Chapter 9

Solutions to the Linear Equations

9.1 Scalar Equations

Consider the scalar first order, linear, ordinary differential equation with constant coefficients a and b:

$$\dot{x} = ax(t) + bu(t)$$

A LaPlace transformation of both sides of the equation yields

$$sx(s) - x(t_0) = ax(s) + bu(s)$$

This is then solved for $x(s)$, with $t_0 = 0$:

$$x(s) = (s-a)^{-1}x(0) + b(s-a)^{-1}u(s)$$

The two parts on the right correspond to the *initial condition response* and the *forced response.* If $u(t) = 0$, then we have the initial condition response

$$x(s) = \frac{1}{s-a}x(0)$$

The inverse transformation yields the initial condition response in the time domain,

$$x(t) = e^{at}x(0)$$

Clearly we require $a \le 0$ or the response will diverge. If $x(0) = 0$ then we have the forced response

$$x(s) = \frac{b}{s-a}u(s)$$

The inverse transformation of the right hand side obviously depends upon $u(s)$. For instance, if $u(t)$ is the unit step at $t = 0$ then $u(s) = 1/s$ and

$$x(s) = \frac{b}{s(s-a)}$$
$$x(t) = \frac{b}{a}\left(e^{at} - 1\right)$$

The forced response is sometimes written as

$$\frac{x(s)}{u(s)} = \frac{b}{s-a}$$

The quantity on the right hand side is identified as the *transfer function* of $u(s)$ to $x(s)$. One may also speak of $1/(s-a)$ as the transfer function from $x(0)$ to $x(s)$.

9.2 Matrix Equations

Now we perform the analogous operations with our vectors and matrices. For now we will drop the Δ's in the equation, and just consider

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

Recall that $\dot{\mathbf{x}}$ and $\mathbf{x}$ are n-dimensional vectors, $\mathbf{u}$ is an m-dimensional vector, A is an $n \times n$ matrix, and B is an $n \times m$ matrix. We consider A

and B to be constant matrices. The LaPlace transform of a vector is the element-by-element transform of its components, so

$$\begin{gathered} s\mathbf{x}(s) - \mathbf{x}(0) = sI\mathbf{x}(s) - \mathbf{x}(0) = A\mathbf{x}(s) + B\mathbf{u}(s) \\ [sI - A]\,\mathbf{x}(s) = \mathbf{x}(0) + B\mathbf{u}(s) \end{gathered}$$

$$\mathbf{x}(s) = [sI - A]^{-1}\,\mathbf{x}(0) + [sI - A]^{-1}\,B\mathbf{u}(s) \tag{9.1}$$

The time-domain response is just the element-by-element inverse LaPlace transform of $\mathbf{x}(s)$, or $\mathbf{x}(t) = \mathcal{L}^{-1}\mathbf{x}(s)$.

Now consider the state transition matrix $[sI - A]^{-1}$. It may be represented as

$$[sI - A]^{-1} = \frac{C(s)}{d(s)}$$

The $n \times n$ matrix $C(s)$ is the *adjoint* of $[sI - A]$, and each entry will be a polynomial of order $n - 1$ or less. The polynomial $d(s)$ is the determinant of $[sI - A]$ and is of order n. Thus each entry in $[sI - A]^{-1}$ is a ratio of polynomials in the complex variable s. A ratio of polynomials in which the order of the numerator is less than or equal to the order of the denominator is called *proper*, and if the numerator's order is strictly less the ratio of polynomials is called *strictly proper*. It is easy to show that each of the ratios of polynomials in $[sI - A]^{-1}$ is strictly proper (because each of the numerator polynomials is the determinant of a matrix of order $n - 1$ with first-order polynomials in s along its diagonal). Thus we may write

$$[sI - A]^{-1} = \left\{ \frac{c_{ij}(s)}{d(s)} \right\}, \quad i,\ j = 1 \ldots n$$

The polynomial $d(s)$ is referred to as the *characteristic polynomial*, and $d(s) = 0$ is the *characteristic equation* of the system. The denominator polynomial $d(s)$ may be factored into a product of first order polynomials in s of the form $(s - \lambda_i)$ in which $\lambda_i,\ i = 1 \ldots n$, are the roots of $d(s) = 0$, or the *eigenvalues* of A.

$$d(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$

If the eigenvalues are distinct then the ratio of polynomials may be further expressed as a sum of ratios in which each denominator is one of the first order factors of the denominator polynomial. That is, for an arbitrary numerator polynomial $n(s)$ of order less than n,

$$\frac{n(s)}{d(s)} = \frac{n(s)}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)}$$
$$= \frac{n_1}{(s - \lambda_1)} + \frac{n_2}{(s - \lambda_2)} + \cdots + \frac{n_n}{(s - \lambda_n)}$$

This is the familiar partial-fraction expansion method of solving inverse LaPlace transform problems. The inverse transform of such an expression is the sum of terms involving exponentials $n_i e \lambda_i t, \ i = 1 \ldots n$:

$$\mathcal{L}^{-1}\left\{\frac{n(s)}{d(s)}\right\} = n_1 e^{\lambda_1 t} + n_2 e^{\lambda_2 t} + \cdots + n_n e^{\lambda_n t}$$

This result applies to each entry of $[sI - A]^{-1}$.

9.3 Initial Condition Response

9.3.1 Modal Analysis

System Modes

For the unforced, or initial-condition response ($\mathbf{u}(t) = \mathbf{u}(s) = 0$) we have

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(s) = [sI - A]^{-1}\,\mathbf{x}(0)$$

Since $\mathbf{x}(0)$ is just a vector of constants, each element of $\mathbf{x}(s)$ will be a linear combination of terms like $n(s)/d(s)$, e.g.,

$$x_i(s) = \frac{v_{i,1}}{s-\lambda_1} + \frac{v_{i,2}}{s-\lambda_2} + \cdots + \frac{v_{i,n}}{s-\lambda_n}$$

Each element of $x_i(t)$ will therefore be a sum of terms like $v_{i,j}e^{\lambda_j t}$, or

$$x_i(t) = v_{i,1}e^{\lambda_1 t} + v_{i,2}e^{\lambda_2 t} + \cdots + v_{i,n}e^{\lambda_n t}$$

The entire vector $\mathbf{x}(t)$ may be represented as

$$\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t} + \cdots + \mathbf{v}_n e^{\lambda_n t} \tag{9.2}$$

Each of the components $\mathbf{v}_i e^{\lambda_i t}$ is one of the *modes* of the system response. In practice the eigenvalues will be a mixture of real and complex numbers. The complex roots will occur in complex conjugate pairs (because A has real components), say $\mathbf{v}_i e^{\lambda_i t} + \mathbf{v}_i^* e^{\lambda_i^* t}$ (the asterisk denotes the complex conjugate). The result is, of course, a real oscillatory response. In such cases the combined pair $\mathbf{v}_i e^{\lambda_i t} + \mathbf{v}_i^* e^{\lambda_i^* t}$ is considered a single mode. It is clearly necessary that each of the eigenvalues have negative real parts for all of the modes, and hence the system, to be stable.

Now evaluate $\dot{\mathbf{x}} - A\mathbf{x} = 0$ using equation 9.2 for $\mathbf{x}$,

$$\begin{aligned}
\dot{\mathbf{x}} &= \lambda_1 \mathbf{v}_1 e^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + \lambda_n \mathbf{v}_n e^{\lambda_n t} \\
A\mathbf{x} &= A\mathbf{v}_1 e^{\lambda_1 t} + A\mathbf{v}_2 e^{\lambda_2 t} + \cdots + A\mathbf{v}_n e^{\lambda_n t}
\end{aligned}$$

Hence,

$$\dot{\mathbf{x}} - A\mathbf{x} = (\lambda_1 \mathbf{v}_1 - A\mathbf{v}_1)\, e^{\lambda_1 t} + \cdots + (\lambda_n \mathbf{v}_n - A\mathbf{v}_n)\, e^{\lambda_n t} = 0 \tag{9.3}$$

Since $e^{\lambda_i t}$ is never zero, equation 9.3 requires that each of the terms in parentheses vanish independently, or

$$(\lambda_i \mathbf{v}_i - A\mathbf{v}_i) = (\lambda_i I_i - A)\, \mathbf{v}_i = 0, \quad i = 1 \ldots n$$

This means that the vectors $\mathbf{v}_i$ are the *eigenvectors* of A, each associated with an eigenvalue λ_i. Since the non-zero multiple (including multiplication by complex numbers) of an eigenvector is also an eigenvector, we may write the initial condition response as

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + \alpha_n e^{\lambda_n t}\mathbf{v}_n \tag{9.4}$$

In equation 9.4, for a given set of eigenvectors $\mathbf{v}_i$, the mulipliers α_i are chosen to satisfy the initial condition

$$\mathbf{x}(0) = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$$

If we denote each of the scalar terms $q_i(t) \equiv \alpha_i e^{\lambda_i t}$, then

$$\dot{q}_i(t) = \lambda_i \alpha_i e^{\lambda_i t} = \lambda_i q_i(t)$$

We then recognize $\alpha_i e^{\lambda_i t}$ as the initial condition response of the differential equation $\dot{q}_i(t) = \lambda_i q_i(t)$ with $\alpha_i = q_i(0)$, or $q_i(t) = q_i(0)e^{\lambda_i t}$. Now define the vector $\mathbf{q}(t) \equiv \{q_i(t)\}$, $i = 1 \ldots n$, and we may write the initial condition response

$$\mathbf{x}(t) = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n]\,\mathbf{q}(t) \equiv M\mathbf{q}(t) \tag{9.5}$$

The Modal Matrix

Here we have defined the matrix M as consisting of columns which are the eigenvectors of A. M is frequently called the *modal matrix*. Note that M is in general not a direction cosine matrix, and $M^{-1} \neq M^T$. However, since we have assumed distinct eigenvalues, the eigenvectors are linearly independent and $|M| \neq 0$. Thus M^{-1} exists and $\mathbf{q}(t) = M^{-1}\mathbf{x}(t)$, so

$$\begin{aligned}\dot{\mathbf{q}}(t) &= M^{-1}\dot{\mathbf{x}}(t) = M^{-1}A\mathbf{x}(t) \\ &= M^{-1}AMM^{-1}\mathbf{x}(t) = \left(M^{-1}AM\right)M^{-1}\mathbf{x}(t)\end{aligned}$$

Define $\Lambda \equiv M^{-1}AM$, so that

$$\dot{\mathbf{q}}(t) = \Lambda\mathbf{q}(t) \tag{9.6}$$

Since $\dot{q}_i(t) = \lambda_i q_i(t)$, then

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = diag\{\lambda_i\}$$

The solution of $\dot{\mathbf{q}} = \Lambda \mathbf{q}(t)$ is therefore

$$\mathbf{q}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \mathbf{q}(0)$$

Define

$$e^{\Lambda t} \equiv \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Whence

$$\mathbf{q}(t) = e^{\Lambda t}\mathbf{q}(0) \tag{9.7}$$

Thus, for a system with distinct eigenvalues, we may solve the initial condition response of $\dot{\mathbf{x}} = A\mathbf{x}$ with given initial conditions $\mathbf{x}(0)$ as follows:

1. Determine the eigenvalues and eigenvectors of A.
2. For the given initial conditions $\mathbf{x}(0)$, determine $\mathbf{q}(0) = M^{-1}\mathbf{x}(0)$.
3. Write down the solution $\mathbf{q}(t)$ using equation 9.7.
4. Evaluate the solution $\mathbf{x}(t) = M\mathbf{q}(t)$.

In short, then,

$$\mathbf{x}(t) = Me^{\Lambda t}M^{-1}\mathbf{x}(0) \tag{9.8}$$

Argand Diagrams

With equation 9.8 we may calculate the time-history of each of the states in response to given initial conditions. Each real eigenvalue will contribute a mode of the form $\alpha_i e^{\lambda_i t}\mathbf{v}_i$, where each α_i is a constant (possibly zero) determined by the initial conditions and $\mathbf{v}_i$ is a constant eigenvector associated with λ_i. The time history of each state in the response of a given mode is a constant component of the vector $\alpha_i \mathbf{v}_i$ multiplied by the exponential term $e^{\lambda_i t}$. At any time, therefore, the magnitudes of the various states will be in the same ratio (as given by $\mathbf{v}_i$) as at any other time.

