Fundamentals of Control Theory

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Lecture 0. Introduction of control theory

Why study control

Q: How to design a stable flight?

Figure: Q: History of control?

Origins and history

- Parts of control theory can be traced to 19th century.
- Builds on classical circuits & systems at Bell labs (1920s on) (transfer functions ...) but with more emphasis on linear algebra;
- Classical control (1900-1960s); modern control (1960-1970s); vast control methods since 1980s.
- First engineering application using classical control: aerospace (?), 1903.
- First engineering application using modern control: aerospace (definitely), 1960s.
- Transitioned from specialized topic to ubiquitous in 1980s (just like digital signal processing, information theory, etc.).

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Have no idea and start feeling boring?

Control is in the nature as well

Q: Who is more maneuverable, bee or bird?

Figure: Q: What is the difference for a bee flight?

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Figure: Q: What is the difference for a dragonfly flight?

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Figure: Q: What is the difference for a dragonfly flight?

Well, interesting, but do NOT know answers, and how to find/develop answers.

Let's study the fundamentals.

Linear dynamical system

Continuous-time linear dynamical system (LDS) has the form

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{A(t)x(t)} + \mathrm{B(t)u(t)}, \quad \mathrm{y(t)} = \mathrm{C(t)x(t)} + \mathrm{D(t)u(t)}.$$

$$\mathrm{Y(S)} = \mathrm{G(S)U(S)}.$$

Introduction

System dynamics

- Control systems are dynamic $\longrightarrow$ transient + steady state.
- Transient motion is important. In the case of a flight, slow transient motion makes passenger impatient, rapid motion is uncomfortable. If the aircraft oscillates this is disconcerting.
- Steady state is also important, it can have an error, i.e. the step up or down on the exit form a lift.

Introduction

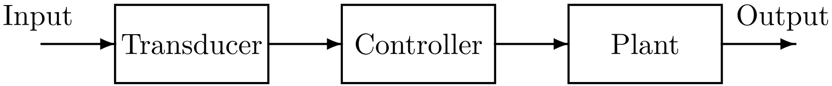

Figure: Open loop system diagram.

e.g. heaters, fan, etc.
Q: What is the diagram for a fan in a classroom?

Transfer function

- A mathematical representation of the relation between the input and output of a linear time-invariant system (will be defined later).
- Q: If above system is linear, what is its transfer function?

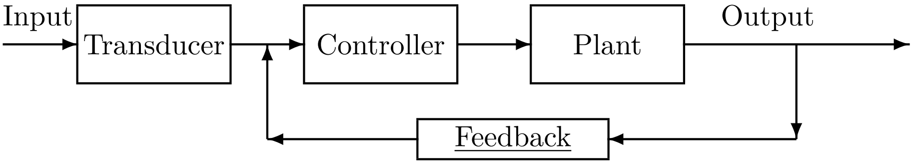

Figure: Closed loop system diagram.

- Q: What is the diagram when we adjust room temperature using a fan? Human-in-the-loop.
- Q: What is its transfer function?
- Q: What is the performance difference between an open- and closed-loop system?

Introduction

Design process

1. Transform requirements to mathematical words;
2. Develop the mathematical model of the plant;
3. Create a schematic of the system;
4. Analyze and design.

Dynamics

We must have a dynamic system:

Total = Natural + Forced

$\downarrow$

describes how the system dissipates energy

This means that the equivalent differential equation equals a homogeneous solution plus a particular solution.

Demo 1

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = [1\ 0]',$$

where $A = [0\ 1; -a^2 - \omega^2\ \ -2a]$; $B = [1\ 0; 0\ 1]$; and $C = [1\ 0; 0\ 1]$.

- Q: What is the block diagram of the above equation?
- Q: Is the above equation linear?

Demo 2

Q: Given desired outputs $y_{des} = [0.36; -0.65]$, how to design a u?
A simple approach: consider static conditions $(u, x, y$ constant):

$$\dot{x} = 0 = Ax + Bu_{static}, \quad y = y_{des} = Cx.$$

solve for u to get:

$$u_{static} = \left(-CA^{-1}B\right)^{-1} y_{des} = \begin{bmatrix} 0.36 \\ -0.65 \end{bmatrix}.$$

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Super cool! It seems we do NOT need to learn this course!

Demo 2

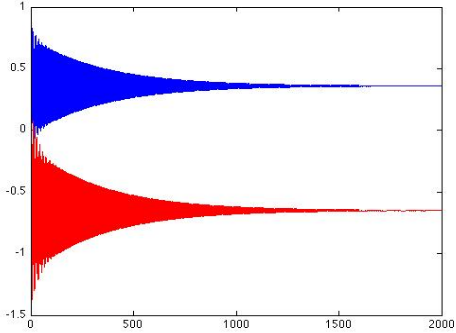

Figure: Forced responses.

