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Eshelby Equivalent Inclusion Method for Composites with Interface Effects

H.L. Duan [a], X. Yi [b], Z.P. Huang [c] and J. Wang [d]

LTCS and Department of Mechanics and Engineering Science, Peking University, Beijing, 100871, P. R. China

[a]hlduan@pku.edu.cn, [b]yixin@pku.edu.cn, [c]huangzp@pku.edu.cn, [d]jxwang@pku.edu.cn

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Abstract. The Eshelby equivalent inclusion method is generalized to calculate the stress fields related to spherical inhomogeneities with two interface conditions depicted by the interface stress model and the linear-spring model. It is found that the method gives the exact results for the hydrostatic loading and very accurate results for a deviatoric loading. The method can be used to predict the effective properties of composites with the interface effects.

Introduction

Interfacial bonding condition is one of the important factors that affect the properties of a composite. Therefore, many models have been proposed to simulate the interface properties in composites, e.g. the linear-spring model (LSM) [1-4]. Recently, the interface stress effect has attracted considerable attention of researchers in materials science and mechanics from different aspects for its importance in nanostructured materials [5-7]. Duan et al. [8-10] gave the Eshelby formalism and micromechanical framework with the interface stress effect and predicted the effective moduli of composites containing spherical inhomogeneities or voids with the surface/interface stress effects.

The Eshelby equivalent inclusion method [11] is very useful in micromechanics. In the previous papers [9,10], the authors have given the stress concentration tensors under remote loading and the Eshelby tensors under eigenstrain for inhomogeneities with the linear-spring interface model and the interface stress model. However, the problems have not been studied within the formalism of the Eshelby equivalent inclusion method. Therefore, in this paper, the Eshelby equivalent inclusion method for spherical inhomogeneities with the interface stress model (ISM) and the linear-spring interface model (LSM) is presented. The accuracy of the elastic fields in the inhomogeneities and in the matrices obtained using the equivalent inclusion method is examined for these two models. The results show that the Eshelby equivalent inclusion method for inhomogeneities with the interface effects in a volume average sense generally gives very accurate stress fields, and thus it is very useful in the prediction of the effective moduli of composites containing inhomogeneities with the interface effects.

Interface and Boundary Conditions

We first consider the *inhomogeneous inclusion* problem, namely, a spherical inhomogeneity embedded in an alien infinite elastic matrix is given a uniform eigenstrain ε^*, where the interface Γ_{12} between the inhomogeneity and the matrix is simulated by the interface stress model (ISM) or the linear-spring model (LSM). For this problem, the Eshelby tensors $\mathbf{S}^k(\mathbf{x})$ (k = 1,2) relate the strains $\boldsymbol{\varepsilon}^k(\mathbf{x})$ in the inhomogeneity (k = 1), denoted by Ω_1, and the matrix (k = 2), denoted by Ω_2, to the prescribed uniform eigenstrain $\boldsymbol{\varepsilon}^*$ in the inhomogeneity, i.e.

$$\boldsymbol{\varepsilon}^k(\mathbf{x}) = \mathbf{S}^k(\mathbf{x}) : \boldsymbol{\varepsilon}^* \qquad (\mathrm{k} = 1, 2), \qquad \forall \mathbf{x} \in \Omega_1 + \Omega_2 \tag{1}$$

where $\mathbf{x}$ is the position vector. The interface and boundary conditions for the interface stress model (ISM) and linear-spring model (LSM) subjected to $\boldsymbol{\varepsilon}^*$ are as follows:

$$\begin{cases} \mathbf{u}^1 + \boldsymbol{\varepsilon}^* \cdot \mathbf{x} = \mathbf{u}^2, (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \cdot \mathbf{n} = \nabla_S \cdot \boldsymbol{\tau}, \quad \text{(ISM) at } \Gamma_{12} \\ \boldsymbol{\sigma}^k \cdot \mathbf{n} = \boldsymbol{\gamma} \cdot [\mathbf{u}^2 - \mathbf{u}^1 - \boldsymbol{\varepsilon}^* \cdot \mathbf{x}], \boldsymbol{\sigma}^1 \cdot \mathbf{n} = \boldsymbol{\sigma}^2 \cdot \mathbf{n}, \quad \text{(LSM) at } \Gamma_{12} \\ \mathbf{u}^2 = 0, \boldsymbol{\sigma}^2 = 0, \quad |\mathbf{x}| \to +\infty \end{cases} \tag{2}$$