That is, if the initial-condition response of the i^{th} mode is $\mathbf{x}(t) = \mathbf{v}_i e^{\lambda_i t}$, then in terms of the individual states,

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} = \begin{Bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{Bmatrix} e^{\lambda_i t} = \begin{Bmatrix} v_{i1}e^{\lambda_i t} \\ v_{i2}e^{\lambda_i t} \\ \vdots \\ v_{in}e^{\lambda_i t} \end{Bmatrix}$$

Then the ratio of the j^{th} state to the k^{th} at any time t is

$$\frac{x_j(t)}{x_k(t)} = \frac{v_{ij}e^{\lambda_i t}}{v_{ik}e^{\lambda_i t}} = \frac{v_{ij}}{v_{ik}}$$

Note that this does not mean that the ratio of two states is in general constant, since more than one mode may be involved. For example, if the two modes λ_1 and λ_2 are both excited, then $\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}$, and

$$\frac{x_j(t)}{x_k(t)} = \frac{v_{1j}e^{\lambda_1 t} + v_{2j}e^{\lambda_2 t}}{v_{1k}e^{\lambda_1 t} + v_{2k}e^{\lambda_2 t}} \neq \frac{v_{1j} + v_{2j}}{v_{1k} + v_{2k}}$$

If complex roots occur then an oscillatory response will result. A complex conjugate pair of roots, although representing two distinct eigenvalues, create a single mode when the roots are combined to get a real response.

Say a complex eigenvalue λ_i occurs. Then, if the entries in the A matrix are real, the conjugate eigenvalue λ_i^* will also occur.

$$\lambda_i = \sigma + j\omega, \qquad \lambda_i^* = \sigma - j\omega$$

Moreover, if the eigenvector associated with λ_i is $\mathbf{v}_i$ then the eigenvector associated with λ_i^* will be $\mathbf{v}_i^*$. Any multiple α_i of $\mathbf{v}_i$ must be accompanied by a multiple α_i^* of $\mathbf{v}_i^*$. Considering just the single mode corresponding to λ_i and λ_i^*,

$$\mathbf{x}(t) = e^{\sigma t}\left(\alpha_i e^{j\omega t}\mathbf{v}_i + \alpha_i^* e^{-j\omega t}\mathbf{v}_i^*\right)$$

In general the multiplier α_i and each of the elements of $\mathbf{v}_i$ may be a complex number. When multiplying complex numbers the polar form is preferred,

$$a + jb = Me^{j\phi}$$
$$M = \sqrt{a^2+b^2}, \qquad \phi = \tan^{-1}\frac{b}{a}$$

Represent α_i and the k^{th} component of eigenvector $\mathbf{v}_i$ in polar form as

$$\alpha_i = M_\alpha e^{j\phi_\alpha}, \qquad v_{ik} = M_{ik}e^{j\phi_{ik}}$$

$$\alpha_i\mathbf{v}_i = M_\alpha e^{j\phi_\alpha}\begin{Bmatrix} M_{i1}e^{j\phi_{i1}} \\ \vdots \\ M_{in}e^{j\phi_{in}} \end{Bmatrix}$$

The contribution of the eigenvalue leads to

$$\alpha_i e^{j\omega t}\mathbf{v}_i = M_\alpha \begin{Bmatrix} M_{i1}e^{j(\omega t+\phi_\alpha+\phi_{i1})} \\ \vdots \\ M_{in}e^{j(\omega t+\phi_\alpha+\phi_{in})} \end{Bmatrix}$$

The conjugate part of the response is

$$\alpha_i^* e^{-j\omega t}\mathbf{v}_i^* = M_\alpha \begin{Bmatrix} M_{i1}e^{-j(\omega t+\phi_\alpha+\phi_{i1})} \\ \vdots \\ M_{in}e^{-j(\omega t+\phi_\alpha+\phi_{in})} \end{Bmatrix}$$

The total response for this mode may then be written as

$$\mathbf{x}(t) = e^{\sigma t} M_\alpha \left\{ \begin{matrix} M_{i1}\left(e^{j(\omega t+\phi_\alpha+\phi_{i1})} + e^{-j(\omega t+\phi_\alpha+\phi_{i1})}\right) \\ \vdots \\ M_{in}\left(e^{j(\omega t+\phi_\alpha+\phi_{in})} + e^{-j(\omega t+\phi_\alpha+\phi_{in})}\right) \end{matrix} \right\}$$

$$= 2e^{\sigma t} M_\alpha \left\{ \begin{matrix} M_{i1}\cos\left(\omega t+\phi_\alpha+\phi_{i1}\right) \\ \vdots \\ M_{in}\cos\left(\omega t+\phi_\alpha+\phi_{in}\right) \end{matrix} \right\}$$

Then the k^{th} component of $\mathbf{x}(t)$, $x_k(t)$, is

$$x_k(t) = 2e^{\sigma t} M_\alpha M_{ik} \cos\left(\omega t + \phi_\alpha + \phi_{ik}\right)$$

The magnitude of $e^{\sigma t}$ and M_α is the same for each state, as are the angles ωt and ϕ_α. Thus the relative difference between the response of two different states is contained in the magnitude M_{ik} and the phase ϕ_k, which are determined by the corresponding entries in the eigenvector. That is, if all we care about is the relationship of one state to another during an initial condition response of an oscillatory mode, all that information is contained in the eigenvector.

An *Argand diagram* is the plot in the complex plane of complex entries in an eigenvector for several different states. It is conventional to pick the eigenvector associated with the eigenvalue with the positive imaginary part, $\sigma + j\omega$. Figure 9.1 shows two such entries, corresponding to the two states x_k and x_{k+1}.

In figure 9.1 the phase difference between the two states is the angle between v_k and v_{k+1}. The relationship between the real magnitudes of the two states is visualized as the projections of v_k and v_{k+1} onto the real axis.

The evolution of time in the response of the states in an Argand diagram may be visualized as a counter-clockwise rotation of the vectors about the origin through an angle ωt. Note that the counterclockise rotation occurs because we have picked the eigenvector corresponding to $\sigma + j\omega$. The eigenvector corresponding to $\sigma - j\omega$ would have to be viewed as a clockwise rotation. Thus at some later time t, the situation would be as shown in figure 9.2.

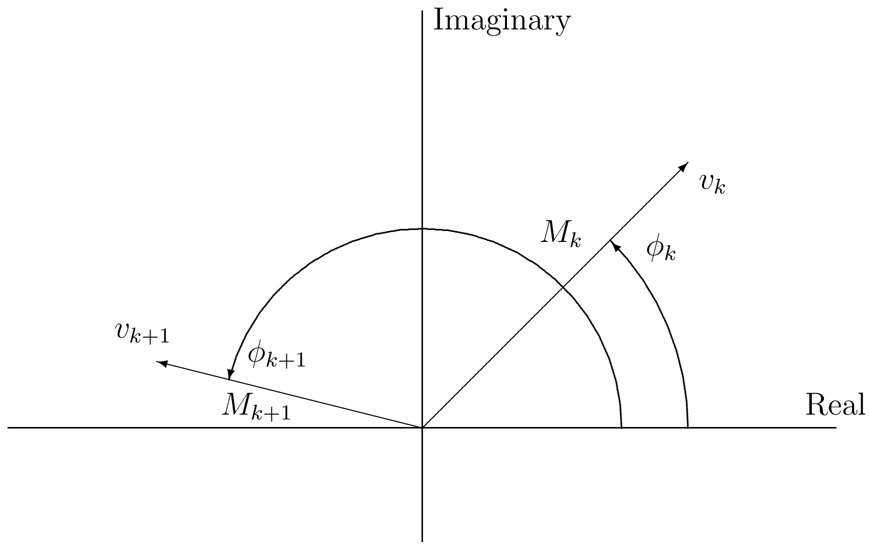

Figure 9.1: Argand Diagram

9.4 Mode Sensitivity and Approximations

9.4.1 Mode Sensitivity

In the long history of flight dynamics there have been ongoing efforts to reduce the larger problem to several of smaller order. The motivation for this effort probably began with the difficulties of analyzing large systems using only a slide rule, pencil, and paper. One major benefit to order reduction is that the smaller problems may be analyzed literally (as opposed to numerically). In many cases this analysis will show that a particular response mode is strongly dependent on a small subset of the parameters used to formulate the equations of motion. These parameters in turn are determined by the physical design of the aircraft. Thus, links between design choices and the dynamic response of an airplane may be established, enabling designers to more directly determine how well their aircraft will fly.

We have already seen some reductions in the order of the problem. Beginning with twelve rigid body equations of motion, it was observed that the geographical coordinates x_E and y_E, as well as the heading angle ψ, do not affect the dynamic response of the aircraft, and these states were ignored. The assumption of constant altitude was not so easily justifiable, but

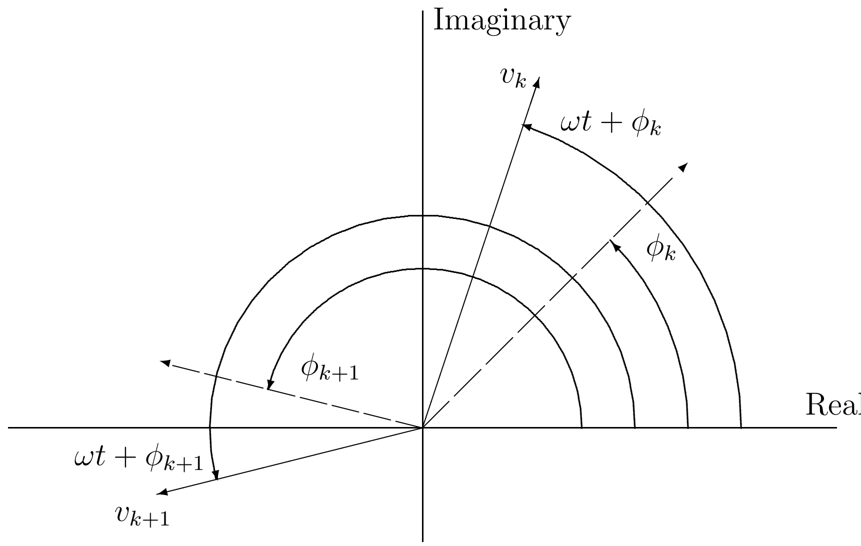

Figure 9.2: Response at time t

experience bears out the validity of neglecting that state. More subtly various assumptions have been made that cause the linearized lateral-directional and longitudinal equations to be uncoupled from each other. Thus, instead of a twelfth-order system of equations, we have two separate fourth-order systems.

For each fourth-order system of linear equations there will be four eigenvalues. We will soon show that the longitudinal eigenvalues typically consist of two complex-conjugate pairs, and the lateral-directional equations consist of one complex-conjugate pair and two real roots. Thus there are two longitudinal modes and three lateral-directional modes. The next level of order reduction seeks to find dynamic systems with the same number of states as the order of the mode, i.e., one state for a real eigenvalue and two for an oscillatory mode.

Eigenvector analysis.

One way to approach the order-reduction problem is to ask if there is a set of initial conditions that will cause only one of the modes to appear in the response. If there is, and if in this set of initial conditions some states are "small", then maybe they can be neglected without affecting the response.

The answer to the question is fairly easy. Since the eigenvectors $\mathbf{v}_i$ have been assumed to be linearly independent, then if $\mathbf{x}(0) = \alpha_i \mathbf{v}_i$, some non-zero multiple of $\mathbf{v}_i$, then all the other $\alpha_j = 0$, $j = 1 \dots n$, $j \neq i$. This can also be shown using $\mathbf{q}(0) = \alpha M^{-1}\mathbf{x}(0) = \alpha M^{-1}\mathbf{v}_i$. Since $M = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n]$, and $M^{-1}M = I$, $M^{-1}\mathbf{v}_i$ is just the i^{th} column of the identity matrix. Then $\mathbf{q}(t) = e^{\Lambda t}\mathbf{q}(0)$ is

$$\mathbf{q}(t) = e^{\Lambda t}\mathbf{q}(0) = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & e^{\lambda_i t} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{Bmatrix} 0 \\ \vdots \\ \alpha \\ \vdots \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ \alpha e^{\lambda_i t} \\ \vdots \\ 0 \end{Bmatrix}$$

From this we see that only the mode corresponding to $e^{\lambda_i t}$ will be present in the response, $\mathbf{x}(t) = \alpha_i e^{\lambda_i t}\mathbf{v}_i$. That is, *If the initial conditions of the states are aligned with one of the eigenvectors, only the mode of the eigenvalue corresponding to that eigenvector will be present in the initial-condition response.* If the mode arises from complex roots, then we may take $\mathbf{x}(0) = \alpha_i \mathbf{v}_i + \alpha_i^* \mathbf{v}_i^*$, which is a vector of real numbers.