Q: Is it acceptable?

Input design

In this course we'll study

- how to synthesize or design such inputs;
- and the tradeoff between size of u and convergence time.

Linear dynamics system

Let us look at the previous demo again,

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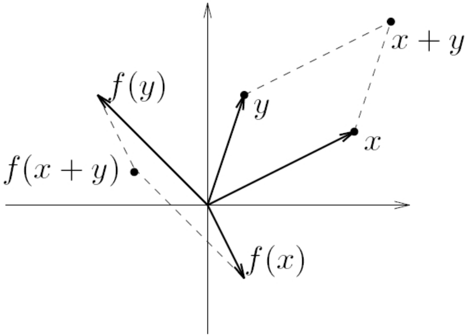
- Q: Is the above equation linear?
- Q: What is a linear dynamical system?

Linear functions

a function $f : \mathfrak{R}^n \rightarrow \mathfrak{R}^m$ is linear if

- $f(x + y) = f(x) + f(y), \forall x, y \in \mathfrak{R}^n$
- $f(\alpha x) = \alpha f(x), \forall x \in \mathfrak{R}^n, \forall \alpha \in \mathfrak{R}$

i.e., superposition holds.



Linear equations

Consider system of linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n, \end{aligned}$$

can be written in matrix form as $y = Ax$, where

$$y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Matrix multiplication function

- Consider function $f : \mathfrak{R}^n \to \mathfrak{R}^m$ given by $f(x) = Ax$, where $A \in \mathfrak{R}^{m\times n}$;
- matrix multiplication function f is linear;
- converse is true: any linear function $f : \mathfrak{R}^n \to \mathfrak{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathfrak{R}^{m\times n}$;
- representation via matrix multiplication is unique: for any linear function f there is only one matrix A for which $f(x) = Ax$ for all x;
- $y = Ax$ is a concrete representation of a generic linear function.

Interpretations of $y = Ax$

- y is measurement or observation; x is unknown to be determined;
- x is ‘input’ or ‘action’; y is ‘output’ or ‘result’;
- $y = Ax$ defines a function or transformation that maps $x \in \mathfrak{R}^n$ into $y \in \mathfrak{R}^m$.

Interpretations of a_{ij}

$$y_i = \sum_{j=1}^{n} a_{ij} x_j,$$

a_{ij} is gain factor from jth input (x_j) to ith output (y_i),
thus, e.g.,

- ith row of A concerns ith output;
- jth column of A concerns jth input;
- $a_{27} = 0$ means 2nd output (y_2) doesn't depend on 7th input (x_7);
- $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means y_3 depends mainly on x_1.

Interpretations of a_{ij}

- $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means x_2 affects mainly y_5;
- A is lower triangular, i.e., $a_{ij} = 0$ for $i < j$, means y_i only depends on $x_1, \ldots, x_i$;
- A is diagonal, i.e., $a_{ij} = 0$ for $i \neq j$, means ith output depends only on ith input.

More generally, sparsity pattern of A, i.e., list of zero / nonzero entries of A, shows which x_j affect which y_i
Consider its meaning in aircraft control...

Linear $\neq$ simple

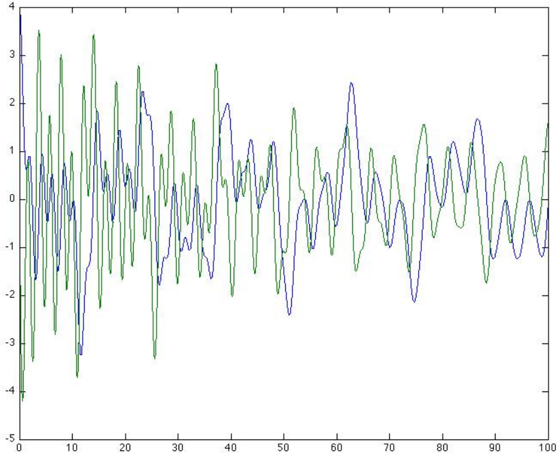


Figure: Q: Is this a linear or nonlinear system?

Linear ≠ simple

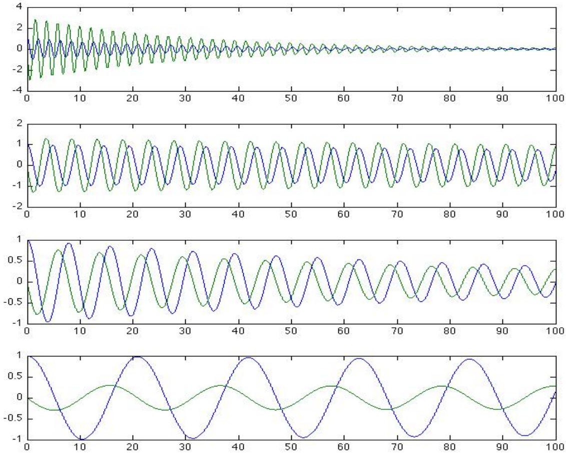


Figure: Linear modes.

