned} &p_j^{(1)}(t_1+1) - p_i^{(1)}(t_1+1) \\ &< D^0 + (T+1)S^0 + S\sum_{l=t_0'}^{t_1} |\sin(\theta_i(l))| + S\sum_{l=t_0'}^{t_1} |\sin(\theta_j(l))| \\ &< D^0 + (T+1)S^0 + S\sum_{l=t_0'}^{t_1} |\theta_i(l)| + S\sum_{l=t_0'}^{t_1} |\theta_j(l)| \\ &\leq \frac{\sqrt{2}R}{2}, \end{aligned}$$

which contradicts Assumption (AC). □

5.4. Robustness against measurement errors

This subsection will show the robustness of protocol (3) against measurement errors. Let $\hat{p}_j(t)|_{\mathcal{O}_i(t)}$ denote the measured relative position of agent j, with a measurement error, obtained by agent i at time t; and assume that these measurement errors are bounded by $e_{\max}^p$, that is, $\|\hat{p}_j(t)|_{\mathcal{O}_i(t)} - p_j(t)|_{\mathcal{O}_i(t)}\|_2 \leq e_{\max}^p$. In this case, we revise the definition of "registered neighbors" and the renewal strategy of "registered neighbor" role with $\hat{p}_j(t)|_{\mathcal{O}_i(t)}$ in place of $p_j(t)|_{\mathcal{O}_i(t)}$ and with $R + e_{\max}^p$ in the place of R, and revise the definitions of $\mathcal{D}_i^{\text{rect}}(t, \cdot)$ and $\mathcal{D}_i(t)$ with $\hat{p}_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(\cdot)}$ in place of $p_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(\cdot)}$. Furthermore, we restate the link recovery assumption by that if $\|p_j(k) - p_i(k)\|_2 \leq R$ for all $k, t \leq k \leq t + T$, then agent i should obtain the relative position of agent j, $\hat{p}_j(t)|_{\mathcal{O}_i(t)}$, with $\|\hat{p}_j(t')|_{\mathcal{O}_i(t')}\|_2 \leq R + e_{\max}^p$, at some time t', $t \leq t' \leq t + T$. Consequently, we can further assume that $\|\hat{p}_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(t-\tau_{ij}(t))}\|_2 \leq R + e_{\max}^p$ for any t. Under the above assumptions, the evolution equation of the studied system can be written as

$$\begin{aligned} p_i^{(k)}(t) = &\frac{1}{\sum_{j\in\mathcal{N}_i(t)\cup\{i\}} \alpha_{ij}^k(t)} \\ &\times \left(\sum_{j\in\mathcal{N}_i(t)} \alpha_{ij}^k(t)\hat{p}_j^{(k)}(t - \tau_{ij}(t)) + \alpha_{ii}^k(t)p_i^{(k)}(t)\right), \\ &i \in \mathcal{I}_n, k \in \{1, 2\}, \end{aligned} \tag{13}$$

where $W_L \leq \alpha_{ij}^k(t) \leq W_U$.

As for Eq. (13), we have the following convergence result.

Theorem 4. *Consider Eq. (13) with $\|\hat{p}_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} - p_j(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)}\|_2 \leq e_{\max}^p$. If registered interaction topology $\mathcal{G}(t)$ always contains a spanning tree, then there exists $0 < \lambda^* < 1$ such that for any $k \in \{1, 2\}$, $V(t) = \max_i p_i^{(k)}(t) - \min_i p_i^{(k)}(t)$ will converge to a value not greater than $\left(1 + n(T+1)(2^{(n+1)^2T^2}+1)\left(\frac{1}{1-\lambda^*}\right)\right)e_{\max}^p$ in the sense of limit superior as time goes on.*

Proof. See the Appendix. □

Remarks.

(1) Theorem 4 just gives a theoretical estimation of the upper bound on the final inter-agent distances; and in practice, the upper bound may not be that large.
(2) A more general result can be derived if the assumption about the registered interaction topology $\mathcal{G}(t)$ is relaxed to that the union of the registered interaction graph contains a spanning tree periodically.