Thus if certain of the entries in $\mathbf{v}_i$ are "small" relative to all the others, then the corresponding states are not influential in the initial conditions that excite only the i^{th} mode. In the extreme case, if an entry is zero then the corresponding state will not respond at all in that mode. In cases of complex conjugate pairs we use the analysis that led to the Argand diagram to determine relative size.

The problem with applying this information is in determining what we mean by a relatively "small" entry. If all the states have the same units this makes some sense. In flight dynamics, however, the states are linear velocities, angular velocities, linear measurements, and angles. So, for example, we would need to decide if a change in velocity of 20 ft/s is large or small compared to a change in pitch rate of 0.2 rad/s.

Sensitivity analysis.

In this analysis the question of different units does not arise, since only one state will be examined at a time. The question to be answered is whether a given mode is more sensitive to changes in some states than others. Put another way, we seek to determine whether a change in the initial condition of a single state will more strongly influence some modes than others. Thus, if a change in the initial condition of a single state, while all others are unchanged, causes one mode to respond more "energetically" than others, then this state may be thought of as "dominant" in the response of the mode. It is convenient to think of this sensitivity analysis as injecting energy into the system through a single state, and looking to see how this energy is then distributed to the various modes.

For our change in initial condition of the j^{th} state we use the notation

$$\Delta \mathbf{x}_j(0) = \{x_i\}\,,\ x_i = \begin{cases} 1,\ i = j \\ 0,\ i \neq j \end{cases} \tag{9.9}$$

This notation means there is a "1" in the j^{th} position, and zeros elsewhere. From equation 9.5 we can easily determine the change $\Delta \mathbf{q}_j(0)$ in $\mathbf{q}(0)$,

$$\Delta \mathbf{q}_j(0) = M^{-1} \Delta \mathbf{x}_j(0) \tag{9.10}$$

The vector $\Delta \mathbf{q}_j(0)$ has components $\{\Delta q_{j_i}\}$, $i = 1 \ldots n$. The change in the initial condition is seen in each mode according to

$$\Delta \mathbf{x}_j(0) = \Delta q_{j_1}(0)\mathbf{v}_1 + \Delta q_{j_2}(0)\mathbf{v}_2 + \cdots + \Delta q_{j_n}(0)\mathbf{v}_n \tag{9.11}$$

Equation 9.11 has all the information we need for our sensitivity analysis. The actual time-varying response is just like equation 9.11 except for the time-varying parts,

$$\Delta \mathbf{x}_j(t) = \Delta q_{j_1}(0)e^{\lambda_1 t}\mathbf{v}_1 + \Delta q_{j_2}(0)e^{\lambda_2 t}\mathbf{v}_2 + \cdots + \Delta q_{j_n}(0)e^{\lambda_n t}\mathbf{v}_n$$

In equation 9.11 we examine the relative values of the j^{th} component of each of the vectors $\Delta q_i(0)\mathbf{v}_i$. For example, if we varied the initial condition of the first state ($j = 1$), then we look at the first component of each vector on the right-hand side to see how that change was distributed among the various modes.

The process of performing the sensitivity analysis is easier than it looks. Note first in equation 9.9 that it would not have mattered whether we put zeros in the non-j^{th} positions, since in the end we just looked at the j^{th} component of the result. This means we can analyze all of the states simultaneously. Now note that equation 9.10 has the effect of assigning to $\Delta\mathbf{q}_j(0)$ the j^{th} column of the matrix M^{-1}. In analyzing all the states simultaneously we form the matrix M^{-1} and interpret its columns as the vectors $\Delta\mathbf{q}_j(0)$:

$$M^{-1} = \begin{bmatrix} \Delta\mathbf{q}_1(0) & \Delta\mathbf{q}_2(0) & \cdots & \Delta\mathbf{q}_n(0) \end{bmatrix} \tag{9.12}$$

This method of modal sensitivity analysis can be summarized as follows:

1. Calculate the eigenvalues and eigenvectors of the system, form the modal matrix M, and calculate M^{-1}.

2. Denote the rows of M as $r_1,\ r_2,\ \ldots$ and the columns of M^{-1} as $c_1,\ c_2,\ \ldots$

$$M = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

Form diagonal matrices $C_1,\ C_2,\ \ldots$ with the entries from $c_1,\ c_2,\ \ldots$ forming the diagonal.

$$c_i = \begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{bmatrix}, \quad C_i \equiv \begin{bmatrix} c_{1i} & 0 & 0 & \cdots & 0 \\ 0 & c_{2i} & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & c_{ni} \end{bmatrix}$$

Now evaluate the $n \times n$ sensitivity matrix S, defined as

$$S = \begin{bmatrix} r_1 C_1 \\ r_2 C_2 \\ \vdots \\ r_n C_n \end{bmatrix} \tag{9.13}$$

3. Retain just the magnitudes of the entries in S (absolute values of real numbers, magnitudes of complex numbers). Normalize each row by summing the entries, then dividing each entry by that sum.

4. Each row of S corresponds to a state, and each column corresponds to a mode. The modes are enumerated in the same order as the eigenvalues that determined the eigenvectors used to form M. Examine each column of S to assess the relative magnitudes of the numbers in that column. An entry of zero means the state in question does not respond at all in the mode, and it may be ignored in analyzing that mode. At the other extreme, if all the numbers in a column are the same, or nearly so, no states can be ignored in the analysis that mode. Between these extremes decisions must be made. Experience has shown that states whose entry is no larger than 10% of the largest entry may be safely ignored.

9.4.2 Approximations

Once a state has been declared ignorable, it and its effects may be removed from the equations of motion to get the approximation. Mathematically we assume that the ignorable variable becomes a constant. If the state x_i is ignorable, then we take $\dot{x}_i = 0$. Now, the variable itself still appears in the remaining equations of motion, to be multiplied by the appropriate entries in those equations. The only remaining question, then, is what constant value should be assigned to the ignorable variable for use in the other equations.

There has long been a notion of "slow" and "fast" variables that is often applied to this sort of analysis. The idea is that a slow ignorable variable does not change as rapidly as the variables of interest, and its initial value should be used as its constant value. Since we are dealing with perturbations in the variable, that constant value is zero. On the other hand, fast ignorable variables are thought of as having finished all their dynamics before the problem of interest has a chance to get started. In this case, the constant value is the "steady-state" response of the ignorable variable.

The problem with slow and fast variables is that it is hard to find a rigorous definition of what these adjectives mean, or a methodology to decide whether a given variable is one or the other. Our approach will be to rank the modes from fastest to slowest according to the associated eigenvalue's real part. The larger (in magnitude) negative the real part the more quickly $e^{\sigma t}$ tends to zero. For a given mode's approximation, every state associated with faster modes will be considered a fast variable, and every state associated with slower modes will be a slow variable.

The method of constructing the approximation to a given mode therefore is:

1. Perform a sensitivity analysis and find the ignorable states for the mode in question.

2. For each ignorable state, decide if it is fast or slow according to the mode in which the state is not ignorable.

3. If an ignorable state is slow, set it to zero in the approximating equations.

4. If an ignorable state is fast, determine its "steady-state" value in its mode, and algebraically evaluate its contribution to the approximating equations.

5. Remove the rows of the A and B matrices associated with the ignorable variable.

6. Remove the columns of A associated with the ignorable variable.

7. The remaining non-trivial equations are the approximation to the mode in question.

To illustrate this process, take a generic third-order system

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Assume that x_1 is dominant in mode 1, x_2 is dominant in mode 2, and x_3 is dominant in mode 3. Further assume that mode 1 is slowest and mode 3 is fastest. We are to approximate mode 2. Relative to mode 2, x_3 is fast and x_1 is slow. First we set $\dot{x}_1 = \dot{x}_3 = 0$. x_1 is slow so set $x_1 = 0$. Next we need to solve for $x_{3_{ss}}$. The equations at this point are

$$\begin{Bmatrix} 0 \\ \dot{x}_2 \\ 0 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ x_2 \\ x_{3_{ss}} \end{Bmatrix}$$

We evaluate $x_{3_{ss}}$ in its mode by solving the third equation for $x_{3_{ss}}$ as a function of x_2,

$$\dot{x}_3 = 0 = a_{32}x_2 + a_{33}x_{3_{ss}} \Rightarrow x_{3_{ss}} = -\frac{a_{32}}{a_{33}}x_2$$

The information is applied to the mode 2 equation to evaluate its approximation:

$$\dot{x}_2 = a_{22}x_2 + a_{23}x_{3_{ss}} = a_{22}x_2 - a_{23}\frac{a_{32}}{a_{33}}x_2 = \left(a_{22} - a_{23}\frac{a_{32}}{a_{33}}\right)x_2$$

The approximation to the second mode is therefore

$$\dot{x}_2 = \left(\frac{a_{22}a_{33} - a_{23}a_{32}}{a_{33}}\right)x_2$$

9.5 Forced Response

9.5.1 Transfer Functions

For the forced response we take $\mathbf{x}(0) = 0$, and

$$\mathbf{x}(s) = [sI - A]^{-1} B\mathbf{u}(s)$$

Calculating the inverse transformation of the right-hand side obviously depends on $\mathbf{u}(t)$ which determines $\mathbf{u}(s)$. At this point $\mathbf{u}(t)$ represents the time-history of the pilot's control inputs, which assuredly are not easy to place in analytical form. We therefore resort to considering simple analytical forms of $\mathbf{u}(t)$: impulses, steps, ramps, and sinusoids.

The matrix $[sI - A]^{-1} B$ consists of ratios of polynomials, each with the characteristic polynomial $d(s) \equiv |sI - A| = d_0 + d_1 s + \cdots + d_n s^n$ as its denominator. Each entry is a transfer function that relates the j^{th} input to the i^{th} state. This matrix of transfer functions is often denoted as

$$[sI - A]^{-1} B \equiv G(s) = \{g_{ij}(s)\}$$

$$\mathbf{x}(s) = G(s)\mathbf{u}(s) \tag{9.14}$$

The relationship is $x_i(s) = \sum_j g_{ij}(s) u_j(s)$. Each of the $g_{ij}(s)$ may be represented as

$$g_{ij}(s) = \frac{n_{ij}(s)}{d(s)}$$

In terms of the factors of the numerator and denominator polynomials,

$$g_{ij}(s) = \frac{n_{ij}(s)}{d(s)} = \frac{k_{ij}(s - z_1)(s - z_2)\cdots(s - z_{n_z})}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

The numerator roots are called *zeros*; n_z is the number of zeros in a particular transfer function, and is generally different for each of the $g_{ij}(s)$. The denominator roots are, of course, the same as the eigenvalues, but in the analysis of transfer functions they are commonly called *poles*. If a pole and a zero are identical they may be cancelled; in that case the remaining poles are not the same as the eigenvalues.

