Duct spinning mode’s particle velocity imaging with in-duct circular microphone array

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Nowadays, the measurements within a duct have to be conducted using in-duct microphone array, which is unable to provide information of complete acoustic solutions across the test section. In this work, an imaging method of acoustic spinning modes propagating within a circular duct simply with surface pressure information is introduced. The proposed method is developed in a theoretical way and is demonstrated by a numerical simulation case. The fundamental idea behind the testing method was originally developed in control theory for ordinary differential equations. Spinning mode propagation, however, is formulated in partial differential equations. A finite difference technique is used to reduce the associated partial differential equations to a classical form in control theory. The observer method can thereafter be applied straightforwardly. The algorithm is recursive and thus could be operated in real-time. The observer error of the method is analyzed then. A numerical simulation for a straight circular duct is conducted. The acoustic solutions on the test section can be reconstructed with a good agreement to analytical solutions. The results suggest the potential and applications of the proposed method.

I. Introduction

A new method to analytically deduce the associated particle velocity based on the information of sound pressure was developed in this work. The particular attention was focused on the sound propagation and radiation from a straight duct [1–3]. The proposed method can find applications in low noise turbofan engine tests [4] in particular and should be applicable to other duct acoustic applications in general.

The rotating fan and stator assembly is mainly responsible for the generation of tonal spinning modes in engines [5, 6]. A theoretical model to predict sound radiation from annular, straight pipes with a jet flow has been developed previously, based on the Wiener-Hopf technique [7]. In addition, formulae describing spinning mode reflecting into a straight, cylindrical duct have been presented [1]. The problem of sound propagation and radiation from a cylindrical duct has also been investigated with a wide range of numerical methods, using finite elements [2, 8], mode-matching strategy [9], based on linearised Euler equations (LEE) [10] and the variants of LEE [11–13].

On experimental side, the development of suitable testing methods for duct acoustics is of particular interest. Nowadays, a series of experimental techniques have been developed mainly for pressure sensor apparatuses, including in-duct microphone array [14, 15] and outside-duct microphone array [4].

Multiple modes could simultaneously propagate within a duct. As a first step, one needs to isolate each spinning mode using the so-called mode detection technique [16, 17]. Next, sound pressure contours at the region of noise sources can be visualised using beamforming technique [16, 18].

Most existing techniques only produce sound pressure outputs. In this work, we proposed a theoretical method that could analytically yield the associated particle velocity from readily available sound pressure information. One

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may regard this method as an imaging technique that can simultaneously visualise sound pressure and the associated acoustic particle velocity. The method was developed and validated in this work particularly for duct acoustic applications.

It is essentially expected that the proposed method could infer unknown states (e.g. the associated acoustic particle velocity) based on some partially available information (e.g. sound pressure), given the mathematical relationship between unknown states and available information. A similar problem has been resolved for linear systems, which are normally described by ordinary differential equations (ODE), using a so-called observer method [19].

On the other hand, the observability of PDE has long been of interest to mathematicians [20–22]. One of the most recent developments is for stochastic PDE by Galerkin approximations [23]. Readers with interest can refer to above citations and references therein for a more exhaustive literature list. The analysis adopted in this investigation builds upon the previous work [20] that firstly studied observability of discrete wave equations. It was followed by theoretical discussions and pioneering insights to spherical wave systems [22].

In this paper, the fundamental idea behind the proposed method shares the same theoretical background of previous works in real-time acoustic beamforming [18, 24] of coherent noise sources. [25] However, the previous beamforming works were conducted in frequency domain and noise sources were explicitly regarded as simple monopoles. The noise sources discussed in this work are spinning modes, having a relatively more complex propagation model. The model was formulated in a modern control form and a so-called observer was designed to infer acoustic velocities and pressures from the in-duct circular array measurements. Section II briefly introduces the preliminary knowledge of spinning modes and observer, and straightforwardly sets up the theoretical model of the proposed method. A numerical case was conducted to validate the proposed method in Sec. III. Finally, Sec. IV summarizes the present work.

II. Method Introduction

A. Observer theory

A so-called observer based algorithm has been previously proposed [18, 24] for acoustic array beamforming. The basic idea was based on observer theory in classical control. The acoustic imaging method developed in this work shares the similar theoretical background. The essence of the proposed method is to recursively estimate time varying acoustic parameters (density, velocities, etc.) of a spinning mode with the knowledge of its dynamic model and measurements (sound pressure for the case). It is also worthwhile to notice the differences from the previous observer based beamforming method. Firstly, the present problem of interest is transmission acoustic field imaging instead of noise source imaging. Secondly, the present problem is solved in time domain and the resultant model is governed by linear partial differential equations instead of ordinary differential equations. The issues due to those differences are to be addressed in the following paper. For convenience of readers, we firstly introduce the fundamental idea of observer theory in this section. More details can be found in any linear control textbook [19].