To show the effectiveness of connectivity preservation, we need to ensure that Rules (R1, R2) can be guaranteed in the evolution of the system. Note that each agent uses the value of $\hat{p}_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)}$ but not $p_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)}$ in the computation of $\mathcal{D}_i(t)$ and that $p_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \geq D$ implies that $\hat{p}_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \geq D - e_{\max}^p$ and $p_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \leq -D$ implies that $\hat{p}_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \leq -(D - e_{\max}^p)$. So it is more convenient to change registering radius D to $D - e_{\max}^p$ ($e_{\max}^p < D$), and use Rule (R2′) instead of (R2) in the selection of $u_i(t)$:

(R2′): for any $i \in \mathcal{I}_n, t \in \mathbb{N}, j \in \mathcal{N}_i(t)$, and any $k \in \{1, 2\}$, if $\hat{p}_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \geq D - e_{\max}^p$, then $u_i^{(k)}(t) \geq 0$, and if $\hat{p}_j^{(k)}(t - \tau_{ij}(t))|_{\mathcal{O}_i(t)} \leq -(D - e_{\max}^p)$, then $u_i^{(k)}(t) \leq 0$, where $e_{\max}^p$ should be smaller than D.

As corollaries of Theorems 2 and 3, we present the following compatible conditions and connectivity preservation result for the case with measurement errors.

Corollary 1 (*Compatibility*). *Assume that for $t' < t$, the control law $u_i(t')$ is chosen according to Rule (R1); i.e., $\|u_i(t')\|_\infty \leq S, t' < t$; assume that the maximum measurement error $e_{\max}^p < D$; the maximum axis alignment error $e_{\max}^{\theta} < \arctan\left(\min\left\{\frac{D - e_{\max}^p}{2(R^0 + e_{\max}^p)}, \frac{S}{4(R^0 + e_{\max}^p)}\right\}\right)$; and parameters W_U and W_L meet the following condition:*

(C2):

$$\frac{W_U}{W_L} \geq \max\left\{ N\left(\frac{R^0 + e^p_{\max}}{S - 2(R^0 + e^p_{\max})\tan(e^\theta_{\max})} - 1\right), \frac{(N-1)(R^0 + e^p_{\max}) + 2(R^0 + e^p_{\max})N\tan(e^\theta_{\max})}{D - e^p_{\max} - 2(R^0 + e^p_{\max})\tan(e^\theta_{\max})} \right\}.$$

Then there exists $u_i(t) \in \mathcal{D}_i(t)$, satisfying Rules (R1, R2′) *and thus satisfying Rule* (R2).

Corollary 2 (*Connectivity Preservation*)**.** *Assume that the maximum measurement error $e^p_{\max} < D$; the maximum axis alignment error*

$$e^\theta_{\max} < \arctan\left(\min\left\{\frac{D - e^p_{\max}}{2(R^0 + e^p_{\max})}, \frac{S}{4(R^0 + e^p_{\max})}\right\}\right);$$

and Eq. (11) *holds. Then if the system evolves under Rules* (R1, R2′), *and $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t_0))$ for some time t_0, then $\{(v_i, v_j), (v_j, v_i)\} \subset \mathcal{E}(\mathcal{G}(t))$ for all $t \geq t_0$.*

Remarks.

(1) Corollary 2 does not give sufficient conditions for the case with time-invariant local coordinate orientations. This is mainly because the moving region of agents may be unbounded in some extreme cases with measurement errors. The related topics are currently under investigation.

(2) The maximum measurement error $e^p_{\max}$ is bounded by D. However, if $e^p_{\max}$ is sufficiently close to D, then the registering radius $D - e^p_{\max}$ will become very small, and in this case, if the initially registered interaction topology $\mathcal{G}(0)$ is connected, then we can guess that all agents are gathered in a small space and thus connectivity-preserving Rules (R1, R2′) can be dropped. Therefore, there is a tradeoff between the maximum measurement error $e^p_{\max}$ and the effective range of Rules (R1, R2′).

(3) It also can be observed that the larger the maximum measurement error $e^p_{\max}$ is, the larger $\frac{W_U}{W_L}$ is required to be.

