# Fundamentals of Control

## Frequency-domain methods for performance and stability analysis

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- Laplace and its inverse transforms.
- Transfer function vs State space model.
- Given $\dot{x} = Ax$, we will have $x(t) = e^{At}x(0)$. Since $A = T\Lambda T^{-1}$, then

$$e^{At} = Te^{\Lambda t}T^{-1}.$$

Why?

## Review

$$e^{At} = Te^{\Lambda t}T^{-1} = [v_1 \cdots v_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & \cdots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} w_1^T \\ \cdot \\ w_n^T \end{bmatrix}.$$

- The following derivation is given on the blackboard...
- The eigenvalues determine the performance and stability of the system! (See note)
- The corresponding concept in transfer function are pole...

## Poles, Zeros and System Response

- Solutions of a problem can be time consuming and laborious, so often a short cut is taken to find the qualitative behaviour. The use of poles and zeros is one such method.
- Poles are where the Laplace Transform becomes infinite. For example,
  $\frac{1}{(s+2)}$ at $s = -2$ has a value of $\infty$. This means $s = -2$ is a pole.
- Zeros are where the Laplace Transform is zero! $s + 4 \longrightarrow s = -4$ is a zero.

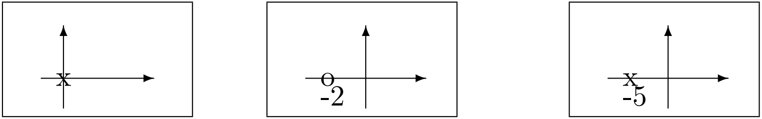
$$R(s) \longrightarrow \boxed{\frac{(s+2)}{(s+5)}} \longrightarrow C(s)$$

Figure: Simple Input / Output Control System.

## Poles, Zeros and System Response

Take for example a system, described by the Laplace variable s, with the following expression.

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{\frac{2}{5}}{s} + \frac{\frac{3}{5}}{(s+5)}.$$




Input Pole | System Zero | System Pole

Figure: Poles and zero positions.

## First Order Systems

We can model the effects of a simple system, often refereed to as a first order system by displacement and velocity by the following diagram.

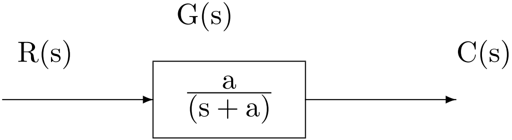



Figure: First order system.

## First Order Systems

If we assume a unit input, then $\mathrm{R(s)} = \frac{1}{\mathrm{s}}$,

$$\therefore \mathrm{C(s)} = \frac{\mathrm{a}}{\mathrm{s(s+a)}},$$

$$\therefore \mathrm{C(s)} = \frac{1}{\mathrm{s}} - \frac{1}{\mathrm{(s+a)}},$$

which means that in the time domain that the solution is:

$$\mathrm{c(t)} = \mathrm{c_f(t)} + \mathrm{c_n(t)},$$

$$\therefore \mathrm{c(t)} = 1 - \mathrm{e}^{-\mathrm{at}}.$$

## Second Order Systems

Demo: trim_linearize_xun
Obtain:

$$G(s) = \frac{w(s)}{\delta_{elevator}} = \frac{-0.09s - 7}{s^2 + 1.3s + 4.5}. \tag{1}$$

Then, we consider a more general second order system defined as

$$G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

We define the damping ratio $\zeta$, the natural frequency $\omega_n$, and the damped frequency,

$$\omega_d = \omega_n\sqrt{1-\zeta^2}.$$

Q: What are $\zeta$ and $\omega_n$ for Eq. (1).

## Second Order Systems

The we can characterize the systems as

1. $\zeta = 0$ is the no damping case. Poles at $\mathrm{s} = \pm \mathrm{i}\omega_\mathrm{n}$.
2. $0 < \zeta < 1$ then the system is under-damped. Poles at $\mathrm{s} = -\zeta\omega_\mathrm{n} \pm \mathrm{i}\omega_\mathrm{d}$.
3. $\zeta = 1$ the system is critically damped. Poles at $\mathrm{s} = -\zeta\omega_\mathrm{n}$ (twice).
4. $\zeta > 1$ the system is over-damped. Poles at $\mathrm{s} = -\zeta\omega_\mathrm{n} \pm \omega_\mathrm{n}\sqrt{\zeta^2 - 1}$.