In classical control, a dynamic process can be generally described as a state space model

\[
\begin{align*}
\frac{d}{dt} x(t) &= A x(t) + B u(t), \\
y(t) &= C x(t),
\end{align*}
\]

where $t$ is time, $x$ and $u$ represent internal states and inputs of the model, $y$ denotes model outputs, and $A$, $B$ and $C$ are dynamic, control and output matrices, respectively. In this work, $A$, $B$ and $C$ are time invariant, and $y$ is the circular sensor array measurements, from which the acoustic variables $x$ are hopefully inferred. The so-called observability in linear control theory defines a measure of how well those internal states could be deduced [19]. The observability of the state space model [Eqs. (1)-(2)] can be investigated by forming an observability matrix

\[
O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{M-1} \end{pmatrix},
\]

where $M$ is the rank of $A$. It has been proved that a system is observable if the rank of $O$ equals $M$. Once a system is observable, it is possible to construct an observer to approximate the state $X$, only with the knowledge of $y$. The
observer has the form of

\[
\frac{d}{dt} \hat{x}(t) = A\hat{x}(t) + Bu(t) + L(y - \hat{y}),
\]

(4)

\[
y(t) = C\hat{x}(t),
\]

(5)

where \(\hat{(}\) denotes the approximation and \(L\) is the observer gain. It is straightforwardly that the estimation error \(e = x - \hat{x}\) satisfies

\[
\frac{d}{dt} e(t) = (A - LC)e,
\]

(6)

which is achieved by subtracting Eq. (1) from Eq. (4). It can be seen that \(e\) converges to zero when \(t \to \infty\), as long as the real parts of all eigenvalues of the matrix \((A - LC)\) are negative. It is worthwhile to mention that for the corresponding discrete system, \(e\) converges if all eigenvalues of the matrix \((A - LC)\) drop in the unit circle. The problem of finding a proper \(L\) can be straightforwardly resolved for most cases in a couple of steps of trial and error.

B. Analytical solution of spinning mode

Viscous dissipation and heat conduction are neglected for the studied sound propagation within a subsonic mean flow. The compressible Euler equations in cylindrical coordinates are used to model fluids around an axisymmetric duct, written in the conservative form as follows

\[
\frac{\partial p}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\rho v) + \frac{1}{r} \frac{\partial (\rho w)}{\partial \theta} = 0,
\]

(7)

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\rho u v) + \frac{1}{r} \frac{\partial (\rho u w)}{\partial \theta} + \frac{\partial p}{\partial x} = 0,
\]

(8)

\[
\frac{\partial \rho v}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\rho v^2) + \frac{1}{r} \frac{\partial (\rho v w)}{\partial \theta} + \frac{\partial p}{\partial r} = 0,
\]

(9)

\[
\frac{\partial \rho w}{\partial t} + \frac{\partial (\rho u w)}{\partial x} + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\rho v w) + \frac{1}{r} \frac{\partial (\rho w^2)}{\partial \theta} + \frac{1}{r} \frac{\partial p}{\partial \theta} = 0,
\]

(10)

where \(x\) is the axial coordinate, \(r\) is the radial coordinate, \(\theta\) is the azimuthal angle, \(\rho\) is the density, \(p\) is the pressure, \(u\) is the axial velocity, \(v\) is the radial velocity and \(w\) is the azimuthal velocity.

The perturbations of acoustic pressure, density and velocities, \((p', \rho', u', v', w')\) are generally small compared with the background mean flow variables \((p_0, \rho_0, u_0, v_0, w_0)\). Sound wave propagation can thus be simply modeled by linearized Euler equations (LEE). The acoustic disturbances can be represented by Fourier series in terms of azimuthal modes \(m\). For example, the acoustic pressure has the form of \(p' = \sum_{m=-\infty}^{\infty} p'_m(x, r, \theta)e^{-im\theta}\). For a uniform mean flow \((u_0, 0, 0)\) in the engine duct, the three-dimensional LEE are

\[
\frac{\partial p'_m}{\partial t} + u_0 \frac{\partial p'_m}{\partial x} + \rho_0 \left( \frac{\partial u'_m}{\partial x} + \frac{\partial u'_m}{\partial r} + \frac{v'_m}{r} + \frac{\partial w'_m}{\partial \theta} \right) = 0,
\]

(11)

\[
\frac{\partial u'_m}{\partial t} + u_0 \frac{\partial u'_m}{\partial x} + \frac{\partial u'_m}{\partial r} + \frac{\partial p'_m}{\partial x} = 0,
\]

(12)

\[
\frac{\partial v'_m}{\partial t} + u_0 \frac{\partial v'_m}{\partial x} + \frac{\partial v'_m}{\partial r} + \frac{\partial p'_m}{\partial r} = 0,
\]

