Fundamentals of Control

Transfer function and reference inputs

Professor Xun Huang

huangxun@pku.edu.cn

Review

- State space mode (quiz?);
- Laplace transform.

$$\mathcal{L}[\mathrm{f(t)}] = \mathrm{F(s)} = \int_{0-}^{\infty} \mathrm{f(t)e^{-st}dt}.$$

- Laplace transform:

$$\mathcal{L}[\mathrm{f(t)}] = \mathrm{F(s)} = \int_{0-}^{\infty} \mathrm{f(t)e^{-st}dt}.$$

Inverse:

$$\mathcal{L}^{-\infty}[\mathrm{F(s)}] = \mathrm{f(t)} = \frac{1}{2\pi \mathrm{i}} \mathrm{lim}_{\mathrm{T}\to\infty} \int_{\gamma-\mathrm{iT}}^{\gamma+\mathrm{iT}} \mathrm{F(s)e^{st}ds}.$$

Q: How to understand?

Not easy to do. So, usually, we reply on a LT table...

Laplace table

Function	Laplace Transform
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$

Table: Table of Laplace Transforms

Laplace table

$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2 + \omega^2}$
$e^{-kt}\sin(\omega t)u(t)$	$\frac{\omega}{(s + k)^2 + \omega^2}$
$e^{-kt}\cos(\omega t)u(t)$	$\frac{s + k}{(s + k)^2 + \omega^2}$

Table: Table of Laplace Transforms

Laplace table

$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2 + \omega^2}$
$e^{-kt}\sin(\omega t)u(t)$	$\frac{\omega}{(s+k)^2 + \omega^2}$
$e^{-kt}\cos(\omega t)u(t)$	$\frac{s+k}{(s+k)^2 + \omega^2}$

Table: Table of Laplace Transforms

Q: What is the previous state space model using Laplace transform?

Laplace transform of matrix valued function

Suppose $z : \mathfrak{R}_+ \to \mathfrak{R}^{p\times q}$

Laplace transform: $Z = \mathfrak{L}(z)$, where $Z : D \subseteq \mathfrak{C} \to \mathfrak{C}^{p\times q}$ is defined by

$$Z(s) = \int_0^\infty e^{-st} z(t) dt.$$

- integral of matrix is done term-by-term;
- convention: upper case denotes Laplace transform;
- D is the domain or region of convergence of Z;
- D includes at least $\{s \mid \mathrm{Re}(s) > a\}$, where a satisfies $|z_{ij}(t)| \leqslant \alpha e^{\alpha t}$ for $t \geqslant 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q$.

Derivative property

$\mathfrak{L}(\dot{z}) = sZ(s) - z(0)$

to derive, integrate by parts:

$$\begin{aligned}\mathfrak{L}(\dot{z})(s) &= \int_0^\infty e^{-st}\dot{z}(t)dt \\ &= e^{-st}z(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st}z(t)dt \\ &= sZ(s) - z(0).\end{aligned}$$

Laplace transform solution of $\dot{x} = Ax$

Consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax,$$

for $t \geqslant 0$, where $x(t) \in \mathfrak{R}^n$,

- take Laplace transform: $sX(s) - x(0) = AX(s)$;
- rewrite as $(sI - A)X(s) = x(0)$;
- hence $X(s) = (sI - A)^{-1}x(0)$;
- take inverse transform

$$x(t) = \mathfrak{L}^{-1}\big((SI - A)^{-1}\big)x(0).$$

Resolvent and state transition matrix

- $(sI - A)^{-1}$ is called the resolvent of A.
- Resolvent defined for $s \in \mathfrak{C}$ except eigenvalues of A, i.e., s such that $\det(SI - A) = 0$.
- $\Phi(t) = \mathfrak{L}^{-1}\big((SI - A)^{-1}\big)$ is called the state-transition matrix; it maps the initial state to the state at time t:

$$x(t) = \Phi(t)x(0).$$

  In other words, x(t) is a linear function of initial state x(0).

- Prepare a MATLAB demo of expm command.

Transfer function

Take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s).$$

Hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s),$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau.$$

- $e^{tA}x(0)$ is the unforced or autonomous response;
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix;
- $(sI - A)^{-1}B$ is called the input-to-state transfer function or transfer matrix.

Transfer function

With $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s),$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)d\tau + Du(t).$$

- Output term $Ce^{tA}x(0)$ due to initial condition;
- $G(s) = C(sI - A)^{-1}B + D$ is called the transfer function or transfer matrix;
- $g(t) = Ce^{tA}B + D\delta(t)$ is called the impulse response or impulse matrix (δ is the Dirac delta function)

Transfer function

With zero initial condition we have:

$$Y(s) = G(s)U(s), \quad y = h * u,$$

where $*$ is convolution (of matrix valued functions).

Interpretation:

- G_{ij} is transfer function from input u_j to output y_i.

Inputs

The step function is defined by:

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases}$$

The ramp function is defined as:

$$tH(t) = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{for } t \geq 0 \end{cases}.$$

Q: Why choose these signals as inputs?
Let us go back to Laplace transform table.