# Second Order Systems

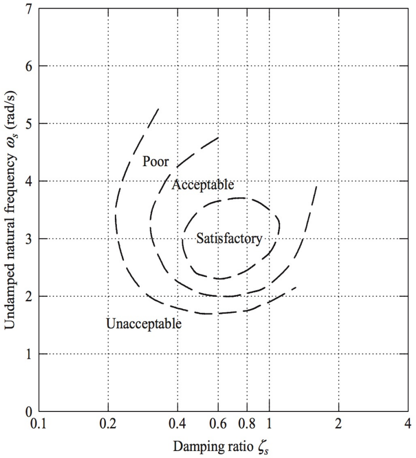



**Figure 10.2** *Longitudinal short period pilot opinion contours – the thumb print criterion.*

Why?

## Second Order Systems

In the case of under damped response we further define the following properties;

1. Rise time $T_r$ is the time between the output reaching 10% and 90% of the final value.
2. Time to the first peak, $T_P$ in the response is overshoot peak.

$$T_p = \frac{\pi}{\omega_n \sqrt{\zeta^2 - 1}}.$$

3. The percentage overshoot of the response at the first peak is given by,

$$OS\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}.$$

4. The settling time is the time to the third crossing of the 90% of final value.

$$T_s = \frac{4}{\zeta\omega_n}.$$

# Second Order Systems

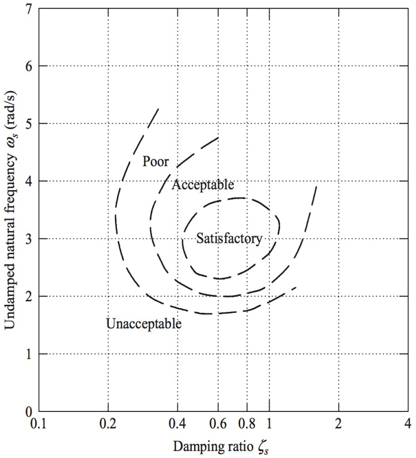



**Figure 10.2** *Longitudinal short period pilot opinion contours – the thumb print criterion.*

Show pzmap of the demo: trim_linearize_xun. What to do?

## Closed-Loop Control

Closed-loop system transfer function:

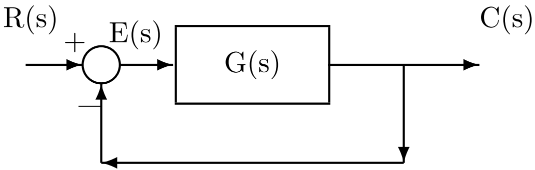



Figure: Closed loop system.

$$T(s) = \frac{C(s)}{R(s)},$$

$$\therefore T(s) = \frac{G(s)}{1 + G(s)}.$$

## Closed-Loop Systems Design

$$\mathrm{G(s)} = \frac{25}{\mathrm{s(s+5)}}, \therefore \mathrm{T(s)} = \frac{25}{\mathrm{s^2+5s+25}}.$$

This means that the natural frequency is

$$\omega_\mathrm{n} = \sqrt{25} = 5,$$

and the damping coefficient can be found from:

$$2\zeta\omega_\mathrm{n} = 5, \therefore \zeta = 0.5.$$

The time to first peak $\mathrm{T_p}$ is;

$$\mathrm{T_p} = \frac{\pi}{\omega_\mathrm{n}\sqrt{1-\zeta^2}} = 0.726, \therefore \mathrm{T_s} = \frac{4}{\zeta\omega_\mathrm{n}} = 1.6.$$

$$\therefore \%\mathrm{OS} = \mathrm{e}^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 16.303\%.$$

Q: How to $\leq 10\%$.

## Closed-Loop System Design

The above method tells the stability of a system, e.g. G(s). But the close loop gain k can be varied. How to analyze the whole closed-loop along with the gain?

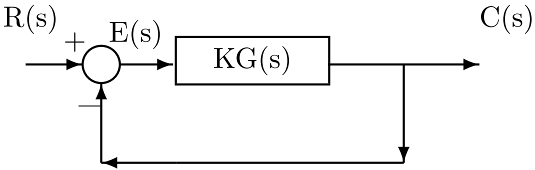



Figure: Closed loop system.