where $\boldsymbol{\gamma} = \alpha\mathbf{P} + \beta\mathbf{n}\otimes\mathbf{n}$, and α and β are two parameters describing the bonding conditions in the tangential and normal directions, respectively. $\mathbf{P} = \mathbf{I}^{(2)} - \mathbf{n}\otimes\mathbf{n}$, $\mathbf{I}^{(2)}$ is the second-order identity tensor in three-dimensional space, and $\mathbf{n}$ is the unit normal vector to the interface Γ_{12}. $\nabla_S \cdot \boldsymbol{\tau}$ denotes the interface divergence of $\boldsymbol{\tau}$ at Γ_{12}. For an elastically isotropic interface, the constitutive equation, which relates the interface stress $\boldsymbol{\tau}$ to the interface strain $\boldsymbol{\varepsilon}^s$ can be expressed as [7-10, 12] $\boldsymbol{\tau} = 2\mu_s\boldsymbol{\varepsilon}^s + \lambda_s(\text{tr}\boldsymbol{\varepsilon}^s)\mathbf{1}$, where λ_s and μ_s are the interface moduli, and $\mathbf{1}$ is the second-order unit tensor in two-dimensional space. For a coherent interface, the interface strain $\boldsymbol{\varepsilon}^s$ is equal to the tangential strain in the abutting bulk materials.

Unlike the classical counterpart for an ellipsoidal inhomogeneity without the interface effects, the interior Eshelby tensors with the interface stress and the linear-spring interface are generally position-dependent [9]. However, the volume average Eshelby tensors in the inhomogeneities are isotropic tensors, which can be expressed as follows:

$$\overline{\mathbf{S}}^1 = \xi_1\mathbf{J} + \varsigma_1\mathbf{K} \tag{3}$$

where $\mathbf{J} = \frac{1}{3}\mathbf{I}^{(2)} \otimes \mathbf{I}^{(2)}$, $\mathbf{K} = -\frac{1}{3}\mathbf{I}^{(2)} \otimes \mathbf{I}^{(2)} + \mathbf{I}^{(4s)}$. $\mathbf{I}^{(4s)}$ is the fourth-order symmetric identity tensor, and the two constants ξ_1 and ς_1 are given in the paper of Duan et al. [9]. When the inhomogeneity has the same elastic constants as those of the matrix, the *inhomogeneous inclusion* problem degenerates into an *inclusion* problem.

Eshelby Equivalent Inclusion Method with Interface Effects

The Eshelby equivalent inclusion method is convenient for solving inhomogeneity problems when the elastic fields in the inhomogeneities are uniform; however, it is difficult to obtain exact closed-form solutions using this method for general non-uniform elastic fields in inhomogeneities, e.g. the inhomogeneity problems with the interface effects. In this section, we will apply the Eshelby equivalent inclusion method in a volume average sense to the spherical inhomogeneities with the above-mentioned two interface effects.

To this end, we calculate the average stress concentration tensors and the exterior (in the matrix) stress fields by applying the Eshelby equivalent inclusion method in a volume average sense to the region 1 shown in Fig. 1(a), i.e. the spherical inhomogeneity in an infinite matrix. Assume that the volume average stress in the inhomogeneity with either of the interface effects is $\overline{\boldsymbol{\sigma}}^1$ while the remote stress is $\boldsymbol{\sigma}^0$. For the same remote stress, when we replace the spherical inhomogeneity with the stiffness tensor C1 (compliance tensor D1) by the matrix material with the stiffness tensor C2 (compliance tensor D2), the volume average stress in the same region is denoted by $\overline{\boldsymbol{\sigma}}_m$, as shown in Fig. 1(b). The boundary-value problem in Fig. 1(b) has been solved in the paper of Duan et al. [10]. This volume average stress $\overline{\boldsymbol{\sigma}}_m$ can be related to the remote stress by

$$\overline{\boldsymbol{\sigma}}_m = \mathbf{B} : \boldsymbol{\sigma}^0 \tag{4}$$

where the fourth-order tensor $\mathbf{B}$ is given in the paper of Duan et al. [10].