(13)

\[
\frac{\partial w'_m}{\partial t} + u_0 \frac{\partial w'_m}{\partial x} + \frac{\partial w'_m}{\partial \theta} + \frac{\partial p'_m}{\partial \theta} = 0,
\]

(14)

where all variables are nondimensionalized using a reference length \((R, \text{the radius of the duct})\), a reference speed (the speed of sound), and a reference density (air density). For the idealized geometry with a straight and semi-infinite unflanged duct, the solutions of Eqs. (11)–(14) have analytical forms for \(m\)th azimuthal mode at a single frequency \(k\)

\[
\rho'_m(x, r, \theta, t) = c J_m(kr)e^{i(kt - k_x x - m\theta)},
\]

(15)

\[
u'_m(x, r, \theta, t) = \frac{ck_x}{k - k_x M_j} J_m(kr)e^{i(kt - k_x x - m\theta)},
\]

(16)

\[
v'_m(x, r, \theta, t) = \frac{c}{k - k_x M_j} \frac{dJ_m(kr)}{dr}e^{i(kt - k_x x - m\theta)},
\]

(17)

\[
w'_m(x, r, \theta, t) = \frac{cm}{r(k - k_x M_j)} J_m(kr)e^{i(kt - k_x x - m\theta)},
\]

(18)
where $M_j = u_0/C_0$, $C_0$ is the speed of sound, $c$ is the amplitude of the acoustic perturbation (the nondimensional value is normally less than $10^{-2}$), and $J_m$ is the $m$th-order Bessel function of the first kind. The real parts of the above formulations are acoustic solutions. The $n$th radial wavenumber $k_r$ of the $m$th spinning mode is the $n$th solution of the following equation determined by the hard-wall boundary conditions of the duct,

$$
\frac{d[J_m(Rk_r)]}{dr} = 0,
$$

(19)

where $R$ is the radius of the outer duct wall. The axial wavenumber $k_x$ of the $m$th mode can be subsequently calculated using

$$
k_x = \frac{k}{1 - M_j^2} \left(-M_j \pm \sqrt{1 - \frac{k^2(1 - M_j^2)}{k^2}}\right).
$$

(20)

The chosen of $(\pm)$ in Eq. (20) is determined by the upstream/downstream direction of the spinning wave.

C. State space model

The problem of a spinning mode radiation of a duct is of particular interest in this work. The following work develops a model in the form of Eqs. (1–2) for spinning modes propagation. It is worthwhile to notice that all formulations are presented in a continuous time version, which is presumably regarded more natural to readers. In this work, the numerical implementation actually adopted a discrete time form that is a strictly discrete analogue of the proposed method. In addition, to simplify the problem, the modeling error and measurement error are omitted here, which enable us to focus on central theoretical issues.

It will be of practical importance to obtain acoustic images of spinning modes propagating within ducts. The image plane considered in this work is the duct cross section at the $x$ location of sensor arrays. The imaging plane is descretized into an $A \times B$ grid, where $A$ and $B$ are the gridpoint numbers in $r$ and $\theta$ coordinates, respectively. Both values should satisfy Nyquist-Shannon sampling theorem, i.e., $A > 2n$ and $B > 2m$, where $n$ is radial mode and $m$ circumferential mode. For simplicity, $B$ is set to the sensor number of the circular array. For example, Fig. 1 shows the grids for a $(m = 3; n = 1)$ spinning mode; the number of gridpoints in the radial direction is 8; and the number of gridpoints in the circumferential direction is 30. It is implicitly assumed that 30 pressure sensors are located on the outer ring to instantaneously capture sound pressure.

![Figure 1. The grids in polar coordinates for the duct test section.](image)

For brevity, the subscript $m$ in Eqs. (11)–(18) is omitted in the rest of this paper. To construct a state space model for each single spinning mode, the following relations and approximations are adopted for Eqs. (11)–(14) at each gridpoint $(a, b)$, where $1 \leq a \leq A$, and $1 \leq b \leq B$:

1. $\partial/\partial x = -ik_x$, using the spectral method;

