Problem 1: Spreading of an axisymmetric drop. A finite volume V of liquid (with density ρ and viscosity μ) is released on a horizontal rigid plate situated at $z = 0$. We neglect the effect of surface tension and consider the subsequent axisymmetric spreading of the drop under gravity. By considering the lubrication approximation in this geometry, show that the conservation of mass and kinematic boundary condition gives that the drop profile evolves according to

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{1}{r} \frac{\partial}{\partial r} \left(r h^3 \frac{\partial h}{\partial r} \right).$$

Using this equation and the total mass constraint

$$V = 2\pi \int_0^{R(t)} r h(r,t)\, \mathrm{d}r$$

to show

$$R(t) \sim \left(\frac{\rho g V^3}{\mu} t \right)^{1/8}$$

by a scaling argument where $R(t)$ is the drop radius. Read the notes I sent you to see how to find out similarity solutions to this system from which you may find

$$R(t) = \left(\frac{2^{10}}{3^5 \pi^3} \right)^{1/8} \left(\frac{\rho g V^3}{\mu} t \right)^{1/8}.$$

Problem 2: Thin film equations with Marangoni flow. In class we derived the thin film equation for unsteady flows driven by surface tension and gravitational forces. Recall, this thin film equation is

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu} \nabla \cdot \left[h^3 \left(\rho g \nabla h - \gamma \nabla \nabla^2 h \right) \right]$$

where h is the thickness of the film, t is time, μ is viscosity, rho is density, g is the gravitational acceleration, and γ is surface tension. We wish to re-derive this equation to include the additional effect of Marangoni stresses.

(a) Starting from the lubrication assumption for the flow field, what is the local velocity field in the case of a Marangoni stress condition on the upper surface of the thin film?

(b) Integrate the local velocity field and use it in the continuity equation to re-derive the thin film equation.

(c) If the Marangoni flow is driven by thermal gradients, what additional equation for thin films is necessary to model the behavior?

(d) Suppose the temperature on the surface of the film is prescribed to be $T = T_0 \sin(\beta x)$. What is the height as a function of position on the interface if you linearize the equations?

Problem 3: Marangoni-driven Spreading of a thin film. Consider the two-dimensional spreading of a fixed volume per unit width of fluid q due to Marangoni forces. The fluid is spreading on a plate whose surface is heated in a way such that the plate is hot near the source of the fluid and is relatively cooler further downstream. Assume that the temperature gradient varies linearly and that the temperature on the surface of the fluid is identical to that on the plate. Further, assume a linear relationship between surface temperature and surface tension.

(a) Assuming there is only flow in the x-direction, what is the unsteady thin film equation? You should simplify the general equation you derived from your previous homework.

(b) Look for a solution of the form $h(x,t) = At^{\alpha} f(\eta)$ where $\eta = x/x_N(t)$, x and t are position and time, respectively, A and α are constants and f is some function of η. Further, the front of the spreading film is $x_N(t) = Bt^{\beta}$, where B and β are constants. The exponents α and β are determined so that the resulting differential equation only depends on the variable η and the constraint of conservation of mass, i.e., $\int_0^{x_N(t)} h(x,t)\,\mathrm{d}x = q$, is satisfied. What is the resulting differential equation and integral condition describing the flow and of course the exponents and other constants? Note one of the constants B should be expressed in terms of the integral constraint.

(c) Solve the equation for $f(\eta)$ and $x_N(t)$.

(d) From a scaling analysis, where mass is conserved and viscous forces are balanced by Marangoni forces, what is the position of the front as a function of time?