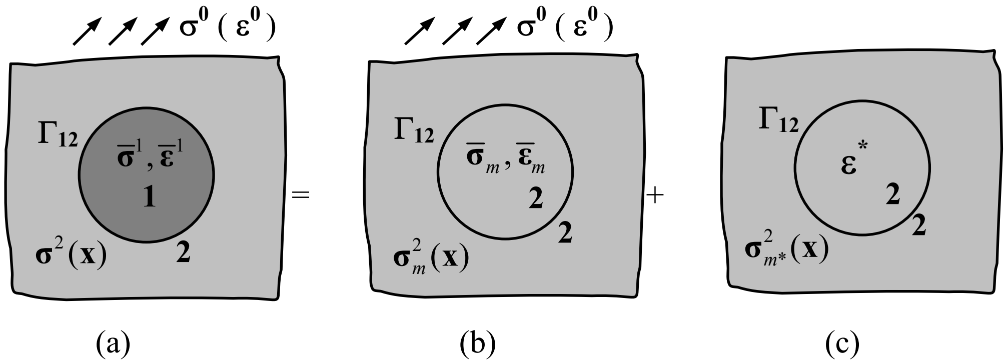


Fig. 1. A spherical inhomogeneity with an interface effect in an infinite medium (a), and the Eshelby equivalent inclusion method in a volume average sense ((b) and (c)). Region 1 denotes the inhomogeneity; region 2 represents the matrix.

Generally, $\bar{\boldsymbol{\sigma}}_m$ is different from $\bar{\boldsymbol{\sigma}}^1$. As in the classical Eshelby equivalent inclusion method, the spherical matrix region 1 is further given a uniform eigenstrain $\boldsymbol{\varepsilon}^*$ (Fig. 1(c)) such that the following equivalency condition is satisfied in a volume average sense:

$$\bar{\boldsymbol{\sigma}}^1 = \mathbf{C}_1 : (\bar{\boldsymbol{\varepsilon}}_m + \bar{\boldsymbol{\varepsilon}}^{'}) = \mathbf{C}_2 : (\bar{\boldsymbol{\varepsilon}}_m + \bar{\boldsymbol{\varepsilon}}^{'} - \boldsymbol{\varepsilon}^*) \tag{5}$$

where the disturbed volume average strain $\bar{\boldsymbol{\varepsilon}}^{'}$ is related to the interior average Eshelby tensor $\bar{\mathbf{S}}^1$ through

$$\bar{\boldsymbol{\varepsilon}}^{'} = \bar{\mathbf{S}}^1 : \boldsymbol{\varepsilon}^* \tag{6}$$

It should be pointed out that the eigenstrain problem in Fig. 1(c) is solved using the interface and boundary conditions in Eq. 2. Therefore, $\bar{\mathbf{S}}^1$ in Eq. 6 is given in Eq. 3 for an *inclusion* problem. Following the standard procedure of the Eshelby equivalent inclusion method, we get

$$\bar{\boldsymbol{\sigma}}^1 = [\mathbf{I}^{(4s)} - \mathbf{C}_1 : \bar{\mathbf{S}}^1 : (\mathbf{D}_1 - \mathbf{D}_2)]^{-1} : \mathbf{C}_1 : \mathbf{D}_2 : \mathbf{B} : \boldsymbol{\sigma}^0 \equiv \mathbf{T}^* : \boldsymbol{\sigma}^0 \tag{7}$$

For an isotropic spherical inhomogeneity, the volume average stress concentration tensor $\mathbf{T}^*$ can be expressed as

$$\mathbf{T}^* = \alpha^* \mathbf{J} + \beta^* \mathbf{K} \tag{8}$$

The exact volume average stress concentration tensor, denoted by $\mathbf{T}^1$, can be obtained by solving the inhomogeneity problem in Fig. 1(a) directly, which has been previously obtained by Duan et al. [10]. In the following section, the approximate volume average stress concentration tensor $\mathbf{T}^*$ will be compared with the exact one $\mathbf{T}^1$.

Similarly, there are two ways to determine the local stress field in the matrix with each of the interface effects. The first is to solve the boundary-value problem directly, and the local stress field in the matrix obtained by this way is the exact one, denoted by $\boldsymbol{\sigma}^2(\mathbf{x})$ (Fig. 1(a)), which can be related to the remote stress by $\boldsymbol{\sigma}^2(\mathbf{x}) = \mathbf{T}^2(\mathbf{x}) : \boldsymbol{\sigma}^0$, where $\mathbf{T}^2(\mathbf{x})$ is given in the paper of Duan et al. [10]. The second is to obtain the local stress field $\boldsymbol{\sigma}_*^2(\mathbf{x})$ in the matrix by the Eshelby equivalent inclusion method, and the result obtained by this way is an approximate one, denoted by $\boldsymbol{\sigma}_*^2(\mathbf{x}) = \bar{\mathbf{T}}^2(\mathbf{x}) : \boldsymbol{\sigma}^0$. $\bar{\mathbf{T}}^2(\mathbf{x})$ can be obtained using the superposition procedure shown in Figs. 1(b) and 1(c), with the equivalent eigenstrain $\boldsymbol{\varepsilon}^*$ solved from Eqs. 5 and 6.