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(2) \( \rho' = p' \) in the nondimensional equations;
(3) \( \partial v'_{a,b}/\partial r \) is approximated by a backward difference, \( (v'_{a,b} - v'_{a-1,b})/\delta r \);
(4) \( \partial w'_{a,b}/\partial \theta \) is approximated by a central difference, \( (w'_{a,b+1} - w'_{a,b-1})/(2\delta \theta) \);
(5) \( \partial p'_{a,b}/\partial r \) and \( \partial p'_{a,b}/\partial \theta \) are approximated in similar ways, respectively;
(6) \( \partial v'_{1,b}/\partial r \approx 0, \partial p'_{1,b}/\partial r \approx 0, \) \( v'_{a,1} = v'_{a,b} \), and \( p'_{a,-1} = p'_{a,b} \);

where \( v'_{a,b} \triangleq v'(a, b) \), \( \partial v'_{a,b}/\partial r \triangleq \partial p'(a, b)/\partial r \), and so forth. All spatially differential terms in Eqs. (11)–(14) are replaced and the partial differential equations are reformulated to the ordinary differential equations. For example, the governing equations of one gridpoint at \( (a, b) \) are

\[
\frac{d}{dt} \begin{bmatrix}
\rho'_{a,b} \\
u'_{a,b} \\
v'_{a,b} \\
w'_{a,b}
\end{bmatrix} = \begin{bmatrix}
& ik_x u_0 & -\frac{\rho_0}{r} - \frac{\rho_0}{r} & 0 & 0 \\
& ik_x u_0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & ik_x u_0 \\
0 & 0 & 0 & ik_x u_0
\end{bmatrix} \begin{bmatrix}
\rho'_{a,b} \\
u'_{a,b} \\
v'_{a,b} \\
w'_{a,b}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & \frac{\rho_0}{r} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p_{a-1,b} \\
u_{a-1,b} \\
v_{a-1,b} \\
w_{a-1,b}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & \frac{\rho_0}{r} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p_{a+1,b} \\
u_{a+1,b} \\
v_{a+1,b} \\
w_{a+1,b}
\end{bmatrix},
\]

(21)

\[
\begin{bmatrix}
p'_{a,b} \\
x'_{a,b} \\
v'_{a,b} \\
w'_{a,b}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p'_{a,b} \\
u'_{a,b} \\
v'_{a,b} \\
w'_{a,b}
\end{bmatrix},
\]

(22)

where the nondimensional \( \rho' \) equals \( p' \). The complete equations of the state space model on the entire mesh (see Fig. 1) are

\[
\frac{d}{dt} \begin{bmatrix}
x_{a,b}
\end{bmatrix} = \begin{bmatrix}
\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D}
\end{bmatrix} \begin{bmatrix}
x_{a-1,b} \\
x_{a,b-1} \\
x_{a,b} \\
x_{a,b+1}
\end{bmatrix}
+ \begin{bmatrix}
x'_{a,b}
\end{bmatrix},
\]

(23)

\[
\begin{bmatrix}
y_{a,b}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & \mathbf{C}
\end{bmatrix} \begin{bmatrix}
x_{a-1,b} \\
x_{a,b-1} \\
x_{a,b} \\
x_{a,b+1}
\end{bmatrix}
+ \begin{bmatrix}
y_{a,b}
\end{bmatrix},
\]

(24)

In most practical tests, Eq. (24) only contains the contributions from those pressure sensors that are surface mounted on the hard wall of the duct, where \( v' \) and \( \partial p'/\partial r \) equal 0. It is easy to examine that the radial, axial and circumferential
velocities within the duct cross section are normally not observable only with measurements from those on-surface pressure sensors. The analytical solution of Eq. (15) is used to extend the system outputs, i.e. the sound pressure at \( r \) (\( 0 \leq r \leq R \)) can be inferred from the measurements at outer radius \( R \), according to the relation,

\[
\rho'(r) = \rho'(R) \frac{J_m(k_r r)}{J_m(k_r R)},
\]

where the nondimensional value \( R \) is 1. According to control theory, the observability of the system [Eqs. (23)-(24)] can be determined by constructing a matrix using Eqs. (3) and subsequently by examining its rank. If the system is observable, an observer of Eqs. (4)–(5) can be constructed with a carefully chosen observer gain.

In summary, Eqs. (11)–(14) describe an ideal mathematical model that is obtained by applying natural laws to the acoustical physics. In contrast, Eqs. (23)–(24) represent a reduced mathematical model that replaces partial differentials by finite difference terms. The particular finite difference schemes chosen in the reformulation could affect the final accuracy and efficiency. It is also worthwhile to note that the present output \( y \) only contains pressure information. It is also possible to regard velocity as known information, using high speed particle image velocimetry, and to infer the unknown sound pressure on the duct cross section. The theoretical method is still the same.

D. Observation errors

Firstly, we reformulate the previous wave equation and the related control-oriented model in a more consistent and succinct form.

The acoustic system of interest is autonomous and written as

\[
\frac{d}{dt} v_c(t, x_c) = A_c v_c(t, x_c), y_c(t, x_c) = C_c v_c(t, x_c),
\]

where \( v = (\rho', u', v', w') \) and \( y = \rho' \); the subscript \( c \) denotes that the associated variables are continuous in time; \( A_c \) represents spatial differential operators; and \( C_c = [1 0 0 0]^T \), where \( T \) is the transpose.