For some given $\mathbf{u}(t)$ for which $\mathbf{u}(s)$ is known, the forced reponse may be calculated from

$$\mathbf{x}(t) = \mathcal{L}^{-1}[G(s)\mathbf{u}(s)] \tag{9.15}$$

9.5.2 Steady-State Response

The steady-state reponse (if it exists) of a system to given control inputs whose LaPlace transforms are known is given by the Final Value Theorem. For a given $x_i(s)$, $u_j(s)$, and $g_{ij}(s)$,

$$\lim_{t\to\infty} x_i(t) = \lim_{s\to 0}\left[s x_i(s)\right] = \lim_{s\to 0}\left[s g_{ij}(s) u_j(s)\right]$$

Given the LaPlace transforms of each of the inputs, $u_j(s)$,

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{s\to 0}\left[s \mathbf{x}(s)\right] = \lim_{s\to 0}\left[s G(s) \mathbf{u}(s)\right]$$

9.6 Example: Longitudinal Dynamics

9.6.1 System Matrices

See Appendix C for data. At a particular flight condition the A-4 *Skyhawk* has the following linearized, dimensional longitudinal system and control matrices (the Δ's have been dropped):

$$\dot{\mathbf{x}}_{Long} = A_{Long}\mathbf{x}_{Long} + B_{Long}\mathbf{u}_{Long}$$

$$\mathbf{x}_{Long}^T = \{u,\ \alpha,\ q,\ \theta\}$$
$$\mathbf{u}_{Long}^T = \{\delta_T,\ \delta_m\}$$

$$A_{Long} = \begin{bmatrix} -1.52\times 10^{-2} & -2.26 & 0 & -32.2 \\ -3.16\times 10^{-4} & -0.877 & 0.998 & 0 \\ 1.08\times 10^{-4} & -9.47 & -1.46 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_{Long} = \begin{bmatrix} 20.5 & 0 \\ 0 & -1.66\times 10^{-4} \\ 0 & -12.8 \\ 0 & 0 \end{bmatrix}$$

9.6.2 State Transition Matrix and eigenvalues

Using Fedeeva's algorithm (Appendix D) we calculate the state transition matrix $[sI - A_{Long}]^{-1}$, which results in

$$[sI - A_{Long}]^{-1} = \frac{C(s)}{d(s)} = \frac{\{c_{ij}(s)\}}{d(s)},\ i = 1\ldots 4,\ j = 1\ldots 4$$

The terms $c_{ij}(s)$ in the numerator are

$$\begin{aligned}
c_{11}(s) &= s^3 + 2.34s^2 + 10.7s \\
c_{12}(s) &= -2.26s^2 - 3.29s + 305 \\
c_{13}(s) &= -34.4s - 28.2 \\
c_{14}(s) &= -32.2s^2 - 75.2s - 345 \\
c_{21}(s) &= -3.16 \times 10^{-4}s^2 - 3.53 \times 10^{-4}s \\
c_{22}(s) &= s^3 + 1.47s^2 + 2.21 \times 10^{-2}s + 3.48 \times 10^{-3} \\
c_{23}(s) &= 0.998s^2 + 1.51 \times 10^{-2}s + 1.02 \times 10^{-2} \\
c_{24}(s) &= 1.02 \times 10^{-2}s + 1.14 \times 10^{-2} \\
c_{31}(s) &= 1.08 \times 10^{-4}s^2 + 3.09 \times 10^{=3}s \\
c_{32}(s) &= -9.47s^2 - 0.144s \\
c_{33}(s) &= s^3 + 0.892s^2 + 1.26 \times 10^{-2}s \\
c_{34}(s) &= -3.48 \times 10^{-3}s - 9.93 \times 10^{-2} \\
c_{41}(s) &= 1.08 \times 10^{-4}s + 3.09 \times 10^{-3} \\
c_{42}(s) &= -9.47s - 0.144 \\
c_{43}(s) &= s^2 + 0.892s + 1.26 \times 10^{-2} \\
c_{44}(s) &= s^3 + 2.35s^2 + 10.8s + 0.162
\end{aligned}$$

The characteristic polynomial is

$$d(s) = s^4 + 2.35s^3 + 10.76s^2 + 0.1652s + 0.0993$$

In factored form,

$$\begin{aligned}
d(s) = (s + 1.17 - j3.06)\,(s + 1.17 + j3.06) \\
(s + 0.0067 - j0.096)\,(s + 0.0067 + j0.096)
\end{aligned}$$

Thus the eigenvalues of the system are

$$\lambda_{1,2} = -1.17 \pm j3.06$$
$$\lambda_{3,4} = -0.0067 \pm j0.096$$

The eigenvalues are in the form $\lambda = \sigma \pm j\omega_d$, in which σ is the damping term and ω_d is the damped frequency of the response. Both modes are stable (negative damping terms), although the second mode has a real part very near zero, indicating that it is only marginally stable. In standard second order form the two modes are

$$d(s) = \left(s^2 + 2.34s + 10.7\right)\left(s^2 + 0.0134s + 0.00925\right)$$

Each of the oscillatory modes may be compared with $s^2+2\zeta\omega_n s+\omega_n^2$, from which we learn that the first mode has natural frequency $\omega_{n_{1,2}} = 3.27 rad/s$ and damping ratio $\zeta_{1,2} = 0.357$; and the second mode has $\omega_{n_{3,4}} = 0.0962 rad/s$ and $\zeta_{3,4} = 0.0696$. These results are qualitatively typical of "conventional" aircraft: one mode is characterized by relatively large natural frequency and damping, and the other by relatively small natural frequency and damping. We may evaluate the time to half amplitude, period, and number of cycles to half amplitude associated with each of the responses:

Metric	$\lambda_{1,2}{=}\lambda_{SP}$ (Short Period)	$\lambda_{3,4}{=}\lambda_{Ph}$ (Phugoid)
$t_{1/2} = \frac{ln(1/2)}{\sigma}$	$0.592s$	$103s$
$T = \frac{2\pi}{\omega_d}$	$2.05s$	$65.4s$
$N_{1/2} = \frac{t_{1/2}}{T}$	0.289	1.57

Table 9.1: Longitudinal metrics.

The first mode, associated with $\lambda_{1,2}$, is seen to have a relatively short period. This gives rise to the unimaginative name for this mode: it is the *short period mode*. The other mode has a much more imaginative name, the *phugoid mode*. The origin of this name is historically reported as being due to F.W. Lanchester in *Aerodonetics* (1908), who thought he had the root of the Greek word "to fly" but erroneously picked the root of the word "to flee". Whatever its origins, the word *phugoid* is firmly esconced in the argot of aeronautical engineers.

9.6.3 Eigenvector Analysis

We now examine the eigenvectors of the longitudinal modes. Using software such as MATLAB, the modal matrix M is determined to be

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{Bmatrix} 3.66 \times 10^{-2} + j0.946 \\ 8.01 \times 10^{-2} + j5.18 \times 10^{-2} \\ -0.182 + j0.231 \\ 8.56 \times 10^{-2} + j2.68 \times 10^{-2} \end{Bmatrix}, \ \mathbf{v}_2 = \mathbf{v}_1^*$$

$$\mathbf{v}_3 = \begin{Bmatrix} 4.90 \times 10^{-2} - j0.999 \\ -3.76 \times 10^{-6} + j3.27 \times 10^{-5} \\ 9.08 \times 10^{-6} - j2.88 \times 10^{-4} \\ -2.99 \times 10^{-3} + j1.14 \times 10^{-4} \end{Bmatrix}, \ \mathbf{v}_4 = \mathbf{v}_3^*$$

The first two columns of M ($\mathbf{v}_{1,2}$) correspond to the short period roots, and the last two ($\mathbf{v}_{3,4}$) to the phugoid roots. In polar form, the short period eigenvector is

$$\mathbf{v}_1 = \begin{Bmatrix} 0.947\angle 87.8\,\text{deg} \\ 9.54 \times 10^{-2}\angle 32.9\,\text{deg} \\ 0.294\angle 128.3\,\text{deg} \\ 8.97 \times 10^{-2}\angle 17.4\,\text{deg} \end{Bmatrix}$$

From the eigenvector the Argand diagram for the short period response is drawn (figure 9.3), with dashed lines indicating the direction of states too small to be represented.

The Argand diagram may be used to visualize the behavior the states relative to one another. The pitch rate leads changes in angle-of-attack and pitch attitude by about 90 *deg*, reaching its maximum and minimum values about one-quarter cycle before the two angles. The undamped changes in angle-of-attack and pitch attitude are nearly the same magnitude and very close in phase. This means they will each reach their minimum and maximum

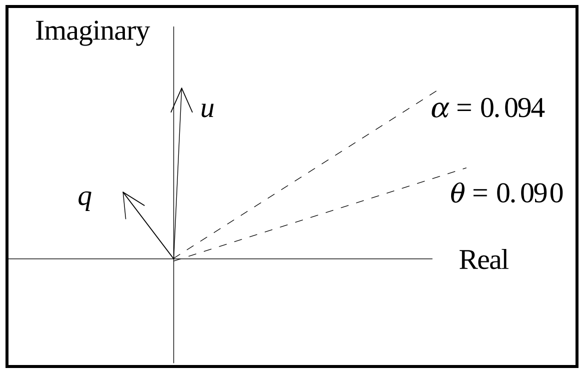

Figure 9.3: Argand Diagram for the Short Period Mode

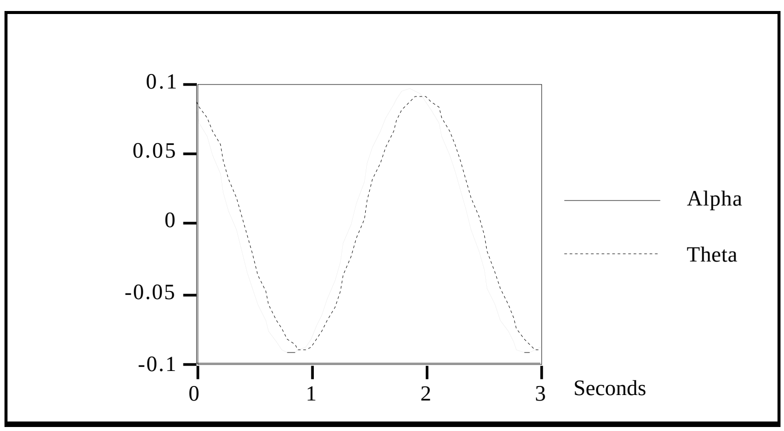

Figure 9.4: Undamped Time Histories, Alpha and Theta

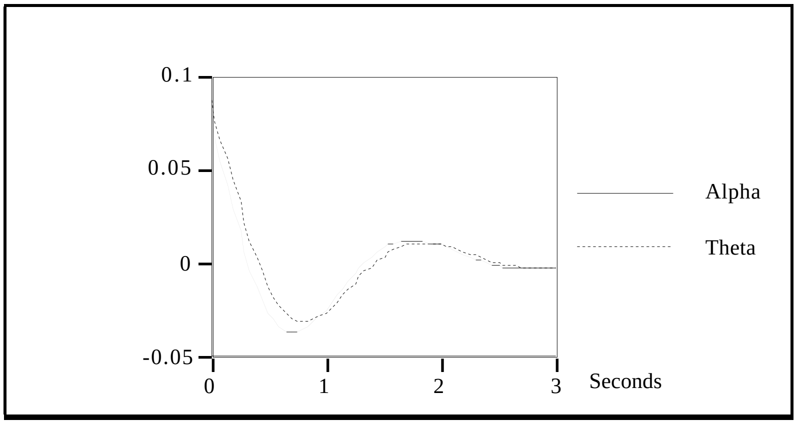

Figure 9.5: Damped Time Histories, Alpha and Theta

values at about the same time. The undamped time histories of α and θ show this result (figure 9.4).

If we include damping, and let the constant multipliers be unity, we have the results shown in figure 9.5.

The damped time histories show that this particular short period mode dies out very quickly. The total response to arbitrary initial conditions will, of course, be the sum of the short period and phugoid modes. The responses associated with the phugoid, with its period of over a minute, will have barely changed during the time it takes the short period to dampen, suggesting that the short period may be viewed in isolation from the phugoid. On the other hand, the short period influences will have long since vanished from the phugoid response by the time large changes have taken place in that mode: the short period will appear as a small wrinkle at the beginning of the phugoid response.