Nonlinear dynamical systems

Many dynamical systems are nonlinear (a fascinating topic) so why study linear systems?

- Most techniques for nonlinear systems are based on linear methods.
- Methods for linear systems often work unreasonably well, in practice, for nonlinear systems.
- If you don't understand linear dynamical systems you certainly can't understand nonlinear dynamical systems.

Linearization of nonlinear dynamic systems

- If $f : \Re^n \to \Re^m$ is differentiable at $x_0 \in \Re^n$, then

$$x \quad \text{near } x_0 \Longrightarrow f(x) \quad \text{very near} \quad f(x_0) + Df(x_0)(x - x_0),$$

  where

$$Df(x_0)_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_0},$$

  is derivative (Jacobian) matrix,
- with $y = f(x), y_0 = f(x_0)$, define input deviation $\delta x := x - x_0$, output deviation $\delta y := y - y_0$
- then we have $\delta y \approx Df(x_0)\delta x$,
- when deviations are small, they are (approximately) related by a linear function.

Why study linear systems?

Applications arise in many areas, e.g.

- automatic control systems;
- signal processing;
- communications;
- economics, finance;
- circuit analysis, simulation, design;
- mechanical and civil engineering;
- aeronautics;
- navigation, guidance.

Linear dynamical system

continuous-time linear dynamical system (LDS) has the form

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t),$$

where:

- $t \in \mathfrak{R}$ denotes time
- $x(t) \in \mathfrak{R}^n$ is the state (vector)
- $u(t) \in \mathfrak{R}^m$ is the input or control
- $y(t) \in \mathfrak{R}^p$ is the output

Please note the above system is time variant!

Linear dynamical system

- $A(t) \in \mathfrak{R}^{n\times n}$ is the dynamics matrix
- $B(t) \in \mathfrak{R}^{n\times m}$ is the input matrix
- $C(t) \in \mathfrak{R}^{p\times n}$ is the output or sensor matrix
- $D(t) \in \mathfrak{R}^{p\times m}$ is the feedthrough matrix

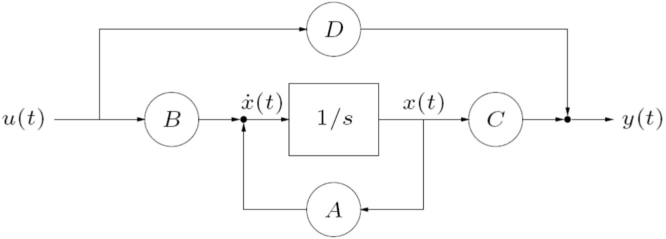
For lighter (simpler) appearance, equations are often written

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

- LDS is a first order vector differential equation,
- also called state equations, or 'm-input, n-state, p-output' LDS.

Q: Draw the block diagram of the above system.

Block diagram

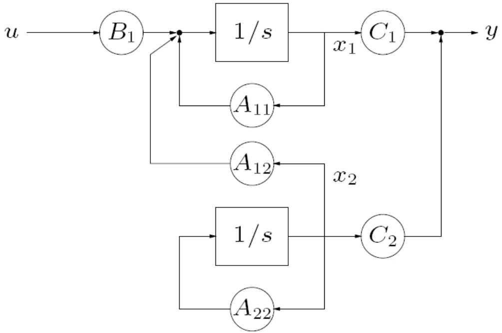


- A_{ij} is gain factor from state x_j into integrator i;
- B_{ij} is gain factor from input u_j into integrator i;
- C_{ij} is gain factor from state x_j into output y_i;
- D_{ij} is gain factor from input u_j into output y_i.

Block diagram

Q: Draw block diagram for

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



- x_2 is not affected by input u, i.e., x_2 propagates autonomously;
- x_2 affects y directly and through x_1.

Some LDS terminology

- Most linear systems encountered are time-invariant: A, B, C, D are constant, i.e., don't depend on t.
- When there is no input u (hence, no B or D) system is called autonomous.
- Very often there is no feedthrough, i.e., $D = 0$.
- When $u(t)$ and $y(t)$ are scalar, system is called single-input, single-output (SISO); when input & output signal dimensions are more than one, MIMO.

Summary

- Linear vs Nonlinear.
- Continuous vs Discrete.
- SISO vs MIMO.
- Time domain vs Frequency domain.

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- SISO vs MIMO.
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We will go through linear, continuous, SISO and MIMO, and both time domain and frequency domain systems.