6. Simulations

This section presents simulation results for a group of agents with minimum sensing radius $R = 20$. To show the effectiveness of the proposed protocol, we consider the case that most of the agents are gathered randomly in the four corners of a rectangle region and the rest of them are sparsely distributed, and assume that the initial position measurements are all successful and all initially possible neighbors are registered.

Fig. 4 shows the trajectories of agents under protocol (3) with Rules (R1, R2), where $T = 2, S = 1$; $D = 10, e^\theta_{\max} = 0.007$, and axis alignment errors are uniformly distributed between $[-e^\theta_{\max}, e^\theta_{\max}]$ and time-invariant. Fig. 5 shows the evolution of the maximum distance between initially mutually registered neighbors. For comparison, Figs. 6 and 7 show the trajectories of agents and the evolution of the same variable as in Fig. 5 under the consensus protocol with equal weighting factors (see Ren & Beard, 2005) and under the assumption that $T = 0$. It is clear that the connectivity can be preserved under our proposed protocol, but the convergence is slowed down due to the restriction on the maximum step length.

For the case with time-varying local coordinate orientations, assume that each $\theta_i(t)$ converges asymptotically to zero at a vanishing speed $\mu = 0.8, e^\theta_{\max} = 0.01, e^p_{\max} = 1$, and $T = 1$; and choose $D = 11$. Figs. 8 and 9 show the trajectories of agents and the evolution of the maximum distance between initially registered neighboring agents under protocol (3) with Rules (R1, R2′) respectively.

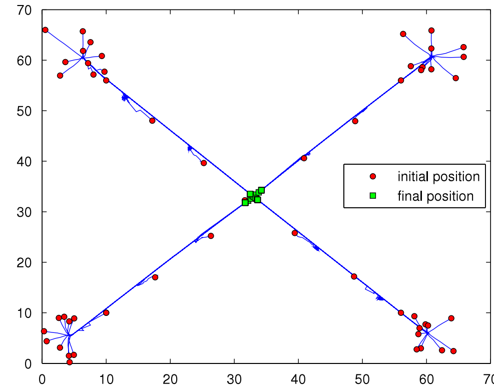


Fig. 4. Trajectories of agents with time-invariant local coordinate orientations under protocol (3) with Rules (R1, R2).

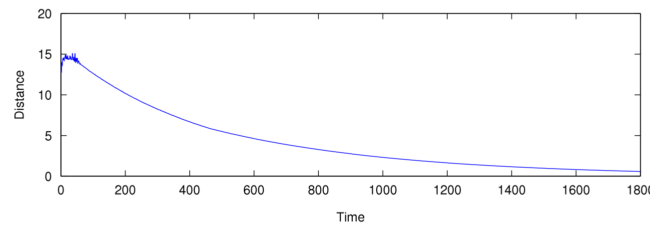


Fig. 5. Evolution of the maximum distance between initially mutually registered neighbors with time-invariant local coordinate orientations under protocol (3) with Rules (R1, R2).

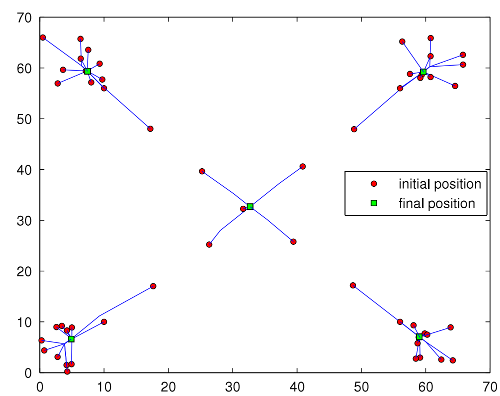


Fig. 6. Trajectories of agents under the consensus protocol with equal weighting factors.

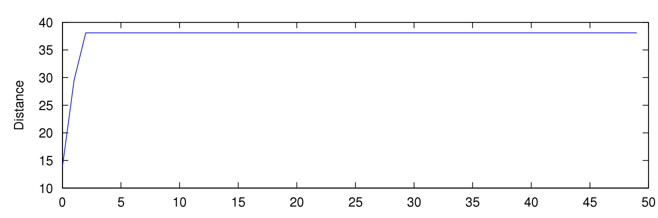


Fig. 7. Evolution of the same variable as in Fig. 5 under the consensus protocol with equal weighting factors.