Comparison of Exact and Approximate Stress Fields

For both of the interface stress model (ISM) and the linear-spring model (LSM), it is found that the dilatational component α^* of the approximate average stress concentration tensor $\mathbf{T}^*$ is identical to that of the exact average stress concentration tensor $\mathbf{T}^1$. The expression of the deviatoric component β^* is different from that of the exact counterpart tensor. Thus, the accuracy of the deviatoric component β^* is examined by comparing it with its exact counterpart, denoted by β_1. Generally, β^* is a function of μ_1/μ_2, ν_1, ν_2, λ_s, μ_s and *R*, where *R* is the radius of the spherical inhomogeneity. Here we take $\nu_1 = \nu_2 = 0.3$. As it has been proved that the interface stress model can accurately approximate a thin and stiff interphase [13,14]. When the interphase is thin and stiff, λ_s and μ_s can be calculated from the formulas $\lambda_s = 2\mu_I \nu_I t/(1-\nu_I)$, $\mu_s = \mu_I t$, where μ_I and ν_I are the shear modulus and the Poisson ratio of the interphase, respectively, and *t* is the interphase thickness. Fig. 2 shows the variations of the ratios β_1/β_0 and β^*/β_0 with the stiffness ratio $\eta = \log_{10}(\mu_1/\mu_2)$ of the inhomogeneity and matrix for two sets of interphase properties, namely, I: $t = 0.01R$, $\nu_I = 0.3$, $\mu_I = 20\mu_1$; II: $t = 0.01R$, $\nu_I = 0.3$, $\mu_I = 5\mu_1$. β_0 is the deviatoric component of the stress concentration tensor without the interface effect. It is seen that the values of β^* and β_1 are practically identical for the two cases. The numerical results for other cases also exhibit this feature. Therefore, we can conclude that the approximate average stress concentration tensor $\mathbf{T}^*$ obtained using the Eshelby equivalent inclusion method in the above volume average sense is very accurate.

Now we compare the local elastic fields $\boldsymbol{\sigma}_*^2(\mathbf{x})$ and $\boldsymbol{\sigma}^2(\mathbf{x})$ in the matrix for the spherical inhomogeneity with the interface stress model (ISM). It is found that under remote hydrostatic loading $\boldsymbol{\sigma}^0 = \sigma^0 \mathbf{I}^{(2)}$ the approximate stress field $\boldsymbol{\sigma}_*^2(\mathbf{x})$ and the exact solution $\boldsymbol{\sigma}^2(\mathbf{x})$ in the matrix are identical; they are different under remote deviatoric loading. Therefore, we compare the radial stresses σ_{rr}^2 and σ_{rr*}^2 in the matrix under remote shear loading $\sigma_{xy}^0 = \sigma^0$. In this case, it is expedient to express them in the following forms in the spherical coordinate system:

$$\sigma_{rr}^2 = T_{rr} \sin^2\theta \sin 2\varphi, \quad \sigma_{rr*}^2 = T_{rr}^* \sin^2\theta \sin 2\varphi \tag{9}$$

where T_{rr} and T_{rr}^* are the amplitudes of σ_{rr}^2 and σ_{rr*}^2, respectively. Fig. 3 shows the variations of T_{rr} and T_{rr}^* along the radial direction for two sets of interphase properties, namely, I: $t = 0.01R$, $\nu_I = 0.3$, $\mu_I = 100\mu_1$; II: $t = 0.01R$, $\nu_I = 0.3$, $\mu_I = 20\mu_1$. λ_s and μ_s are obtained in the same way as for Fig. 2. It can be seen from Fig. 3 that the exact and approximate solutions of T_{rr} and T_{rr}^* are very close to each other for each of the considered interphase properties. The numerical results for other stress components in the matrix also exhibit this feature. Therefore, we can conclude that the approximate stress field $\boldsymbol{\sigma}_*^2(\mathbf{x})$ is also very accurate.