One is forced to conduct a spatially discrete approximation to approximate spatially partial differentials in \( A_c \). The resultant control-oriented model on one specified \( r - \theta \) cross section is

\[
\frac{d}{dt} v_d(t, x_d) = A_d v_d(t, x_d), y_d(t, x_d) = C_d v_d(t, x_d),
\]

where the subscript \( d \) denotes that the associated variables are in discrete-time domain. We define a bounded linear operator, \( \Pi : \mathcal{C} \to \mathcal{C}^N \) that satisfies \( \Pi x_c = x_d \) [20], where \( N = n_r \times n_\theta \) is the number of discretised gridpoints on the spatial domain. Physically, \( \Pi \) projects infinite dimensional space to finite dimensional space. An observer has the following form,

\[
\frac{d}{dt} \hat{v}_d = A_d \hat{v}_d + \mathbf{K}(\Pi y_c - \hat{y}_d),
\]

\[
\hat{y}_d = C_d \hat{v}_d,
\]

where \((t, x_d)\) is omitted for brevity. The estimation error is defined as

\[
e_d \triangleq \Pi v_c - \hat{v}_d,
\]

which represents the evolving difference between the original infinite-dimensional dynamics and the corresponding discretised finite-dimensional dynamics.

**Proposition 1:** The spatial discretization methods for the discrete linear wave system solely determine the stability of the observer. Nevertheless, the convergence rate of the observation error is subject to the original continuous-time system.

**Proof:** By simple algebraic manipulations, one obtains

\[
\frac{d}{dt} e_d = \Pi (A_c v_c) - A_d \hat{v}_d - \mathbf{K}(\Pi y - \hat{y}_d)
\]

\[
= (A_d - \mathbf{KC}_d)e_d + (\Pi A_c - A_d \Pi) v_c
\]

\[
\triangleq \mathbf{A} e_d + \mathbf{B} v_c,
\]
where the relationship $\Pi C_e v_e = C_d \Pi v_e$ is assumed. And $\mathfrak{B} \to 0$, as $\delta r, \delta \theta \to 0$, if the adopted finite difference method is consistent. Given an initial estimation error $e_d(0)$, the explicit formulation of the estimation error over time is

$$e_d(t) = e^{\mathfrak{A}t} e_d(0) + \int_0^t e^{\mathfrak{A}(t-\tau)} \mathfrak{B} v_e(\tau) d\tau. \quad (32)$$

Conducting Laplace transform ($\mathcal{L}$) on Eq. (31), one can achieve

$$E_d(s) = \frac{\mathfrak{B} V_e(s)}{sI - \mathfrak{A}}, \quad (33)$$

where $s$ is a complex number and $E_d(s) = \mathcal{L} \{e_d(t)\} = \int_0^\infty e_d(t) e^{-st} dt$. $V_e(s) = \mathcal{L} \{v_e(t)\}$; and $I$ is an identity matrix with an appropriate rank. □

**Remark:** Firstly, the stability of the observer only depends on $\mathfrak{A}$, which depends on the adopted finite difference schemes and the observer gain. Once $A_d$ and $C_d$ are known, one can design a suitable $K$ to ensure the stability of the observer, i.e., to make sure that the eigenvalues of $\mathfrak{A}$ are on the left complex plane.

Secondly, the approximation error due to $\mathfrak{B}$ is proportional to the state ($v_e$) from the original, continuous-time system. Physically, it means that if sound properties (e.g., pressure and particle velocity) are larger at one position, the observation error will be larger at the same place. To suppress this error, one needs a smaller $\mathfrak{B} = \Pi A_e - A_d \Pi$, which suggests that a high-order finite difference scheme for $A_d$ is preferable to approximate $A_c$.

Thirdly, Eq. (33) helps to develop asymptotic estimations. At high frequency range ($|s| \gg ||\mathfrak{A}||$), $e_d(t) \approx \int_0^t \mathfrak{B} v_e(\tau) d\tau$; while at low frequency range ($|s| \ll ||\mathfrak{A}||$), $e_d(t) \approx ||\mathfrak{B} v_e(t)/\mathfrak{A}||$. □

One can easily check that the alternative definition, $e_d \triangleq v_d - \dot{v}_d$, holds little physical meaning and is therefore not adopted.