7. Conclusion

This paper proposed a new protocol for multi-agent rendezvous and connectivity preservation with directed interaction and time delayed information. We showed its effectiveness and robustness by rigorous mathematical arguments. However, the proposed algorithm is a generalized convex combination algorithm with connectivity-preserving restrictions, and these restrictions may

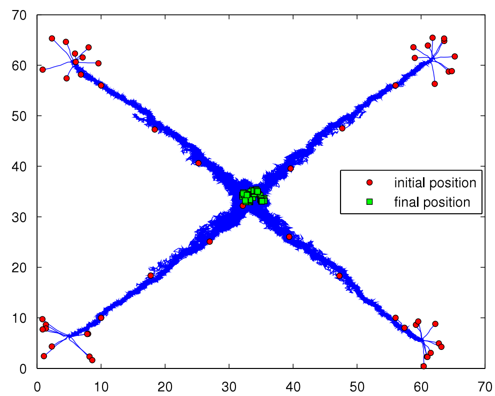


Fig. 8. Trajectories of agents with time-varying local coordinate orientations and measurement errors under protocol (3) with Rules (R1, R2).

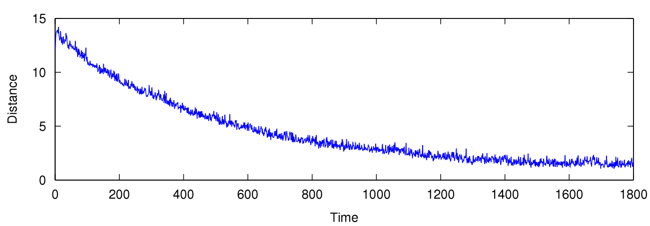


Fig. 9. Evolution of the maximum distance between initially mutually registered neighbors with time-varying local coordinate orientations and measurement errors under protocol (3) with Rules (R1, R2′).

negatively affect the performance of the system. Therefore, how to remove these restrictions and improve the convergence performance is a future research topic. Future work also includes formation control with connectivity preservation by similar ideas as employed in this paper.

Appendix

This Appendix will give the proof of Theorem 4. To reach this end, we first present some preliminary notion and results in nonnegative matrix theory (Wolfowitz, 1963).

A stochastic matrix A is called indecomposable and aperiodic (SIA) (or *ergodic*) if there exists a column vector ν such that $\lim_{k\to\infty} A^k = \mathbf{1}\nu^T$, where $\mathbf{1} = [1, 1, \ldots, 1]^T$ with compatible dimensions. Let $A = [a_{ij}]$ be any stochastic matrix and let $\delta(A) = \max_j \max_{i_1,i_2} |a_{i_1 j} - a_{i_2 j}|$. Thus $\delta(A)$ measures how different the rows of A are. If the rows of A are identical, $\delta(A) = 0$, and vice versa. Define $\lambda(A) = 1 - \min_{i_1,i_2} \sum_j \min(a_{i_1,j}, a_{i_2,j})$, where A is stochastic. If $\lambda(A) < 1$, A is called a *scrambling matrix*. Let $\prod_{i=1}^{l} A_i = A_l A_{l-1} \cdots A_1$, denoting the left product of matrices.

Lemma 6 (*Hajnal, 1958, Theorem 2*). *For any stochastic matrices* $A_1, A_2, \ldots, A_l$,

$$\delta\left(\prod_{i=1}^{l} A_i\right) \le \prod_{i=1}^{l} \lambda(A_i).$$

Proof of Theorem 4. Denote $q^0(t) = [p_1^{(k)}(t), p_2^{(k)}(t), \ldots, p_n^{(k)}(t)]^T$ and denote $q(t) = [q^0(t)^T, q^0(t-1)^T, q^0(t-T)^T]^T$. Then Eq. (13) can be equivalently represented in the following matrix form:

$$q(t+1) = \Xi(t)q(t) + \Delta q(t), \quad t \ge T, \tag{14}$$

where $\Xi(t)$ is stochastic and $\|\Delta q(t)\|_\infty < e^p_{\max}$. Let $\mathcal{X}$ denote the closure of the set consisting of all possible state matrices $\Xi(t)$. Then by Lemma 5 in Xiao and Wang (2008), it can be shown that for any $A_1, A_2, \ldots, A_l \in \mathcal{X}$ (repetitions permitted), $\prod_{i=1}^{l} A_i$ is an SIA matrix, and if $l > 2^{n^2(T+1)^2}$, then $\prod_{i=1}^{l} A_i$ is a scrambling matrix, see Xiao and Wang (2008) for detailed discussions. For notational simplicity, let $m = 2^{n^2(T+1)^2} + 1$. Define

$$\lambda^* = \sup_{A_i \in \mathcal{X}} \lambda\left(\prod_{i=1}^{m} A_i\right).$$

It follows from the compactness of $\mathcal{X}$ that $0 < \lambda^* < 1$. For any t, let $\left\lfloor \frac{t}{m} \right\rfloor$ denote the maximum integer not greater than $\frac{t}{m}$ and let $\left(\frac{t}{m}\right)^+ = t - \left\lfloor \frac{t}{m} \right\rfloor m$. Then $0 \le \left(\frac{t}{m}\right)^+ < m$, and

$$t = \left\lfloor \frac{t}{m} \right\rfloor m + \left(\frac{t}{m}\right)^+.$$

To study the convergence property of difference Eq. (14), we introduce the following Lyapunov-like function:

$$V'(q(t)) = \max_{i \in \mathcal{I}_n, 0 \le t' \le T} p_i^{(k)}(t - t') - \min_{i \in \mathcal{I}_n, 0 \le t' \le T} p_i^{(k)}(t - t').$$

Clearly,

$$V'(q(t)) \le 2\|q(t)\|_\infty.$$

Moreover, it can be shown that for any stochastic matrix A in $\mathbb{R}^{n(T+1)\times n(T+1)}$,

$$V'(Aq(t)) \le n(T+1)\delta(A)\|q(t)\|_\infty, \tag{15}$$

and

$$\|Aq(t)\|_\infty \le \|q(t)\|_\infty. \tag{16}$$

By Eq. (14), we get that

$$\begin{aligned} q(t+T+1) &= \Xi(t+T)\Xi(t+T-1)\cdots\Xi(T)q(T) \\ &\quad + \Xi(t+T)\Xi(t+T-1)\cdots\Xi(T+1)\Delta q(T) \\ &\quad + \cdots + \Xi(t+T)\Delta q(t+T-1) + \Delta q(t+T). \end{aligned}$$

With respect to the first item on the right side of the above equation, we have

$$\begin{aligned} V'\left(\prod_{i=0}^{t} \Xi(T+i)q(T)\right) &\le n(T+1)\delta\left(\prod_{i=\left(\frac{t}{m}\right)^+ + 1}^{t} \Xi(T+i)\right) \\ &\quad \times \left\| \prod_{i=0}^{\left(\frac{t}{m}\right)^+} \Xi(T+i)q(T) \right\|_\infty \\ &\le n(T+1)(\lambda^*)^{\left\lfloor \frac{t}{m} \right\rfloor}\|q(T)\|_\infty, \end{aligned}$$

where the first inequality follows from inequality (15) and the last inequality follows from Lemma 6 and inequality (16). With the same arguments, it can be shown that for any t' with $1 \le t' \le t$,

$$V'\left(\prod_{i=t'}^{t} \Xi(T+i)\Delta q(T+t'-1)\right) \le n(T+1)(\lambda^*)^{\left\lfloor \frac{t-t'}{m} \right\rfloor} e^p_{\max}.$$

Therefore,

$$\begin{aligned} V'(q(t+T+1)) &\le n(T+1)(\lambda^*)^{\left\lfloor \frac{t}{m} \right\rfloor}\|q(T)\|_\infty \\ &\quad + \Bigg(1 + n(T+1)m\Big(1 + \lambda^* + (\lambda^*)^2 + \cdots \\ &\quad + (\lambda^*)^{\left\lfloor \frac{t-1}{m} \right\rfloor}\Big)\Bigg) e^p_{\max}. \end{aligned}$$