From the Argand diagram one might be tempted to conclude that the short period mode is characterized by large changes in velocity u and pitch rate q. However, a $1 ft/s$ change in speed is not very large when compared to $V_{Ref} = 446.6 ft/s$. Note that if u is scaled by the reference velocity (effectively yielding $\hat{V}$) quite a different picture results, in which the short period occurs at almost constant speed, and is dominated by changes in α, θ, and q. Also note that if we use $\hat{V}$ and the nondimensional pitch rate, $\hat{q} = q\bar{c}/2V_{Ref}$ ($\bar{c} = 10.8 ft$) then the interpretation might be that the short period is dominated by changes in α and θ. Analysis such as this often

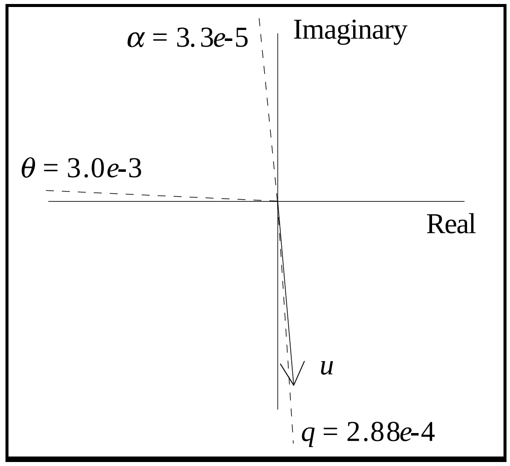

Figure 9.6: Argand Diagram for the Phugoid Mode

depends upon knowing the right answer before beginning, in which some combination of scaling can be found to show it is right.

Now examining the phugoid response, the polar form of its eigenvector is

$$\mathbf{v}_3 = \left\{ \begin{array}{c} 1.0\angle -87.2\,\text{deg} \\ 3.30\times 10^{-5}\angle 96.5\,\text{deg} \\ 2.88\times 10^{-4}\angle -88.2\,\text{deg} \\ 3.00\times 10^{-3}\angle 177.8\,\text{deg} \end{array} \right\}$$

From this information the Argand diagram is drawn, as shown in figure 9.6.

The inference from the Argand diagram is that the phugoid mode is dominated by large changes in velocity u. Even if u is scaled by $V_{Ref} = 446.6 ft/s$ it is still much larger than α and q, and only slightly smaller than θ. It will turn out that this is largely true. Had we included altitude as a fifth longitudinal state we would have found its component of the phugoid eigenvector to be comparable to that of velocity and almost 180 deg out of phase with it. The phugoid mode corresponds to cyclic tradeoffs in kinetic energy (velocity) and potential energy (height), performed at nearly constant angle-of-attack. A time history of changes in velocity, figure 9.7, shows part of this relationship.

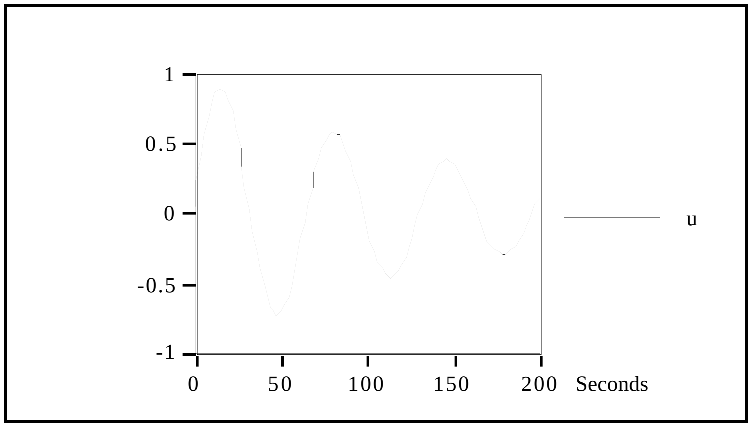

Figure 9.7: Time History of u for the Phugoid

9.6.4 Longitudinal Mode Sensitivity and Approximations

Applying the steps described in section 9.4.1 yields the sensitivity matrix for the longitudinal dynamics, shown in table 9.2.

	Short Period		Phugoid	
u	0.0005	0.0005	0.4995	0.4995
α	0.4952	0.4952	0.0048	0.0048
q	0.4961	0.4961	0.0039	0.0039
θ	0.0004	0.0004	0.4996	0.4996

Table 9.2: Longitudinal mode senstivities.

The entries in table 9.2 are unambiguous. Each mode has two states that dominate in that mode. We should be able to approximate the short period using the states α and q, and approximate the phugoid mode using the states u and θ.

The objective of this analysis is to formulate two second-order systems, one for each mode, in terms of the stability and control parameters. Before doing that, however, we will apply the approximations to the numerical results and see how well they do.

Short Period

Numerical formulation. The short period is clearly a faster mode than the phugoid, (real part of −1.17, compared to −0.067 for the phugoid). To approximate the short period mode, we assume that $\dot{u} = \dot{\theta} = u = \theta = 0$. As a result,

$$\tilde{\mathbf{x}}_{SP} \equiv \begin{Bmatrix} \alpha \\ q \end{Bmatrix}$$

$$\tilde{A}_{SP} = \begin{bmatrix} -0.877 & 0.998 \\ -9.47 & -1.46 \end{bmatrix}$$

Denoting the short period approximation eigenvalues by $\tilde{\lambda}_{SP}$,

$$\tilde{\lambda}_{SP} = -1.17 \pm j3.06$$

This result is identical to that from the full system, and is a very good approximation indeed, at least for the current example.

Literal formulation. In terms of the parameters from which the example was derived (neglecting $Z_{\dot{w}}$ and $M_{\dot{w}}$), we have

$$\tilde{\mathbf{x}}_{SP} \equiv \begin{Bmatrix} \alpha \\ q \end{Bmatrix} \tag{9.16a}$$

$$\tilde{A}_{SP} = \begin{bmatrix} \frac{Z_w}{m} & \frac{Z_q + mV_{Ref}}{mV_{Ref}} \\ \frac{M_w V_{Ref}}{I_{yy}} & \frac{M_q}{I_{yy}} \end{bmatrix} \tag{9.16b}$$

The eigenvalues of the short period approximation are found from

$$\left| sI - \tilde{A}_{SP} \right| = s^2 + c_1 s + c_0 = 0$$

where

$$\left| sI - \tilde{A}_{SP} \right| \cong s^2 - \left(\frac{I_{yy} Z_w + m M_q}{m I_{yy}} \right) s + \frac{Z_w M_q - M_w \left(Z_q + mV_{Ref} \right)}{m I_{yy}}$$

Applying this assumption to our example yields

$$\tilde{A}_{SP} = \begin{bmatrix} -0.879 & 1.00 \\ -9.77 & -1.12 \end{bmatrix}$$

$$\tilde{\lambda}_{SP} = -1.00 \pm j3.12$$

This approximation still compares favorably to $\lambda_{SP} = -1.17 \pm j3.06$, especially in terms of natural frequency (3.28 vs. 3.27 rad/s) and somewhat so with respect to damping ratio (0.307 vs. 0.357). Considering that $Z_{\dot{w}}$ and $M_{\dot{w}}$ are difficult to determine experimentally and thus subject to fairly large uncertainty, the approximation is probably not bad.

Phugoid

Numerical formulation. Now considering the phugoid approximation, we set $\dot{\alpha} = \dot{q} = 0$. Because α and q are associated with a faster mode, we treat them as fast variables. The procedure asks us to solve the $\dot{\alpha}$ and $\dot{q}$ for steady-state values to be used in the phugoid approximation. This yields

$$\begin{aligned} \dot{\alpha} = 0 &= -\left(3.16 \times 10^{-4}\right) u - 0.877\alpha_{ss} + 0.998 q_{ss} \\ \dot{q} = 0 &= \left(1.08 \times 10^{-4}\right) u - 9.47\alpha_{ss} - 1.46 q_{ss} \end{aligned}$$

Solving this system of equations for the steady-state values of α and q as a function of u yields

$$\alpha_{ss} = -\left(3.295 \times 10^{-5}\right) u, \quad q_{ss} = \left(2.877 \times 10^{-4}\right) u$$

The resulting equations for $\dot{u}$ and $\dot{\theta}$ become

$$\begin{aligned} \dot{u} &= -\left(1.51 \times 10^{-2}\right) u - 2.26\alpha_{ss} - 32.2\theta = -\left(1.50 \times 10^{-2}\right) u - 32.2\theta \\ \dot{\theta} &= q_{ss} = \left(2.877 \times 10^{-4}\right) u \end{aligned}$$

The phugoid approximation is then

$$\tilde{\mathbf{x}}_{Ph} = \begin{Bmatrix} u \\ \theta \end{Bmatrix}$$

$$\tilde{A}_{Ph} = \begin{bmatrix} -1.50 \times 10^{-2} & -32.2 \\ 2.877 \times 10^{-4} & 0 \end{bmatrix}$$

The eigenvalues of this system are

$$\hat{\lambda}_{Ph} = -0.0075 \pm j0.096$$

Compared with the actual eigenvalues, $\lambda_{Ph} = -0.0067 \pm j0.096$, this approximation does well on the damped frequency, but overestimates the damping term by almost 15%.

Literal formulation. In terms of the dimensional stability derivatives, we have

$$\begin{aligned} \dot{\alpha} = 0 &= \frac{Z_u}{mV_{Ref}}u + \frac{Z_w}{m}\alpha_{ss} + \left(\frac{Z_q}{mV_{Ref}} + 1\right) q_{ss} \\ \dot{q} = 0 &= \frac{M_u}{I_{yy}}u + \frac{M_w V_{Ref}}{I_{yy}}\alpha_{ss} + \frac{M_q}{I_{yy}}q_{ss} \end{aligned}$$

The derivatives Z_q and M_u are usually negligible, which results in

$$\begin{aligned} \begin{Bmatrix} \alpha_{ss} \\ q_{ss} \end{Bmatrix} &= \begin{bmatrix} Z_w V_{Ref} & mV_{Ref} \\ M_w V_{Ref} & M_q \end{bmatrix}^{-1} \begin{Bmatrix} -Z_u u \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} -M_q \\ M_w V_{Ref} \end{Bmatrix} \frac{Z_u u}{V_{Ref}\left(M_q Z_w - m M_w V_{Ref}\right)} \end{aligned}$$

The contribution of α_{ss} appears only in the equation for $\dot{u}$, and is normally quite small (see the numerical example above, in which the factor of u was changed from -1.51×10^{-2} to -1.53×10^{-2}). The q_{ss} term is, however, critical to the $\dot{\theta}$ equation; without it we have $\dot{\theta} = 0$ and the system is not oscillatory at all. We therefore ignore α_{ss} but retain q_{ss}, leaving:

$$\dot{u} = \frac{X_u + T_u}{m} u - g\theta$$
$$\dot{\theta} = q_{ss} = \frac{M_w Z_u}{M_q Z_w - m M_w V_{Ref}} u$$

We have let $Z_{\dot{w}} = 0$ (since we have let $\dot{\alpha} = 0$), $\gamma_{Ref} = \epsilon_T = 0$ (they are usually quite small), leaving us with the phugoid approximation:

$$\tilde{\mathbf{x}}_{Ph} = \begin{Bmatrix} u \\ \theta \end{Bmatrix} \tag{9.17a}$$

$$\tilde{A}_{Ph} = \begin{bmatrix} \frac{X_u + T_u}{m} & -g \\ \frac{M_w Z_u}{M_q Z_w - m M_w V_{Ref}} & 0 \end{bmatrix} \tag{9.17b}$$

Further simplifications. In the phugoid approximation it is normally taken that $|m M_w V_{Ref}| \gg |M_q Z_w|$, so that the system matrix becomes

$$\tilde{A}_{Ph} = \begin{bmatrix} \frac{X_u + T_u}{m} & -g \\ \frac{-Z_u}{m V_{Ref}} & 0 \end{bmatrix} \tag{9.17c}$$

The derivatives X_u and Z_u can be expressed in terms of the lift and drag of the aircraft. Moreover, since the example aircraft is a jet, $T_u = 0$. Finally, we focus on subsonic flight and therefore ignore Mach effects. The evaluation of X_u proceeds as follows:

$$\begin{aligned} X_u &= \left(\frac{\bar{q}_{Ref} S}{V_{Ref}}\right) \left(2 C_W \sin\gamma_{Ref} - 2 C_{T_{Ref}} \cos\epsilon_T - M_{Ref} C_{D_M}\right) \\ &= -2 \left(\frac{\bar{q}_{Ref} S}{V_{Ref}}\right) C_{T_{Ref}} \\ &= -2 \left(\frac{\bar{q}_{Ref} S}{V_{Ref}}\right) C_{D_{Ref}} \\ &= -2 \left(\frac{D_{Ref}}{V_{Ref}}\right) \end{aligned}$$

