Why the Inertial Matrix Takes the Current Form?

Supplementary Materials

Xun Huang

2011-2016

Angular Momentum of a Particle

$$d\mathbf{h} = \left(\mathbf{r} \times dm\mathbf{v}\right) = \left(\mathbf{r} \times \mathbf{v}_m\right)dm$$
$$= \left(\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right)dm$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- Cross Product

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right)i + \left(zv_x - xv_z\right)j + \left(xv_y - yv_x\right)k$$

Cross-Product: Equivalent Matrix

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)i + (zv_x - xv_z)j + (xv_y - yv_x)k$$

$$= \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} (yv_z - zv_y) \\ (zv_x - xv_z) \\ (xv_y - yv_x) \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v}$$

- **Cross-product-equivalent matrix**

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Angular Momentum of AC

$$\mathbf{h} = \int_{Body} (\mathbf{r} \times \mathbf{v}_o)\,dm + \int_{Body} (\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}))\,dm$$

$$= 0 - \int_{Body} (\mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega}))\,dm = -\int_{Body} (\mathbf{r} \times \mathbf{r})\,dm \times \boldsymbol{\omega}$$

$$\equiv -\int_{Body} (\tilde{\mathbf{r}}\tilde{\mathbf{r}})\,dm\boldsymbol{\omega}$$

Q: How to prove the highlighted derivations?

The Inertia Matrix

$$\mathbf{h} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,\boldsymbol{\omega}\,dm = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm\,\boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- where

$$\boldsymbol{I} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm$$

- **Inertia matrix** derives from **equal effect of angular rate** on all particles of the aircraft

Inertia matrix

$$\boldsymbol{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$