Since $0 \le \lambda^* < 1$,

$$\limsup_{t\to\infty} V'(q(T+t+1)) \le \left(1 + n(T+1)m\left(\frac{1}{1-\lambda^*}\right)\right) e^p_{\max},$$

which implies that

$$\limsup_{t\to\infty} V(t) \le \left(1 + n(T+1)m\left(\frac{1}{1-\lambda^*}\right)\right) e^p_{\max}. \quad \square$$

References

Ando, H., Oasa, Y., Suzuki, I., & Yamashita, M. (1999). Distributed memoryless point convergence algorithm for mobile robots with limited visibility. *IEEE Transactions on Robotics and Automation*, *15*(5), 818–828.

Conte, G., & Pennesi, P. (2010). The rendezvous problem with discontinuous control policies. *IEEE Transactions on Automatic Control*, *55*(1), 279–283.

Cortés, J. (2009). Global and robust formation-shape stabilization of relative sensing netwoks. *Automatica*, *45*, 2754–2762.

Dimarogonas, D. V., & Kyriakopoulos, K. J. (2007). On the rendezvous problem for multiple nonholonomic agents. *IEEE Transactions on Automatic Control*, *52*(5), 916–922.

Fagnani, F., Johansson, K. H., Speranzon, A., & Zampieri, S. (2004). On multi-vehicle rendezvous under quantized communication. In *Proceedings of international symposium of mathematical theory of networks and systems. Leuven, Belgium*.

Godsil, C., & Royal, G. (2001). *Algebraic graph theory*. New York: Springer-Verlag.

Gustavi, T., Dimarogonas, D. V., Egerstedt, M., & Hu, X. (2010). Sufficient conditions for connectivity maintenance and rendezvous in leader–follower networks. *Automatica*, *46*, 133–139.

Hajnal, J. (1958). Weak ergodicity in non-homogeneous Markov chains. *Proceedings of the Cambridge Philosophical Society*, *54*, 233–246.

Ji, M., & Egerstedt, M. (2007). Distributed coordination control of multiagent systems while preserving connectedness. *IEEE Transactions on Robotics*, *23*(4), 693–703.

Ji, Z., Wang, Z., Lin, H., & Wang, Z. (2009). Interconnection topologies for multi-agent coordination under leader–follower framework. *Automatica*, *45*, 2857–2863.

Lin, Z., Broucke, M., & Francis, B. (2004). Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control*, *49*(4), 622–629.

Lin, Z., Francis, B., & Maggiore, M. (2005). Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, *50*(1), 121–127.

Lin, Z., Francis, B., & Maggiore, M. (2007). State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, *46*(1), 288–307.

Lin, J., Morse, A. S., & Anderson, B. D. O. (2007a). The multi-agent rendezvous problem. Part 1: the synchronous case. *SIAM Journal on Control and Optimization*, *46*(6), 2096–2119.

Lin, J., Morse, A. S., & Anderson, B. D. O. (2007b). The multi-agent rendezvous problem. Part 2: the asynchronous case. *SIAM Journal on Control and Optimization*, *46*(6), 2120–2147.

Litus, Y., Zebrowski, P., & Vaughan, R. T. (2009). A distributed heuristic for energy-efficient multirobot multiplace rendezvous. *IEEE Transactions on Robotics*, *25*(1), 130–135.

Martínez, S. (2009). Practical multiagent rendezvous through modified circumcenter algorithms. *Automatica*, *45*, 2010–2017.

Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, *95*(1), 215–233.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Ren, W., Beard, R. W., & Atkins, E. M. (2007). Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine*, *27*(2), 71–82.

Smith, S. L., Broucke, M. E., & Francis, B. A. (2007). Curve shortening and the rendezvous problem for mobile autonomous robots. *IEEE Transactions on Automatic Control*, *52*(6), 1154–1159.