Plant Outputs–Example One

If the following function is defined from a mechanical analogy, determine the time domain signal. $\mathrm{F_1(s)} = \frac{1}{(\mathrm{s}+3)^2}$.

Use the Laplace Transform table:

$$\mathrm{F(s)} = \frac{1}{\mathrm{s}^2} \Rightarrow \mathrm{tH(t)}.$$

Using the frequency shift theorem:

$$\mathrm{F(s+a)} = \frac{1}{(\mathrm{s}+\mathrm{a})^2} \Rightarrow \mathrm{e}^{-\mathrm{at}}\mathrm{tH(t)}.\ \mathcal{L}^{-1}\left(\frac{1}{(\mathrm{s}+3)^2}\right) \Rightarrow \mathrm{e}^{-3\mathrm{t}}\mathrm{tH(t)}.$$

Q: What is the input signal?

Q: Draw the plot.

Plant Outputs–Example Two

Derive the output time history described by the Laplace function $F_1(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$.

Thus by using the Laplace tables we see:

$f(t) = 2(e^{-t} - e^{-2t})H(t)$.

Q: Draw the plot.

Plant Outputs–Example Three

If the following output time history has been defined, what is the time domain signal? $F_1(s) = \frac{(s^3 + 2s^2 + 6s + 7)}{(s^2 + s + 5)}$.

$$F_1(s) = s + 1 + \frac{2}{(s^2 + s + 5)}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathcal{L}^{-1}\left(\frac{2}{(s^2 + s + 5)}\right)$$

By completing the squares:

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathcal{L}^{-1}\left(\frac{2}{(s + 0.5)^2 + 4.75}\right).$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \frac{2}{\sqrt{4.75}}\mathcal{L}^{-1}\left(\frac{\sqrt{4.75}}{(s + 0.5)^2 + (\sqrt{4.75})^2}\right),$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + 0.9177e^{-0.5t}\sin(2.179t)H(t).$$

Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau,$$

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)d\tau + Du(t),$$

for $t \geqslant 0$:

current state ($x(t)$) and output ($y(t)$) depend on past input ($u(\tau)$ for $\tau \leqslant t$)

i.e., mapping from input to state and output is causal (with fixed initial state).

Causality

now consider fixed final state $x(T)$: for $t \leqslant T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau)d\tau,$$

i.e., current state (and output) depend on future input!

So for fixed final condition, the same system is anti-causal.

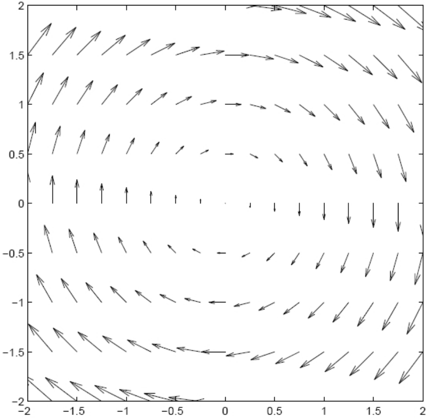
Idea of state

$x(t)$ is called state of system at time t since:

- future output depends only on current state and future input;
- future output depends on past input only through current state;
- state summarizes effect of past inputs on future output;
- state is bridge between past inputs and future outputs.

Math case – harmonic oscillator

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathrm{x}.$$



Math case – harmonic oscillator

$\mathrm{SI} - \mathrm{A} = \begin{bmatrix} \mathrm{s} & -1 \\ 1 & \mathrm{s} \end{bmatrix}$, so resolvent is

$$(\mathrm{sI} - \mathrm{A})^{-1} = \begin{bmatrix} \frac{\mathrm{s}}{\mathrm{s}^2+1} & \frac{1}{\mathrm{s}^2+1} \\ \frac{-1}{\mathrm{s}^2+1} & \frac{\mathrm{s}}{\mathrm{s}^2+1} \end{bmatrix}.$$

(eigenvalues are $\pm \mathrm{i}$).

State transition matrix is

$$\Phi(\mathrm{t}) = \mathfrak{L}^{-1}\left(\begin{bmatrix} \frac{\mathrm{s}}{\mathrm{s}^2+1} & \frac{1}{\mathrm{s}^2+1} \\ \frac{-1}{\mathrm{s}^2+1} & \frac{\mathrm{s}}{\mathrm{s}^2+1} \end{bmatrix}\right) = \begin{bmatrix} \cos\mathrm{t} & \sin\mathrm{t} \\ -\sin\mathrm{t} & \cos\mathrm{t} \end{bmatrix}.$$

A rotation matrix ($-\mathrm{t}$ radians).

so we have $\mathrm{x}(\mathrm{t}) = \begin{bmatrix} \cos\mathrm{t} & \sin\mathrm{t} \\ -\sin\mathrm{t} & \cos\mathrm{t} \end{bmatrix} \mathrm{x}(0)$.

- Transfer function.
- State space.
- From state space to transfer function.