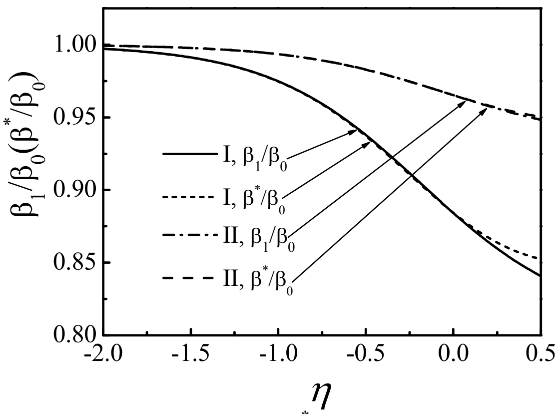


Fig. 2. Comparison of β^* and β_1 for ISM

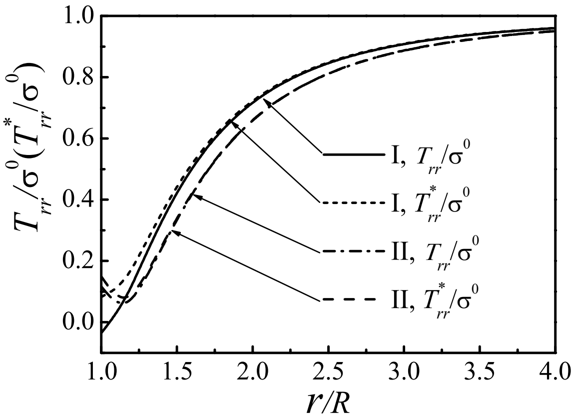
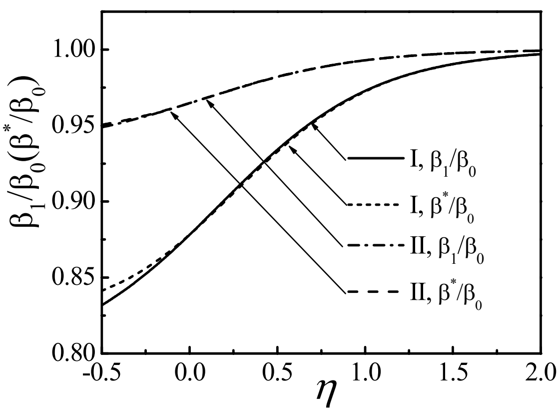


Fig. 3. Comparison of $\boldsymbol{\sigma}_*^2(\mathbf{x})$ and $\boldsymbol{\sigma}^2(\mathbf{x})$ for ISM



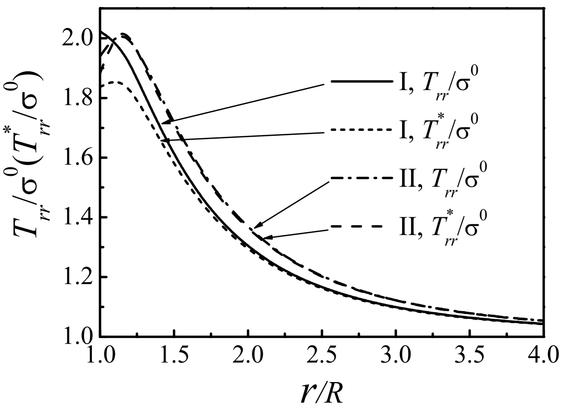


Fig. 4. Comparison of β^* and β_1 for LSM

Fig. 5. Comparison of $\boldsymbol{\sigma}_*^2(\mathbf{x})$ and $\boldsymbol{\sigma}^2(\mathbf{x})$ for LSM

We have also compared the approximate and exact solutions for the linear-spring model (LSM), shown in Figs. 4 and 5. Fig. 4 corresponds to Fig. 2, and Fig. 5 corresponds to Fig. 3. They have the similar features to those of the results for the interface stress model. However, it should be noted that the linear-spring model can be equivalent to a thin and soft interphase [1,14]. Thus, the spring constants α and β in Eq. 2 for the linear-spring interface model can be calculated from the formulas $\alpha = \mu_I / t$, $\beta = 2\mu_I(1-\nu_I)/[t(1-2\nu_I)]$. Fig. 4 shows the variations of the ratios β_1/β_0 and β^*/β_0 with the stiffness ratio $\eta = \log_{10}(\mu_1/\mu_2)$ of the inhomogeneity and matrix for two soft interphases, namely, I: t=0.01R, $\nu_I = 0.3$, $\mu_I = 0.05\mu_1$; II: t=0.01R, $\nu_I = 0.3$, $\mu_I = 0.2\mu_1$. Fig.5 shows the variations of T_{rr} and T_{rr}^* along the radial direction for two soft interphases, I: t=0.01R, $\nu_I = 0.3$, $\mu_I = 0.01\mu_1$; II: t=0.01R, $\nu_I = 0.3$, $\mu_I = 0.05\mu_1$. It can be seen from Fig.5 that the exact solutions and approximate solutions of T_{rr} and T_{rr}^* are very close.

Thus, we can conclude that the equivalent inclusion method for the spherical inhomogeneity with the interface stress model (ISM) and the linear-spring model (LSM) is very accurate, and it can be used to predict the effective properties of composites within the Eshelby formalism.

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