Similarly for Z_u:

$$\begin{aligned} Z_u &= \left(\frac{\bar{q}_{Ref} S}{V_{Ref}}\right)\left(-2C_{L_{Ref}} - M_{Ref} C_{L_M}\right) \\ &= -2\left(\frac{\bar{q}_{Ref} S}{V_{Ref}}\right) C_{L_{Ref}} \\ &= -2\left(\frac{L_{Ref}}{V_{Ref}}\right) \end{aligned}$$

The phugoid system matrix is therefore approximated by

$$\tilde{A}_{Ph} = \begin{bmatrix} -2\frac{D_{Ref}}{mV_{Ref}} & -g \\ 2\frac{L_{Ref}}{mV_{Ref}^2} & 0 \end{bmatrix} \tag{9.17d}$$

The characteristic polynomial of this system is

$$\left|sI - \tilde{A}_{Ph}\right| = s^2 + 2\frac{D_{Ref}}{mV_{Ref}} s + 2\frac{gL_{Ref}}{mV_{Ref}^2}$$

In the last term replace g with mg/m, and note that $L_{Ref} = W = mg$, so that

$$\left|sI - \tilde{A}_{Ph}\right| = s^2 + 2\frac{D_{Ref}}{mV_{Ref}} s + 2\frac{L_{Ref}^2}{m^2 V_{Ref}^2}$$

From this expression we see that the natural frequency of the phugoid approximation is inversely proportional to the trimmed flight speed,

$$\tilde{\omega}_{n_{Ph}} = \sqrt{2}\frac{L_{Ref}}{mV_{Ref}} \tag{9.17e}$$

The damping ratio turns out to be inversely proportional to the famous aircraft performance parameter, the ratio of lift to drag (L/D):

$$\tilde{\zeta}_{Ph} = \frac{D_{Ref}}{\sqrt{2}L_{Ref}} = \frac{1}{\sqrt{2}\,(L/D)_{Ref}} \tag{9.17f}$$

The phugoid is thus least well damped when the aircraft is most efficiently operated, i.e., at $(L/D)_{Max}$.

The phugoid approximation given by equation 9.17c can be derived by treating the aircraft as a point-mass that possesses the lift and drag characteristics of the aircraft. Then, by analyzing the kinetic and potential energy of the system, equation 9.17c results. Note that the assumption that got us to that equation could be reached by letting $M_q = 0$. When the aircraft is treated as a point-mass, there are no pitch dynamics, so the two approaches are equivalent.

9.6.5 Forced Response

We now turn to the forced response. The matrix of transfer functions is easily evaluated from $[sI - A]^{-1} B \equiv G(s) = \{g_{ij}(s)\}$.

$$G(s) = \frac{\begin{bmatrix} (20.5s^3 + 48.0s^2 + 220s) & (44.2s + 363) \\ (-6.49 \times 10^{-3}s^2 - 7.26 \times 10^{-3}s) & (-12.8s^2 - 0.194s - 0.131) \\ (2.22 \times 10^{-3}s^2 + 0.0634s) & (-12.8s^3 - 11.5s^2 - 0.162s) \\ (2.22 \times 10^{-3}s + 0.0634) & (-12.8s^2 - 11.5s - 0.162) \end{bmatrix}}{(s + 1.17 \pm j3.06)\,(s + 0.0067 \pm j0.096)}$$

This result is more useful with the numerator factored to show the zeros:

$$G(s) = \frac{\begin{bmatrix} 20.5s\,(s + 1.17 \pm j3.06) & 44.2\,(s + 0.820) \\ -6.49 \times 10^{-3}s\,(s + 1.12) & -12.8\,(s + 0.0076 \pm j0.101) \\ 2.22 \times 10^{-3}s\,(s + 28.6) & -12.8s\,(s + 0.877)\,(s + 0.0143) \\ 2.22 \times 10^{-3}\,(s + 28.6) & -12.8\,(s + 0.877)\,(s + 0.0143) \end{bmatrix}}{(s + 1.17 \pm j3.06)\,(s + 0.0067 \pm j0.096)}$$

Two interesting results may be directly observed from the matrix of transfer functions. First, with respect to the influence of the throttle on changes in speed,

$$\begin{aligned} g_{11}(s) = \frac{u(s)}{\delta_T(s)} &= \frac{20.5s\,(s + 1.17 \pm j3.06)}{(s + 1.17 \pm j3.06)\,(s + 0.0067 \pm j0.096)} \\ &= \frac{20.5s}{s + 0.0067 \pm j0.096} \end{aligned}$$

The short period mode has been cancelled by an identically placed pair of zeros in the numerator. This cancellation means that changes in throttle setting will have *no* effect on the short period dynamics. The reason for this is that we have modeled the thrust as acting straight along the aircraft x-axis, hence it will create no pitching moment or off-axis forces.

The transfer function of the pitching moment controller (elevator) to angle-of-attack has a similar result, but in this case it is the phugoid mode that is (almost) canceled:

$$g_{22}(s) = \frac{\alpha(s)}{\delta_m(s)} = \frac{-12.8\,(s + 0.0076 \pm j0.101)}{(s + 1.17 \pm j3.06)\,(s + 0.0067 \pm j0.096)}$$
$$\approx \frac{-12.8}{s + 1.17 \pm j3.06}$$

The matrix of transfer functions may be used to evaluate steady-state longitudinal responses to control inputs. There are three considerations we must observe:

1. The input and response must be "small",
2. There must be no coupling of the response with lateral-directional modes, and
3. The steady-state conditions must exist.

The first two requirements mean that we want to stay within the range of validity of the assumptions made in linearizing the equations of motion. Because the equations are linear, the magnitudes of inputs and responses may be uniformly scaled, so if a test input results in large response, we simply reduce inputs and responses by the same factor. The second consideration, no coupling, will be satisfied for the longitudinal equations of motion in straight, symmetric flight because we have assumed away any dependence on lateral-directional variables: there is no mechanism to create sideslip, roll or yaw rates, or bank angle changes. Finally, we will have to apply common sense to determine whether the inputs result in true steady-state conditions. For instance, if the analysis shows steady-state values of pitch rate and pitch angle, we should dismiss the results since this is not possible. And of course, we require that the system be stable, otherwise the steady-state solution will not exist.

Throttle Input

In response to a step throttle input of magnitude 0.1, the steady-state responses of the longitudinal states are

$$\mathbf{x}_{Long}(\infty) = \lim_{s\to 0}(s)\frac{\begin{Bmatrix} 20.5s\,(s+1.17\pm j3.06) \\ -6.49\times 10^{-3}s\,(s+1.12) \\ 2.22\times 10^{-3}s\,(s+28.6) \\ 2.22\times 10^{-3}\,(s+28.6) \end{Bmatrix}}{s^4+2.35s^3+10.76s^2+0.1652s+0.0993}\left(\frac{0.1}{s}\right)$$

$$\mathbf{x}_{Long}(\infty) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0638\ rad \end{Bmatrix} = \begin{Bmatrix} \Delta u_{SS} \\ \alpha_{SS} \\ q_{SS} \\ \theta_{SS} \end{Bmatrix}$$

The steady-state values will be obtained only after the short-period and phugoid modes have subsided.

This result may at first seem counter-intuitive. One might think that increasing the throttle should make the aircraft go faster, but instead the speed and angle-of-attack return to the trim value and the aircraft ends up in a climb. Recall, however, the steady-state requirement $M + M_T = 0$. The functional dependency of the pitching moment coefficient (neglecting altitude dependency) is $C_m\left(M, \alpha, \hat{\dot{\alpha}}, \hat{q}, \delta_m\right)$. In our example $M_T = 0$. There is no Mach dependency, no change in pitching moment control, and $\dot{\alpha}$ and q are zero in steady, straight flight. Therefore if $C_m\left(M, \alpha, \hat{\dot{\alpha}}, \hat{q}, \delta_m\right) = 0$ in the reference condition, and in the steady-state condition, the sole remaining variable—α—must be unchanged. Moreover, from $\dot{\alpha} = 0 \Rightarrow \dot{w} = 0$, every term in the numerator of the $\dot{w}$ equation is zero in the steady state except $Z_u\Delta u$, which must vanish as well:

$$Z_u\Delta u = 0 \Rightarrow \Delta u = 0$$

Elevator Input

In response to a $-1\,\text{deg}$ step input in the elevator (TEU), the steady-state responses of the longitudinal states are

$$\mathbf{x}_{Long}(\infty) = \lim_{s\to 0}(s)\,\frac{\begin{bmatrix} 44.2\,(s+0.820) \\ -12.8\,(s+0.0076\pm j0.101) \\ -12.8s\,(s+0.877)\,(s+0.0143) \\ -12.8\,(s+0.877)\,(s+0.0143) \end{bmatrix}}{s^4+2.35s^3+10.76s^2+0.1652s+0.0993}\left(\frac{-0.01745}{s}\right)$$

$$\mathbf{x}_{Long}(\infty) = \begin{Bmatrix} \Delta u_{SS} \\ \alpha_{SS} \\ q_{SS} \\ \theta_{SS} \end{Bmatrix} = \begin{Bmatrix} -6.37\,ft/s \\ 0.0231\,rad \\ 0 \\ 0.0282\,rad \end{Bmatrix}$$

The elevator input has resulted in the aircraft seeking a new trim airspeed and angle-of-attack. The new value of α_{SS} is relative to the stability-axis value of zero. Since, in wings-level flight, we have $\gamma = \theta - \alpha$, we see that the aircraft is also climbing at an angle of $\gamma_{SS} = 0.292\,\text{deg}$.

9.7 Example: Lateral/Directional Dynamics

9.7.1 System Matrices

See Appendix C for data. At a particular flight condition the A-4 *Skyhawk* has the following linearized, dimensional lateral/directional system and control matrices (the Δs have been dropped):

$$\dot{\mathbf{x}}_{LD} = A_{LD}\mathbf{x}_{LD} + B_{LD}\mathbf{u}_{LD}$$

$$\mathbf{x}_{LD} = \begin{Bmatrix} \beta \\ p \\ r \\ \phi \end{Bmatrix} \quad \mathbf{u}_{LD} = \begin{Bmatrix} \delta_\ell \\ \delta_n \end{Bmatrix}$$

$$A_{LD} = \begin{bmatrix} -0.248 & 0 & -1 & 0.072 \\ -23.0 & -1.68 & 0.808 & 0 \\ 13.5 & -0.0356 & -0.589 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_{LD} = \begin{bmatrix} 0 & 0.0429 \\ 17.4 & -21.9 \\ 4.26 & 0.884 \\ 0 & 0 \end{bmatrix}$$

9.7.2 State Transition Matrix and eigenvalues

Using Fedeeva's algorithm (Appendix D) we calculate the state transition matrix $[sI - A_{Long}]^{-1}$, which results in

$$[sI - A_{Long}]^{-1} = \frac{C(s)}{d(s)} = \frac{\{c_{ij}(s)\}}{d(s)},\ i = 1 \ldots 4,\ j = 1 \ldots 4$$

The terms $c_{ij}(s)$ in the numerator are

$$
\begin{aligned}
c_{11}(s) &= s^3 + 2.27s^2 + 1.02s \\
c_{12}(s) &= 0.108s + 0.0425 \\
c_{13}(s) &= -s^2 - 1.68s + 0.0582 \\
c_{14}(s) &= 0.0720s^2 + 0.164s + 0.0735 \\
c_{21}(s) &= -23.0s^2 - 2.64s \\
c_{22}(s) &= s^3 + 0.837s^2 + 13.6s \\
c_{23}(s) &= 0.808s^2 + 23.2s \\
c_{24}(s) &= -1.66s - 0.190 \\
c_{31}(s) &= 13.5s^2 + 23.5s \\
c_{32}(s) &= -0.0356s^2 - 8.81 \times 10^{-3}s + 0.972s \\
c_{33}(s) &= s^3 + 1.93s^2 + 0.417s + 1.66 \\
c_{34}(s) &= 0.972s + 1.69 \\
c_{41}(s) &= -23.0s - 2.64 \\
c_{42}(s) &= s^2 + 0.837s + 13.6 \\
c_{43}(s) &= 0.808s + 23.2 \\
c_{44}(s) &= s^3 + 25.2s^2 + 15.1s + 23.8
\end{aligned}
$$

The characteristic polynomial is

$$
\begin{aligned}
d(s) &= s^4 + 2.52s^3 + 15.s^2 + 25.s + 0.190 \\
&= (s + 0.340 \pm j3.70)(s + 1.83)\left(s + 7.51 \times 10^{-3}\right) \\
&= \left(s^2 + 0.679s + 13.8\right)(s + 1.83)\left(s + 7.51 \times 10^{-3}\right)
\end{aligned}
$$

The eigenvalues of the system are

$$
\begin{aligned}
\lambda_{1,2} &= -0.340 \pm j3.70 \\
\lambda_3 &= -1.83 \\
\lambda_4 &= -7.51 \times 10^{-3}
\end{aligned}
$$

The complex roots give rise to a stable oscillatory mode with $\omega_n = 3.71\,rad/s$ and $\zeta = 0.0914$. The two real roots are both stable (damped exponentials), but λ_4 is very small and nearly unstable. This distribution of eigenvalues is fairly typical of "conventional" aircraft, and the fourth eigenvalue is sometimes slightly unstable. For reasons that will become more clear following the modal analysis, the oscillatory mode is called the *Dutch roll,* the real root greatest in magnitude is the *Roll mode,* and the last root is the *Spiral mode.* The associated metrics are

Metric	$\lambda_{1,2}=\lambda_{DR}$ $(DutchRoll)$	$\lambda_3=\lambda_R$ $(RollMode)$	$\lambda_4=\lambda_S$ $(SpiralMode)$
$t_{1/2} = \frac{ln(1/2)}{\sigma}$	2.04 s	0.379 s	92.3 s
$T = \frac{2\pi}{\omega}$	1.70 s	—	—
$N_{1/2} = \frac{t_{1/2}}{T}$	1.2	—	—

Table 9.3: Lateral-directional metrics.

