Fundamentals of Control

State space and Laplace transform

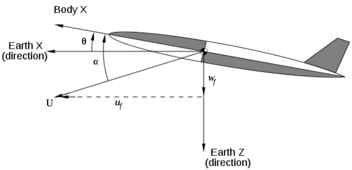
Professor Xun Huang

huangxun@pku.edu.cn

Notation

- l, d: lift and drag;
- $\mathrm{C_l}, \mathrm{C_d}$: non-dimensional lift and drag coefficients;
- $\mathrm{L}, \mathrm{M}, \mathrm{N}$: roll, pitch, yaw moments;
- $\mathrm{C_L}, \mathrm{C_M}, \mathrm{C_N}$: non-dimensional coefficients of roll, pitch, yaw moments;
- $\mathrm{C}_{\mathrm{l}_\alpha} = \partial \mathrm{C_l} / \partial \alpha$.

Review: longitudinal and lateral angles



Longitudinal Equations of Motion

$$\theta = \alpha + \gamma, \quad \xi = \phi + \beta$$

$\mathrm{u_f} = \mathrm{U}\cos(\theta - \alpha); \mathrm{w_f} = \mathrm{U}\sin(\theta - \alpha)$. The forces in inertial axes are:
$\mathrm{X_f} = \mathrm{m}\frac{\mathrm{du_f}}{\mathrm{dt}} = \mathrm{m}\frac{\mathrm{dU}}{\mathrm{dt}}\cos(\theta - \alpha) - \mathrm{mU}\frac{\mathrm{d}(\theta-\alpha)}{\mathrm{dt}}\sin(\theta - \alpha)$,
$\mathrm{Z_f} = \mathrm{m}\frac{\mathrm{dw_f}}{\mathrm{dt}} = \mathrm{m}\frac{\mathrm{dU}}{\mathrm{dt}}\sin(\theta - \alpha) + \mathrm{mU}\frac{\mathrm{d}(\theta-\alpha)}{\mathrm{dt}}\cos(\theta - \alpha)$. The speed variation $\mathrm{m}\frac{\mathrm{dU}}{\mathrm{dt}}$ is negligible over the period of the oscillation, so:
$\mathrm{X_f} = -\mathrm{mU}\frac{\mathrm{d}(\theta-\alpha)}{\mathrm{dt}}\sin(\theta - \alpha)$,
$\mathrm{Z_f} = \mathrm{mU}\frac{\mathrm{d}(\theta-\alpha)}{\mathrm{dt}}\cos(\theta - \alpha)$.

Review: How many variables should be used to describe the complete aircraft?

$$[\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \phi, \theta, \psi, \mathrm{p}, \mathrm{q}, \mathrm{r}]$$

We can further divide the variables into two groups:
The corresponding dynamic modes are:
(range; height; phugoid; short-period)

Review: How many variables should be used to describe the complete aircraft?

$$[\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \phi, \theta, \psi, \mathrm{p}, \mathrm{q}, \mathrm{r}]$$

We can further divide the variables into two groups:
The corresponding dynamic modes are:
(range; height; phugoid; short-period)
and
(cross-range; heading; roll; spiral; dutch roll mode).

Today's objective is to understand the analyse method of the longitudinal dynamics.

Review: The simplest longitudinal model

$$\begin{aligned} m(\dot{w} - qU_0) &= F_z, && (1) \\ I_{yy}\dot{q} &= M. && (2) \end{aligned}$$

Review: The simplest longitudinal model

$$\begin{aligned} m(\dot{w}' - q' U_0) &= Z^c, & (3) \\ I_{yy}\dot{q}' &= M_w w' + M_q q' + M^c. & (4) \end{aligned}$$

Assume we only have one aerodynamic device: rudder, i.e.

$$\delta Z^c = m Z_d \delta r,$$

$$\delta M^c = I_{yy} M_r \delta r.$$

Review: The simplest longitudinal model

$$m(\dot{w}' - q'U_0) = Z^c, \quad (3)$$
$$I_{yy}\dot{q}' = M_w w' + M_q q' + M^c. \quad (4)$$

Assume we only have one aerodynamic device: rudder, i.e.

$$\delta Z^c = m Z_d \delta r,$$

$$\delta M^c = I_{yy} M_r \delta r.$$

How to analysis the dynamics?

To answer this question, we need to use methods in linear system.

Linear dynamical system

Continuous-time linear dynamical system (LDS) has the form

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{A(t)x(t)} + \mathrm{B(t)u(t)}, \quad \mathrm{y(t)} = \mathrm{C(t)x(t)} + \mathrm{D(t)u(t)}.$$

Linear dynamical system

Continuous-time linear dynamical system (LDS) has the form

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{A(t)x(t) + B(t)u(t)}, \quad \mathrm{y(t) = C(t)x(t) + D(t)u(t)}.$$

If $\mathrm{A, B, C, D}$ are time-invariant, we can have the corresponding transfer function:

$$\mathrm{Y(S) = G(S)X(S)}.$$

State space model and more...

1. The above cases are represented by state space models;
2. They are in the time domain and the bare bone of modern control.
3. In the very beginning of control (so-called classical control), frequency domain is more preferred.
4. Time domain is still a part of classical control.
5. Next we will present primitive mathematical knowledge (Laplace transform).
6. Another reason to study Laplace transform is to achieve transfer function for state space model.

Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Laplace Integral, Q: why?

Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Laplace Integral, Q: why?

- Conservation;
- Causality:
  The relationship between an event (the cause) and a second event (the effect), where the second event is understood as a consequence of the first;
  Current state (x(t)) and output (y(t)) depending on past input (u(τ) for $\tau \leq t$) is causal;

Laplace transform

$$\mathcal{L}[\mathrm{f(t)}] = \mathrm{F(s)} = \int_{0-}^{\infty} \mathrm{f(t)e^{-st}dt}$$

Laplace Integral, Q: why?

- Conservation;
- Causality:
  The relationship between an event (the cause) and a second event (the effect), where the second event is understood as a consequence of the first;
  Current state (x(t)) and output (y(t)) depending on past input (u(τ) for $\tau \leq$ t) is causal;
  Current state (and output) depending on future input is anti-causal.

Laplace transform

1. $\mathcal{L}[tf(t)] = -F'(s)$
2. $\mathcal{L}[e^{-at}f(t)] = F(s+a)$ Frequency Shift
3. $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$ Time Shift
4. $\mathcal{L}\left[f(at)\right] = \frac{1}{a}F(\frac{s}{a})$ Scaling
5. $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$ Differentiation
6. $\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$ Integration Theorem
7. $f(\infty) = \lim_{s\to 0}(sF(s))$ Final Value Theorem
8. $f(0+) = \lim_{s\to\infty}(sF(s))$ Initial Value Theorem

Key points

- State space model.
- Given a dynamic model, write down its state space model. Example: Eqs. (3-4).
- Laplace transform.

Homeworks

Prove (2), (7), (8).