Sun, Z., & Huang, J. (2009). A note on connectivity of multi-agent systems with proximity graphs and linear feedback protocol. *Automatica*, *45*, 1953–1956.

Tanner, H. G. (2004). On the controllability of nearest neighbor interconnections. In *Proceedings of the 43rd IEEE conference on decision and control* (pp. 2467–2472).

Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., & Schochet, O. (1995). Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, *75*(6), 1226–1229.

Wang, L., & Xiao, F. (2010). Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control*, *55*(4), 950–955.

Wolfowitz, J. (1963). Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, *14*, 733–737.

Xiao, F., & Wang, L. (2008). Consensus protocols for discrete-time multi-agent systems with time-varying delays. *Automatica*, *44*, 2577–2582.

Xie, G., & Wang, L. (2009). Moving formation convergence of a group of mobile robots via decentralised information feedback. *International Journal of Systems Science*, *40*(10), 1019–1027.

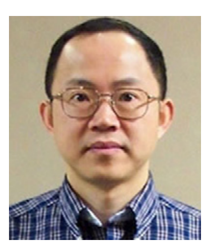
Yang, P., Freeman, R. A., Gordon, G. J., Lynch, K. M., Srinivasa, S. S., & Sukthankar, R. (2010). Decentralized estimation and control of graph connectivity for mobile sensor networks. *Automatica*, *46*, 390–396.

Zavlanos, M. M., & Pappas, G. J. (2007). Potential fields for maintaining connectivity of mobile networks. *IEEE Transactions on Robotics*, *23*(4), 812–816.

Feng Xiao received the B.S. and M.S. degrees in Mathematics from Inner Mongolia University, Huhhot, in 2001 and 2004, respectively, and the Ph.D. degree in Systems and Control from Peking University, Beijing, in 2008. He worked at the School of Automation, Beijing Institute of Technology, Beijing, from 2008 to 2010, and is now a Postdoctoral Fellow in the Department of Electrical and Computer Engineering at the University of Alberta. He was a recipient of the 2010 Izaak Walton Killam Postdoctoral Fellowship, and won the Dorothy J. Killam Memorial Postdoctoral Fellow Prize at the University of Alberta in 2010. His research interests are in the fields of coordination of multi-agent systems, consensus problems, formation control and swarm dynamics. Email: fxiao@ece.ualberta.ca

Long Wang was born in Xian, China on Feb. 13, 1964. He received his Bachelor's, Master's, and Doctorate degrees in Dynamics and Control from Tsinghua University and Peking University in 1986, 1989, and 1992, respectively.

He has held research positions at the University of Toronto, the University of Alberta, Canada, and the German Aerospace Center, Munich, Germany. He is currently a Cheung Kong Chair Professor of Dynamics and Control, and the Director of Center for Systems and Control of Peking University. He is also a Guest Professor at Wuhan University and Beihang University. He serves as the Editor-in-Chief of the International Journal of Information and Systems Science, an Editor of the Journal of Mechanical Science and Technology, an executive member of Chinese Society of Artificial Intelligence and Chinese Society of Systems Simulation, a panel member of the Division of Information Science, National Natural Science Foundation of China, and a member of the IFAC (International Federation of Automatic Control) Technical Committee on Networked Systems. He is on the editorial boards of Science in China, Progress in Natural Science, Journal of Intelligent Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Control and Decision, Information and Control, Journal of Applied Mathematics and Computation, Journal of Intelligent and Robotic Systems, and International Journal of Computer Systems. His research interests are in the fields of complex networked systems, information dynamics, collective intelligence, and bio-mimetic robotics. Email: longwang@pku.edu.cn



Tongwen Chen is presently a Professor of Electrical and Computer Engineering at the University of Alberta, Edmonton, Canada. He received the B.Eng. degree in Automation and Instrumentation from Tsinghua University (Beijing) in 1984, and the M.A.Sc. and Ph.D. degrees in Electrical Engineering from the University of Toronto in 1988 and 1991, respectively.

His research interests include computer and network based control systems, and their applications to the process and power industries. He has served as an Associate Editor for several international journals, including IEEE Transactions on Automatic Control, Automatica, and Systems and Control Letters. He is a Fellow of IEEE.