9.7.3 Eigenvector Analysis

The eigenvectors associated with these modes are as follows. With

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1^* & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{Bmatrix} -0.0601 + j0.126 \\ -0.495 - j0.651 \\ 0.450 + j0.245 \\ -0.162 + j0.149 \end{Bmatrix} \; \mathbf{v}_3 = \begin{Bmatrix} -0.00480 \\ -0.878 \\ 0.0269 \\ 0.479 \end{Bmatrix} \; \mathbf{v}_4 = \begin{Bmatrix} 0.00305 \\ -0.00749 \\ 0.0711 \\ 0.997 \end{Bmatrix}$$

Only one Argand diagram is necessary. For the Dutch roll mode, we have the relationships shown in figure 9.8. The Dutch roll mode appears to have significant components of each of the four lateral-directional states, suggesting (from this analysis) that there is no reasonable approximation. From the Argand diagram we see relatively large roll and yaw rates that are nearly 180 deg out of phase with each other. This means, roughly, that the aircraft will be simultaneously rolling one way and yawing the other. Since the bank angle is about 90 deg out of phase with both roll and yaw rates, we surmise that, for example, as the wings pass through the level position

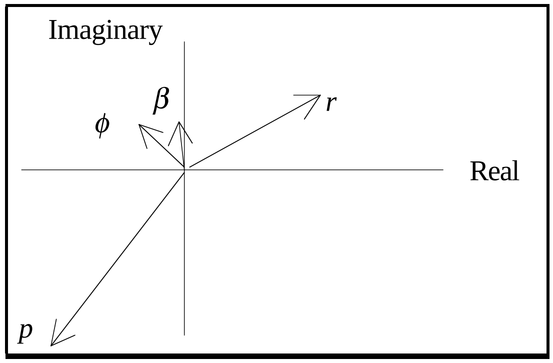

Figure 9.8: Argand Diagram of the Dutch Roll

the aircraft will be rolling to the right while yawing to the left, both at near maximum rates.

In figure 9.9, moving from bottom to top, the aircraft begins with its wings level and a slight amount of positive sideslip. At this instant the yaw rate is positive (nose right) and the roll rate is negative (left wing down). One-quarter cycle later the bank angle has reached its maximum left wing down position and sideslip is near its maximum negative value. Roll and yaw rates are near or just past zero. Half-cycle into the time history the aircraft is again wings-level, but now with some negative sideslip, negative yaw rate (nose left) and positive roll rate (right wing down). The roll continues to the right and the yaw to the left, until the three-quarters cycle position at the top of the figure. Here the bank angle has reached its maximum right wing down position and sideslip is near its maximum positive values. One-quarter cycle later the aircraft is again in the position in the bottom figure, although the amplitudes of all four states will have diminished due to damping. In ice-skating a Dutch roll is executed by gliding with the feet parallel and pressing alternately on the edges of each foot; the similarity of the skater's motion to that shown by the aircraft gives rise to the name of this mode.

Other useful information may be gleaned from the Dutch roll eigenvector. As we shall later see, the ratio of peak roll angle to sideslip angle is of importance in determining the flying qualities of an aircraft—loosely, how easy and pleasant it is to fly. This ratio is determined from the magnitudes of the ϕ and β components of the eigenvector, and for our example it is

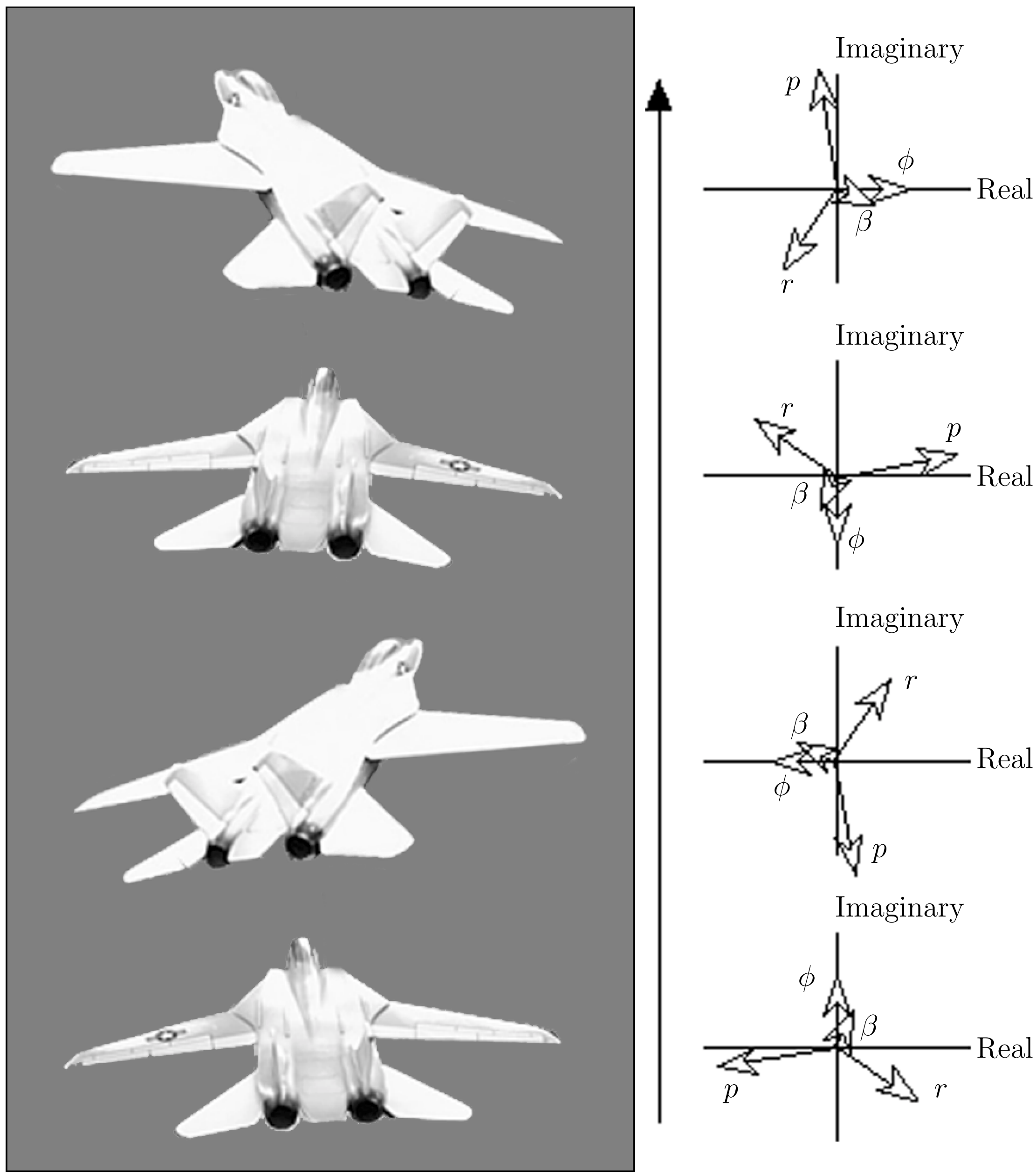

Figure 9.9: Dutch Roll (Bottom to Top) and Argand Diagram

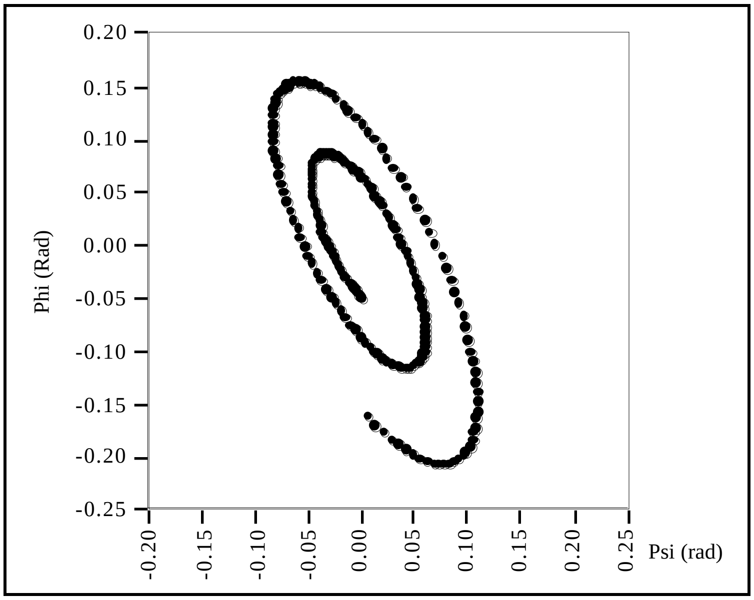

Figure 9.10: Path Traced by the Left Wing Tip

$$\left|\frac{\phi}{\beta}\right| = \frac{0.220}{0.140} = 1.57$$

Thus the magnitude of the roll angle is almost twice as great as that of sideslip; in supersonic flight this relationship is often reversed, giving rise to a "snaking" motion. Except in aircraft with highly swept wings it is fairly easy for a pilot to observe the relationship between ϕ and β during the Dutch roll. The change in heading angle $\Delta\psi$ is almost exactly the negative of the sideslip angle β during the Dutch roll. As the aircraft rolls and yaws the left wing tip will trace a path which, for this example, looks like figure 9.10.

9.7.4 Lateral-Directional Mode Sensitivity and Approximations

Results of the mode sensitivity analysis for the lateral-directional modes are shown in table 9.4. The results are unambiguous and show that β and r are dominant in the Dutch roll mode, p in the roll mode, and ϕ in the spiral mode. So far as the speed of the three modes, the roll mode is fastest with a real eigenvalue of $-1.83\, s^{-1}$, followed by the Dutch roll with a real part of $-0.34\, s^{-1}$, followed by the real spiral root of $-7.51 \times 10^{-3}\, s^{-1}$.

	Dutch roll		Roll	Spiral
β	0.4931	0.4931	0.0135	0.0003
p	0.0207	0.0207	0.9545	0.0041
r	0.4506	0.4506	0.0385	0.0604
ϕ	0.0147	0.0127	0.0522	0.9184

Table 9.4: Lateral-directional mode senstivities.

Roll Mode

Numerical formulation. The roll rate p is dominant in the roll mode. The roll mode is almost purely motion about the body x-axis. The roll mode is the fastest of the lateral-directional modes, so we let $\dot{\beta} = \dot{r} = \dot{\phi} = \beta = r = \phi = 0$, leaving simply

$$\dot{p} = -1.68p$$

This simple first order system has eigenvalue $\tilde{\lambda}_R = -1.68$, which compares favorably with the actual eigenvalue, $\lambda_R = -1.83$.