Q: What is the transfer function of the closed-loop?

## Closed-Loop Systems Design

$$G(s) = \frac{K}{s(s+5)}, \therefore T(s) = \frac{K}{s^2 + 5s + K}.$$

This means that the natural frequency is;

$$\omega_n = \sqrt{K}$$

The damping coefficient is therefore found:

$$2\zeta\omega_n = 5, \text{or} \zeta = \frac{5}{2\sqrt{K}}.$$

Thus is we have a system requirement that the first overshoot is to be a maximum of 10% then we can see that

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 10\%, \therefore \zeta = 0.591,$$

$$\therefore K = 17.982.$$

# Root Locus Method

- Given G(s), root locus method is a graphical method for examining how the roots of a system change with variation of the gain of a feedback system.
- The root locus method is one method for solving this difficult problem. To find the effects of the gain, K, we need to plot the locus of the poles as the gain varies from zero to $\infty$.

## Preliminary knowledge

$$\text{Given } F(s) = \frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{N}(s+p_j)},$$

which can be written to a magnitude and argand form;

$$F(s) = Me^{i\theta}, \text{then}$$

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}}, \theta = \sum_{i=1}^{M}\angle(s+z_i) - \sum_{j=1}^{N}\angle(s+p_j).$$

## Starting points

Here $F(s) = 1 + KG(s)$, where $G(s) = \frac{N_G(s)}{D_G(s)}$.
That is $D_G(s) + KN_G(s) = 0$, as $K \to 0$, $\therefore D_G(s) = 0$. Hence, the root locus starts at the poles of $G(s)$, i.e. the poles of open loop system.

## Ending points

$D_G(s) + KN_G(s) \approx KN_G(s)$, as $K \to \infty$, the ending points are at $N_G(s) = 0$, i.e. the zeros of open loop system.

## Root Locus Method

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Homework: Read the root locus part and learn how to do portrait by hand.

## Closed-Loop System Transfer Function

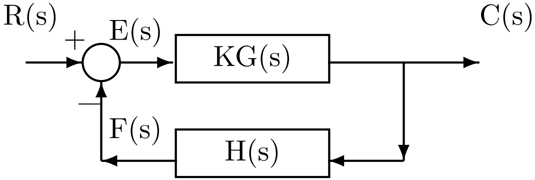



Figure: Closed loop system.

$$C(s) = E(s)KG(s), E(s) = R(s) - F(s) = R(s) - C(s)H(s),$$

$$C(s) = R(s)KG(s) - C(s)H(s)KG(s),$$

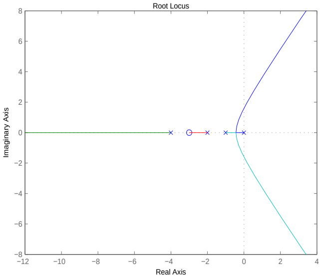
$$\therefore C(s)[1 + KH(s)G(s)] = KR(s)G(s), Q : T(s) =?$$

# Root Locus Method

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}, H(s) = 1$$

The MATLAB code can help us to do root locus,

```
>> numg=[1 3];
>> deng=[1 7 14 8 0];
>> sysS=tf(numg,deng);
>> rlocus(sysS);
```



# Recall: Second Order Systems

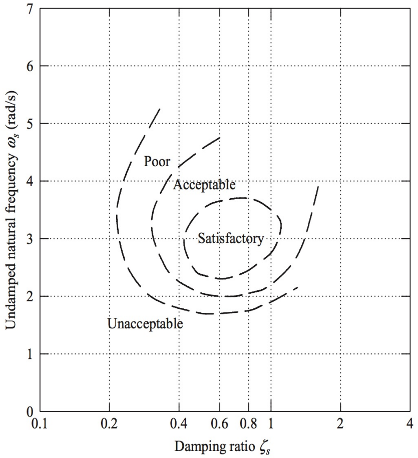



**Figure 10.2** *Longitudinal short period pilot opinion contours – the thumb print criterion.*

Show pzmap of the demo: trim_linearize_xun.

## Root Locus Method

First, examine "rlocus(tf1)".
Then, examine "tfs=tf([1],[1 0]); rlocus(tf1(1)*tfs)".

## Key Points

- 2nd-order system;
- Closed-loop system;
- Root locus method.