Fundamentals of Control

State space and Laplace transform

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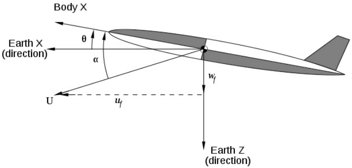
Notation

- l, d: lift and drag;
- C_l, C_d: non-dimensional lift and drag coefficients;
- L, M, N: roll, pitch, yaw moments;
- C_L, C_M, C_N: non-dimensional coefficients of roll, pitch, yaw moments;
- $C_{l_\alpha} = \partial C_l / \partial \alpha$.

Review: longitudinal and lateral angles



Longitudinal Equations of Motion

$$\theta = \alpha + \gamma$$

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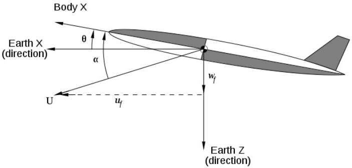

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$$\xi = \phi + \beta$$

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Today's objective is to understand the analyse method of the longitudinal dynamics.

Review: The simplest longitudinal model

$$
\begin{aligned}
\mathrm{m}(\dot{\mathrm{w}} - \mathrm{q}\mathrm{U}_0) &= \mathrm{F}_\mathrm{z}, \qquad (1) \\
\mathrm{I}_{\mathrm{yy}}\dot{\mathrm{q}} &= \mathrm{M}. \qquad (2)
\end{aligned}
$$

Review: The simplest longitudinal model

$$m(\dot{w}' - q'U_0) = Z^c, \tag{3}$$

$$I_{yy}\dot{q}' = M_w w' + M_q q' + M^c. \tag{4}$$

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How to analysis the dynamics?

To answer this question, we need to use methods in linear system.

Linear dynamical system

Continuous-time linear dynamical system (LDS) has the form

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{A(t)x(t)} + \mathrm{B(t)u(t)}, \quad \mathrm{y(t)} = \mathrm{C(t)x(t)} + \mathrm{D(t)u(t)}.$$

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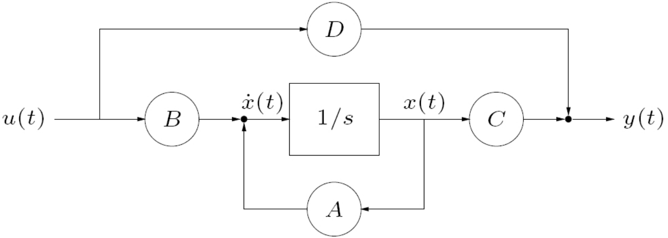
If $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ are time-invariant, we can have the corresponding transfer function:

$$\mathrm{Y(S)} = \mathrm{G(S)X(S)}.$$

Block diagram



- 1/s represent integral;
- A_{ij} is gain factor from state x_j into integrator i;
- B_{ij} is gain factor from input u_j into integrator i;
- C_{ij} is gain factor from state x_j into output y_i;
- D_{ij} is gain factor from input u_j into output y_i.

Block diagram

Q: Draw block diagram for

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

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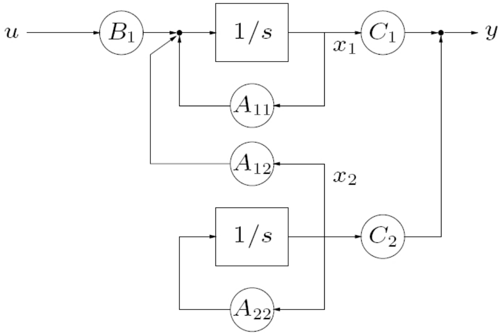

- x_2 is not affected by input u, i.e., x_2 propagates autonomously;
- x_2 affects y directly and through x_1.

Course introduction

- Prerequisite: Linear algebra, Fourier transform. (Read through the related textbooks yourself again!)

Introduction

System dynamics

- Control systems are dynamic $\longrightarrow$ transient + steady state.
- Transient motion is important. In the case of a lift, slow transient motion makes people impatient, rapid motion is uncomfortable. If the lift oscillates this is disconcerting.
- Steady state is also important, it can have an error, i.e. the step up or down on the exit form a lift.

Introduction

Transfer function

- A mathematical representation of the relation between the input and output of a linear time-invariant system (will be defined later).
- Q: If above system is linear, what is its transfer function?

Introduction

Dynamics

We must have a dynamic system:

Total = Natural + Forced

$\downarrow$

describes how the system
dissipates energy

This means that the equivalent differential equation equals a homogeneous solution plus a particular solution.

Some LDS terminology

- Most linear systems encountered are time-invariant: A, B, C, D are constant, i.e., don't depend on t.
- When there is no input u (hence, no B or D) system is called autonomous.
- Very often there is no feedthrough, i.e., $D = 0$.
- When u(t) and y(t) are scalar, system is called single-input, single-output (SISO); when input & output signal dimensions are more than one, MIMO.

State space model and more...

1. The above cases are represented by state space models;
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1. The above cases are represented by state space models;
2. They are in the time domain and the bare bone of modern control.
3. In the very beginning of control (so-called classical control), frequency domain is more preferred.
4. Time domain is still a part of classical control.
5. Next we will present primitive mathematical knowledge (Laplace transform).
6. Another reason to study Laplace transform is to achieve transfer function for state space model.

Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Laplace Integral, Q: why?

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  The relationship between an event (the cause) and a second event (the effect), where the second event is understood as a consequence of the first;

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  The relationship between an event (the cause) and a second event (the effect), where the second event is understood as a consequence of the first;
  Current state (x(t)) and output (y(t)) depending on past input (u(τ) for $\tau \leq t$) is causal;
  Current state (and output) depending on future input is anti-causal.

Laplace transform

1. $\mathcal{L}[tf(t)] = -F'(s)$
2. $\mathcal{L}[e^{-at}f(t)] = F(s+a)$ Frequency Shift
3. $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$ Time Shift
4. $\mathcal{L}\left[f(at)\right] = \frac{1}{a}F(\frac{s}{a})$ Scaling
5. $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$ Differentiation
6. $\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$ Integration Theorem
7. $f(\infty) = \lim_{s\to 0}(sF(s))$ Final Value Theorem
8. $f(0+) = \lim_{s\to\infty}(sF(s))$ Initial Value Theorem

Key points

- State space model.
- Given a dynamic model, write down its state space model.
  Example: Eqs. (3-4).
- Laplace transform.

Homeworks

Prove (2), (7), (8).