Literal formulation. The roll mode approximation is simply

$$\tilde{\mathbf{x}}_R = \{p\} \tag{9.18a}$$

$$\dot{p} = \frac{L_p}{I_{xx}} p \tag{9.18b}$$

We may include the $\dot{\phi}$ equation, $\dot{\phi} = p$, without changing the roll mode eigenvalue.

$$\tilde{\mathbf{x}}_R = \begin{Bmatrix} p \\ \phi \end{Bmatrix}$$
$$\tilde{A}_R = \begin{bmatrix} L_p/I_{xx} & 0 \\ 1 & 0 \end{bmatrix}$$

The approximation treats the aircraft as if it were constrained to motion about the x-axis only, and the only moment that remains is due to the roll-rate damping, L_p. Since only the roll of the aircraft is involved the mode is naturally called the roll mode, or sometimes the *roll-subsidence mode.*

Dutch roll

Numerical formulation. The Dutch roll is dominated by the variables β and r. It is slower than the roll mode but faster than the spiral mode. For the slower mode we take $\dot{\phi} = \phi = 0$. The roll mode is faster so we take $\dot{p} = 0$ but treat p as p_{ss}. This leads to

$$\dot{p} = 0 = -23.0\beta - 1.68p_{ss} + 0.808r \Rightarrow p_{ss} = -13.69\beta + 0.4810r$$

Substituting this result into the equations for $\dot{\beta}$ and $\dot{r}$ effects only $\dot{r}$, since $\dot{\beta}$ does not depend on p:

$$\dot{r} = 13.5\beta - 0.0356p_{ss} - 0.590r = 14.0\beta - 0.607r$$

The Dutch roll approximation (for this example) then becomes

$$\tilde{\mathbf{x}}_{DR} = \begin{Bmatrix} \beta \\ r \end{Bmatrix}$$
$$\tilde{A}_{DR} = \begin{bmatrix} -0.248 & -1 \\ 14.0 & -0.607 \end{bmatrix}$$
$$\tilde{\lambda}_{DR} = -0.428 \pm j3.74$$

Compared to the actual eigenvalues, $\lambda_{DR} = -0.340 \pm j3.70$, this approximation appears reasonable. The estimated damping ratio $\left(\tilde{\zeta}_{DR} = 0.114\right)$ captures the poorly damped characteristics of the Dutch roll ($\zeta_{DR} = 0.0914$). The natural and damped frequencies of the estimate and actual Dutch roll are very nearly the same.

Literal formulation. The literal expression is quite messy unless we neglect I_{xz}. A survey of stability and control data for various aircraft suggests I_{xz} is usually small, so we do assume $I_{xz} = 0$. The expression for p_{ss} becomes

$$p_{ss} = -\frac{L_v}{L_p}v - \frac{L_r}{L_p}r = -\frac{V_{Ref}L_v}{L_p}\beta - \frac{L_r}{L_p}r$$

When this expression is substituted into the equations for $\dot{\beta}$ and $\dot{r}$,

$$\begin{aligned} \dot{\beta} &= \left(\frac{Y_v L_p - Y_p L_v}{m L_p}\right)\beta + \left(\frac{Y_r L_p - Y_p L_r}{m V_{Ref} L_p} - 1\right) r \\ \dot{r} &= \left(\frac{V_{Ref}\left(N_v L_p - N_p L_v\right)}{I_{zz} L_p}\right)\beta + \left(\frac{N_r L_p - N_p L_r}{I_{zz} L_p}\right) r \end{aligned}$$

Most stability and control data take $Y_p = Y_r = 0$, so that

$$\begin{aligned} \dot{\beta} &= \left(\frac{Y_v}{m}\right)\beta - r \\ \dot{r} &= \left(\frac{V_{Ref}\left(N_v L_p - N_p L_v\right)}{I_{zz} L_p}\right)\beta + \left(\frac{N_r L_p - N_p L_r}{I_{zz} L_p}\right) r \end{aligned}$$

The Dutch roll approximation is then given by:

$$\tilde{\mathbf{x}}_{DR} = \begin{Bmatrix} \beta \\ r \end{Bmatrix} \tag{9.19a}$$

$$\tilde{A}_{DR} = \begin{bmatrix} \frac{Y_v}{m} & -1 \\ \frac{V_{Ref}(L_p N_v - L_v N_p)}{I_{zz} L_p} & \frac{L_p N_r - L_r N_p}{I_{zz} L_p} \end{bmatrix} \tag{9.19b}$$

In equation 9.19b, further simplification is possible by making assumptions about the relative magnitudes of the stability derivatives involved. For the example aircraft being used, $|L_p N_r| \gg |L_r N_p|$, and $|L_p N_v| \gg |L_v N_p|$. A suitable approximation for the *Skyhawk's* Dutch roll mode at the given flight condition is therefore

$$\tilde{A}_{DR} = \begin{bmatrix} \frac{Y_v}{m} & -1 \\ \frac{V_{Ref} N_v}{I_{zz}} & \frac{N_r}{I_{zz}} \end{bmatrix}$$

The eigenvalues of this approximation are $\underset{\sim}{\lambda}_{DR} = -0.419 \pm j3.67$, not much different from the previous approximation, $\tilde{\lambda}_{DR} = -0.428 \pm j3.74$. Note that this approximation to the approximation is equivalent to taking $p_{ss} = 0$. This ambiguous effect of the roll rate equation may be due to the fact that the roll mode is only about five times faster than the Dutch roll.

Spiral Mode

Numerical formulation. The spiral mode eigenvector shows a dominance in bank angle. Unlike the roll mode, however, there is no accompanying large roll rate. When viewed in the context of the long time scale associated with this mode, we may visualize the aircraft with some bank angle, slowly rolling back towards the wings level position. If this eigenvalue had been positive then the interpretation would have been of slowly increasing bank angle.

For the spiral mode approximation we let $\dot{\beta} = \dot{p} = \dot{r} = 0$. The steady-state values of β, p, and and r are calculated from the roll and Dutch roll modes,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -0.248 & 0 & -1 & 0.072 \\ -23.0 & -1.68 & 0.808 & 0 \\ 13.5 & -0.0356 & -0.590 & 0 \end{bmatrix} \begin{Bmatrix} \beta_{ss} \\ p_{ss} \\ r_{ss} \\ \phi \end{Bmatrix}$$

$$\begin{bmatrix} -0.248 & 0 & -1 \\ -23.0 & -1.68 & 0.808 \\ 13.5 & -0.0356 & -0.590 \end{bmatrix} \begin{Bmatrix} \beta_{ss} \\ p_{ss} \\ r_{ss} \end{Bmatrix} = \begin{Bmatrix} -0.072 \\ 0 \\ 0 \end{Bmatrix} \phi$$

The only one of the steady-state values needed is p_{ss}, which is determined to be

$$p_{ss} = -0.0081\phi$$

The spiral mode approximation then becomes

$$\dot{\phi} = -0.0081\phi$$

The eigenvalue associated with this equation is $\tilde{\lambda}_S = -0.0081$, which is close to the actual value $\lambda_S = -0.00751$.

Literal formulation. First solving for p_{ss} (with $I_{xz} = 0$),

$$p_{ss} = \frac{\begin{vmatrix} \frac{V_{Ref} L_v}{I_{xx}} & \frac{L_r}{I_{xx}} \\ \frac{V_{Ref} N_v}{I_{zz}} & \frac{N_r}{I_{zz}} \end{vmatrix}}{\begin{vmatrix} \frac{Y_v}{m} & \frac{Y_p}{m V_{Ref}} & \left(\frac{Y_r}{m V_{Ref}} - 1\right) \\ \frac{V_{Ref} L_v}{I_{xx}} & \frac{L_p}{I_{xx}} & \frac{L_r}{I_{xx}} \\ \frac{V_{Ref} N_v}{I_{zz}} & \frac{N_p}{I_{zz}} & \frac{N_r}{I_{zz}} \end{vmatrix}} \left(\frac{g}{V_{Ref}}\right) \phi$$

Again taking $Y_p = Y_r = 0$,

$$p_{ss} = \frac{-\begin{vmatrix} \frac{V_{Ref} L_v}{I_{xx}} & \frac{L_r}{I_{xx}} \\ \frac{V_{Ref} N_v}{I_{zz}} & \frac{N_r}{I_{zz}} \end{vmatrix}}{\begin{vmatrix} \frac{Y_v}{m} & 0 & -1 \\ \frac{V_{Ref} L_v}{I_{xx}} & \frac{L_p}{I_{xx}} & \frac{L_r}{I_{xx}} \\ \frac{V_{Ref} N_v}{I_{zz}} & \frac{N_p}{I_{zz}} & \frac{N_r}{I_{zz}} \end{vmatrix}} \left(\frac{g}{V_{Ref}}\right) \phi$$

$$\tilde{\mathbf{x}}_S = \{\phi\} \tag{9.20a}$$

$$\dot{\phi} = \frac{g\left(L_v N_r - L_r N_v\right)}{\frac{Y_v}{m}\left(L_p N_r - L_r N_p\right) - V_{Ref}\left(L_v N_p - L_p N_v\right)} \phi \tag{9.20b}$$

Further simplifications. In equation 9.20b, further simplification may be possible by making assumptions about the relative magnitudes of the stability derivatives involved. For instance, for the example aircraft being used, the denominator term $V_{Ref}\left(L_v N_p - L_p N_v\right)$ is more than 16 times as large as the other term, and $|L_p N_v| \gg |L_v N_p|$. A suitable approximation for the *Skyhawk's* spiral mode at the given flight condition is:

$$\dot{\phi} = \frac{g\left(L_v N_r - L_r N_v\right)}{V_{Ref} L_p N_v}\phi$$

When the terms in this approximation are calculated, the eigenvalue that results is $\tilde{\lambda}_S = -0.00776$, which is (coincidentally) closer to the actual value $\lambda_S = -0.00751$ than the full approximation.

9.7.5 Forced Response

We now turn to the forced response. Evaluating the matrix of transfer functions $\left[sI - A_{LD}\right]^{-1} B \equiv G(s)$, we have

$$\mathbf{x}_{LD} = \begin{Bmatrix} \beta \\ p \\ r \\ \phi \end{Bmatrix} \qquad \mathbf{u}_{LD} = \begin{Bmatrix} \delta_\ell \\ \delta_n \end{Bmatrix}$$

$$G(s) = \frac{\begin{bmatrix} -4.26\left(s+1.41\right)\left(s-0.165\right) & 0.0429\left(s-22.3\right)\left(s+3.76\right)\left(s+0.243\right) \\ 17.4s\left(s+0.517 \pm j4.36\right) & -21.9\left(s+0.425 \pm j3.54\right) \\ 4.26\left(s+2.52\right)\left(s+0.368 \pm j1.45\right) & 0.884\left(s-1.89\right)\left(s+2.68 \pm j2.17\right) \\ 17.4\left(s+0.517 \pm j4.36\right) & -21.9\left(s+0.425 \pm j3.54\right) \end{bmatrix}}{\left(s+0.340 \pm j3.70\right)\left(s+1.83\right)\left(s+0.00751\right)}$$

Note that there are no pole-zero cancellations (with the possible exception of considering the free s in three of the numerators cancelling the spiral mode). The interpretation of this is that, due to the great amount of cross-coupling between the rolling and yawing moment controls, application of either control will excite all three lateral-directional modes. Application of only the rolling moment control (in this case, aileron) will not generate pure roll, but will also excite the other two modes (primarily the Dutch roll). Pilots adapt to this phenomenon by learning to accompany aileron inputs with simultaneous rudder inputs.

There is no point in investigating the steady-state response of the aircraft to step inputs in rudder and in aileron. Any change in bank angle will incline the lift vector away from the vertical and will cause immediate coupling with the longitudinal modes. Application of rolling moment controller (aileron) will certainly change the bank angle, and so will the yawing moment controller (rudder) through first the change in sideslip, then through dihedral effect. The significance of this may be observed by considering a step input in aileron, which will among other consequences cause the aircraft to roll continuously. During part of the roll the lift vector will be pointing downward, causing the flight path to curve until the aircraft is accelerating in a steep descent.