**E. Observability with real-valued measurements**

One might argue that it is misleading to directly examine the observability of Eq. (27), because the time-domain outputs $y_d$ are complex-valued, which are actually impossible in practical tests. All time-domain (pressure, velocity, etc.) readings in (acoustic) measurements are real-valued. Duct acoustics are actually governed by

$$\frac{d}{dt} \begin{bmatrix} v_d \\ v^*_d \end{bmatrix} = \begin{bmatrix} A_d & A_d^* \\ A_d & v_d \\ A_d & v^*_d \end{bmatrix}, \quad \text{Re}(y_d) = \begin{bmatrix} C_d & C_d^* \\ C_d & v_d \\ C_d & v^*_d \end{bmatrix} \begin{bmatrix} v_d \\ v_d^* \end{bmatrix}, \quad (34)$$

where $(\cdot)^*$ means element-by-element conjugation; $\text{Re}(y_d)$ represents the real-valued outputs in practical measurements; and the constant coefficient (0.5) in $C_e$ is omitted. Next, we are going to prove that the extended dynamic system $(A_e, C_e)$ is observable, if $(A_d, C_d)$ is observable.

**Lemma 1:** If a linear, time invariant, unforced system $(A_d, C_d)$ is observable, the system $(A_d^*, C_d^*)$ is also observable.

The proof is straightforward and omitted for brevity.

**Proposition 2:** If a linear, time invariant system $(A_d, C_d)$ is observable, the extended system $(A_e, C_e)$ with real-valued outputs is also observable.

**Proof:** If the system $(A_d, C_d)$ is observable, according to Lemma 1, $(A_d^*, C_d^*)$ is also observable. Then, there should exist $K_T > 0$ such that

$$K_T ||v_{d0}||^2 \leq \int_0^T ||C_0 v_d(t)||^2 dt, \forall v_{d0} \in V, \quad (35)$$

$$K_T ||v_{d0}^*||^2 \leq \int_0^T ||C_0^* v_d(t)||^2 dt, \forall v_{d0}^* \in V, \quad (36)$$

where the relation $C_d = C_d^*$ for the wave system is adopted, because $C_d$ only contains real-valued vectors of $(1, 0, 0, 0)$. As $v_d, v_d^*$ and $v_e$ are on a Hilbert space, it is easy to know that $||v_{e0}||^2 \leq ||v_{d0}||^2 + ||v_{d0}^*||^2$, and $||C_v(t)||^2 + ||C^*_v(t)||^2 = ||C_v(t) + C^*_v(t)||^2 = ||C_e v_e(t)||^2$, because $v_d$ and $v_d^*$ are orthogonal. After applying these two relations and summing Eqs. (35)-(36), one can have

$$K_T ||v_{e0}||^2 \leq \int_0^T ||C_e v_e(t)||^2 dt, \forall v_{e0} \in V, \quad (37)$$

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which concludes the proof. □

Remark 1: In contrast, the proof based on algebraic observability theorem is not straightforward. The derivation is simply described below.

If \((A_d, C_d)\) is observable, the rank of \(O = [C_d, \cdots, C_dA_d^{M-1}]^T\) is equal to the system order \(M\). According to Lemma 1, the rank of \(O^* = [C_d, \cdots, C_dA_d^{M-1}]^T\) also equals \(M\). One can have the observability test matrix for the extended system,

\[
O_e = \begin{bmatrix}
C_d & C_d^* \\
\vdots & \vdots \\
C_dA_d^{M-1} & C_d^*A_d^{M-1} \\
C_dA_d^M & C_d^*A_d^M \\
\vdots & \vdots \\
C_dA_d^{2M-1} & C_d^*A_d^{2M-1}
\end{bmatrix},
\]

(38)

where \(C_d = C_d^*\). If the system \((A_e, C_e)\) is observable, the rank of \(O_e\) should equal the order of the extended system, that is, \(2M\). Note that \(O_e\) is a matrix having \(2MN\) rows and \(2M\) columns, where \(N\) is the number of outputs. We firstly prove that the row rank of \(O_e\) equals \(2M\).

We notice that the rank of first \(MN\) rows of \(O_e\) is \(M\). If \(A_d\) is nonsingular, the rank of the second \(MN\) rows of \(O_e\) is also \(M\). To prove the algebraic observability condition, we should find at least \(M\) rows from the second \(MN\) rows of \(O_e\) are linearly independent of the first \(MN\) rows of \(O_e\). Suppose the contrary: that at least one submatrix \([e.g. the (M + j)th, 0 \leq j < M]\) is linearly dependent of the first \(MN\) rows of \(O_e\). Then, we must have

\[
C_d \begin{bmatrix} A_d^{M+j} \\ A_d^*^{M+j} \end{bmatrix} = -a_1C_d \begin{bmatrix} A_d^{M-1} \\ A_d^*^{M-1} \end{bmatrix} - a_2C_d \begin{bmatrix} A_d^{M-2} \\ A_d^*^{M-2} \end{bmatrix} - \cdots - a_MC_d \begin{bmatrix} I \\ I \end{bmatrix},
\]

(39)

where \(a_1, \cdots, a_M\) are not all zero. By the Cayley-Hamilton theorem, we have

\[
C_dA_d^{M+j} = -b_1C_dA_d^{M-1} - b_2C_dA_d^{M-2} - \cdots - b_MC_dI,
\]

(40)

where \(b_1, \cdots, b_M\) are not all zero. We can conduct element-by-element conjugation to achieve

\[
C_dA_d^{M+j} = -b_1^*C_dA_d^{M-1} - b_2^*C_dA_d^{M-2} - \cdots - b_M^*C_dI,
\]

(41)

where \(C_d = C_d^*\). Comparing Eq. (39) to Eqs. (40)-(41), we will have \(a_1 = b_1 = b_1^*, \cdots, a_M = b_M = b_M^*\). In other words, \(a_1, \cdots, a_M\) and \(b_1, \cdots, b_M\) are real-valued.

However, if \(j = 0\), then \(b_1 = \sum ik_ju_0\), which is not real for almost all mean flow cases. This is a contradiction.

By repeated using of the Cayley-Hamilton theorem, any \(b_j\) can be expressed as the coefficients of the characteristic polynomial of \(A_d\), \(\det(\lambda I - A_d)\). For generic mean flow cases of interest, \(b_1\) for various \(j\) should not be real. Then, we are forced to conclude that \(M\) submatrices of the second \(MN\) rows of \(O_e\) are linearly independent of the first \(MN\) rows of \(O_e\). As a result, the row rank of \(O_e\) is \(2M\).

In a similar way, we can prove that the column rank of \(O_e\) is also \(2M\). This completes the proof. □

Remark 2: One might prefer to numerically evaluate observability of any system of interest using the algebraic observability theorem, once dynamic and output matrices \((A_e, C_e)\) are known. □

III. Numerical Tests Results

The attention of this work is focused on the theoretical development of a new testing method. A primitive numerical simulation was done to validate the proposed method. A noise source of a single spinning mode \(m = 3, n = 1\) is assumed propagating within a straight duct. The amplitude of sound source is 0.001. The sound is assumed tonal and the frequency is 400Hz. The analytical solutions of sound \((\rho', u', v', w')\) at each nondimensional time are calculated using Eqs. (15)–(18), where \(M_j\) is set to 0.4. The analytical solutions \((\rho', u', v', w')\) when the errors are stable (1000 steps later when \(t = 10\)) are shown in the first row of Fig. 2 here, which can be used to validate the proposed method.

In the simulation, it is assumed that a circular array of 30 pressure sensors is surface mounted at a downstream position, \(x = 0.05\). As a result, only the pressure on the outer duct wall can be measured. All acoustic velocities on the whole cross section (where \(r \leq 1\)) and sound pressure within the duct (where \(r < 1\)) are unknown. An observer in the form of Eqs. (4)–(5) is developed to reconstruct whole acoustic parameters from those pressure measurements
Figure 2. Analytical solutions (first row), observed results (second row) and error distribution (third row) of the modal sound parameters $p'$, $u'$, $v'$, $w'$ at the cross section $x = 0.05$ from the spinning mode sound source, where $t = 10$, $(m, n) = (3, 1)$, $k = 400$.

At $r = 1$. It is worthwhile to mention that Eq. (25) is adopted to generate an initial guess of $p'$ at $r < 1$ from the measurements of $p'$ at $r = 1$.

The observer based method is conducted as follows:

Step 1: Prepare the dynamics matrix $A$ and the sensor matrix $C$ according to Eqs. (23)–(24). It should be noticed that the discrete form with a sampling step $\delta t$ has to be considered for practical cases. The discrete counterpart of $A$ in continuous version is $e^{A\delta t}$.

Step 2: Solve the observer gain $L$ using the formula, $L = (A - P)^{-1}C$, which can be straightforwardly derived from Eq. (6). $P$ represents observer poles that are deliberately chosen to stabilize the observer. In addition, suitable values of $P$ can speed up the convergence of estimation errors. In this simulation, $L$ is empirically set to $0.5I$, where $I$ is the identity matrix with a rank equals the rank of $A$.

Step 3: Measurements of the sound pressure are generated by the circular sensor array installed on the outer wall. The measurements are simulated using Eq. (15) in this work. In addition, the initial guess of the sound pressure at $r < 1$ are approximated by Eq. (25). The resulted sound pressure solutions across the $r - \theta$ section constitute $y$ in Eq. (4).

Step 4: The observer is conducted to extract the approximation of sound parameters, such as $p'$, $u'$ etc, using Eqs. (4)–(5). It should be noticed that the acoustic system of interest is autonomous and thus the control matrix $B$ is null.

Step 5: Step 4 is instantaneously conducted for each samples. The observation error ($y - \hat{y}$) is examined. If the error converges in a unacceptable slow speed, or even worse, the error is divergent, a different observer gain $L$ should be considered.

We should emphasize that the key point in the above algorithm is the pole assignment of $A - LC$. A suitable pole should drop in the unit circle. Otherwise, the approximation of the resultant observer will blow up. When the amplitude of a complex valued pole is close to 1, the observer will converge quickly but be more sensitive to modeling and measurement errors. On the other hand, the convergence speed will slow but be more robust to external noise. Hence, the value of 0.5 was empirically chosen here. In addition, for most cases with large order systems, we have to conduct generalized inversion to achieve $C^{-1}$. The resultant $L$ after the calculation will differ from the desired values.
A robust and efficient pole assignment for a very large system is still an open problem.

The observer results $\tilde{x}$ including $\tilde{\rho}'$, $\tilde{u}'$, $\tilde{v}'$, $\tilde{w}'$ can be plotted on the mesh. Second row of Fig. 2 show the solutions at $t = 10$. Compared to the corresponding analytical solutions in the first row, $\tilde{u}'$ and $\tilde{w}'$ are already well reconstructed from array measurements ($\rho'$). The third row of Fig. 2 show the distribution of error between analytical solution and observed results.

![Figure 3. The convergence history of the observer error for the case.](image)

Figure 3 quantitatively shows the convergence history of the observer error for the case. The approximation error of sound density is defined as $\sum_{1\leq a\leq A, 1\leq b\leq B} |\tilde{\rho}_{a,b}' - \rho_{a,b}'|$, and so forth for other acoustical parameters. The observer errors shown in Fig. 3 are nondimensionalized by the amplitude, which is 0.01 for the case. Each sample step is 0.01s. It can be seen that the errors oscillate but converge in almost 100 steps. The errors of $\tilde{\rho}'$ and $\tilde{w}'$ converge more rapidly. The oscillations of errors could be caused by the imaginary parts in the dynamics matrix $A$.

In addition, it can be seen that stationary errors less than 5% remain. In this work, the observer gain $L$ was carefully chosen to set the eigenvalues ($P$) of $(A - LC)$ to 0.5. In a discrete time real valued system, those eigenvalues ensure the convergence of observer errors to zero as $t \to \infty$. As a result, it seems that those residues of errors violate the observer theory in classical control. The discrepancy can be explained by examining the dynamic matrix $A$ of the acoustic system. The elements of $A$ include imaginary parts and $(A - LC)$ may not be normal. Hence, it could be improper to infer the entire dynamics of $(A - LC)$ from its eigenvalues. Subtle mathematical manipulations could be needed, which calls further investigations.

The acoustic system is originally described by partial differential equations [Eqs. (7)–(10)] and reformulated in ordinary differential equations [Eqs. (23)–(24)] by second-order finite difference methods. A higher-order scheme may be more preferable for acoustic modeling, in terms of accuracy. [26] On the other hand, the hard wall boundary condition ($\tilde{v}' = 0$) is implicitly enforced in Eq. (23). Different boundary condition should be evaluated to improve the observation.

The above simulations are conducted for a tonal sound of a single spinning mode. Practical systems generally include various spinning modes at broadband frequencies, which can be decomposed to a variety of tonal single spinning modes by Fourier transform and mode detection. After that the proposed method can be applied to each component to reconstruct the related acoustic field. Although some issues are still to be addressed, the proposed method satisfactorily reconstructs most acoustic parameters only using measurements from a circular microphone array. In addition, the proposed observer could be performed in real-time in terms of its recursive nature.

### IV. Summary

An imaging method for acoustic spinning modes propagating within a duct has been introduced in this paper. The new method is developed in a theoretical way. The fundamental idea behind the method was originally developed in control theory for ordinary differential equations. Spinning mode propagation in circular duct, however, is formulated
by partial differential equations. For each single spinning mode at a tonal frequency, a finite difference technique is used to reduce the associated partial differential equation to a classical form of state space in control. A so-called observer can thereafter be constructed and applied straightforwardly. The complete acoustic solutions, including fluctuations of velocities and pressure, can be inferred by the observer from the pressure measurements of in-duct circular microphone array.

A numerical simulation of a single spinning mode at a tonal frequency for a straight circular duct has been conducted. The simulation generates pressure measurements of a circular array on the outer wall. The demonstration shows that the whole acoustic solutions on the test section can be estimated with the proposed method. A generally good agreement is achieved, compared with analytical solutions. The results suggest the potential and application of the proposed new method for acoustic imaging of spinning mode sound. For example, it is quite difficult, if not impossible, to measure the instantaneous fluctuations of acoustic velocities within a duct. In addition, the whole in-duct sound pressure distribution has to be measured using a rotating strut, where microphones are installed in the radial direction. Hence, the proposed method can extensively save those experimental efforts.

V. ACKNOWLEDGMENTS

This work is supported by National Science Foundation of China (Grants 11172007 and 11322222) and Aviation Industry Corporation of China, Commercial Aircraft Engine Co., Ltd.
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