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Optimal Motion Planning for Differentially Flat Underactuated Mechanical Systems*

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***Abstract* - It is shown that the flat output of the single input underactuated mechanical system can be obtained by finding a smooth output function such that the system has relative degree equals to the dimension of the state space. Assuming the flat output of an underactuated system can be solved explicitly, an optimization method is proposed for the motion planning of the differentially flat underactuated mechanical systems by constructing a shape adjustable curve with satisfying specific boundary conditions in flat output space. The inertia wheel pendulum is used to verify the proposed optimization method and some numerical simulation results are included.**

Index Terms - Differential flatness\, Underactuated system, Motion planning, Optimization.

I. INTRODUCTION

The underactuated mechanical systems that the degrees of freedom (DOF) are more than the numbers of independent inputs, allow to reduce cost, weight as well as the occurrence of failures, thus can be used in the fields such as space [1], underwater [2], and biomechanical systems [3] etc. The underactuated mechanical systems are generally slaved by first-order [1] or second-order [4] nonholonomic constraints in high nonlinear form. In some cases, the linear approximation of the underactuated mechanical systems lose the controllability [5], thus the motion planning and control problem for the underactuated system is generally nonlinear in nature. In the field of nonlinear control, so far, some effective methods [6-8] are just developed for the nonlinear systems with specially geometric or algebraic structure such as differentially flat [9] or nilpotent [7] systems.

In this paper, optimizing the motion of underactuated mechanical system with differentially flat property is investigated. Differential flatness was first defined by Fliess et al. [9]. Now it is well known that both motion planning and control problem for a nonlinear system with differentially flat property are simple [6] since that the motion planning problem can be transformed to solve a algebraic equation in flat output space, as well as the nonlinear control problem is equivalent to a linear one in flat output space. Unfortunately, so far there is no systematic way to determine if a nonlinear system is differentially flat, or what the flat outputs for a nonlinear system are, with the major exception of the single-input system. For the single-input system, there is a sufficient and necessary condition, while for multi-inputs nonlinear system, only a necessary condition had been presented [6].

If an underactuated mechanical system can be confirmed to be differential flat and the flat output can be expressed explicitly, then the problem of finding a trajectory satisfying the underactuation constraints becomes the relatively simple algebraic problem of finding a curve to fit the start and final constraints on flat output. As to be shown in this paper, by constructing a polynomial with redundant design parameters, optimizing the feasible trajectory for the underactuated mechanical system is also possible. For instance, considering the main application fields of underactuated mechanical system, improving the energy efficiency of the motion of the underactuated system is appealing because that the power of the remote manipulation system is limited generally.

II. THE DYNAMICS OF UNDERACTUATED MECHANICAL SYSTEMS

Consider a mechanical system with n DOF, denote by $\boldsymbol{q} \in R^n$ the vector of generalized coordinates of the system. Assume there are no external constraints on the system, the dynamics of the mechanical system can be calculated by Euler-Lagrange equation given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = F(\boldsymbol{q})\boldsymbol{\tau} \tag{1}$$

where L is the Lagrangian, $\boldsymbol{\tau} \in R^m$ is the vector of generalized forces. If $m < n$, then $F(\boldsymbol{q}) = [0, \boldsymbol{I}_m]^{\mathrm{T}}$, where $\boldsymbol{I}_m \in R^{m\times m}$ is the identity matrix. Partition the generalized coordinates as $\boldsymbol{q} = [\boldsymbol{q}_{\mathrm{p}} \quad \boldsymbol{q}_{\mathrm{a}}] \in R^{n-m} \times R^m$ according to $F(\boldsymbol{q})$, where $\boldsymbol{q}_{\mathrm{p}}$ and $\boldsymbol{q}_{\mathrm{a}}$ are the unactuated and actuated generalized coordinates respectively. After partitioning the inertia matrix of the system accordingly, then the dynamics (1) can be partitioned as

$$\begin{bmatrix} \boldsymbol{m}_{\mathrm{pp}} & \boldsymbol{m}_{\mathrm{pa}} \\ \boldsymbol{m}_{\mathrm{ap}} & \boldsymbol{m}_{\mathrm{aa}} \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{q}}_{\mathrm{p}} \\ \ddot{\boldsymbol{q}}_{\mathrm{a}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ \boldsymbol{c}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\tau} \end{bmatrix} \tag{2}$$

where $\begin{bmatrix} \boldsymbol{m}_{\mathrm{pp}} & \boldsymbol{m}_{\mathrm{pa}} \\ \boldsymbol{m}_{\mathrm{ap}} & \boldsymbol{m}_{\mathrm{aa}} \end{bmatrix}$ is the inertia matrix of the system, $\begin{bmatrix} \boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ \boldsymbol{c}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}$

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includes centrifugal, Coriolis, gravitational and frictional forces. Since there are no external generalized forces, the first $n-m$ rows of equations (2) are given by

$$\boldsymbol{m}_{\mathrm{pp}}\ddot{\boldsymbol{q}}_{\mathrm{p}}+\boldsymbol{m}_{\mathrm{pa}}\ddot{\boldsymbol{q}}_{\mathrm{a}}+\boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}})=0 \tag{3}$$

and it can be seen as "constraints" of the subsystem

$$\boldsymbol{m}_{\mathrm{ap}}\ddot{\boldsymbol{q}}_{\mathrm{p}}+\boldsymbol{m}_{\mathrm{aa}}\ddot{\boldsymbol{q}}_{\mathrm{a}}+\boldsymbol{c}_2(\boldsymbol{q},\dot{\boldsymbol{q}})=\boldsymbol{\tau} \tag{4}$$

The second-order differential "constraints" (3) have first integral if and only if the passive generalized coordinates $\boldsymbol{q}_{\mathrm{p}}$ are cyclic [4], viz., satisfying

$$\frac{\partial L}{\partial \boldsymbol{q}_{\mathrm{p}}}=0 \tag{5}$$

This means the dynamics (1) has conserved quantities

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\mathrm{p}}}=\text{constant} \tag{6}$$

Equations (6) are a set of first-order differential equations that indicate the generalized momentums are constants such as showing in the space free-floating manipulator system [1]. We consider the case that the passive generalized coordinates $\boldsymbol{q}_{\mathrm{p}}$ are not cyclic, viz., the equations (3) absence the first integrals then showing a set of second-order "nonholonomic constraints". This terminology used in this paper emphasizes that the motion planning and control problem for the underactuated mechanical system can indeed be dealt with by the same tools as for the classical nonholonomic systems.

For simplifying the dynamics (3)-(4), by the partial feedback linearization proposed by Spong [12], the dynamics (2) can be transformed to

$$\begin{aligned}\ddot{\boldsymbol{q}}_{\mathrm{p}}&=-\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}}\boldsymbol{u}\\ \ddot{\boldsymbol{q}}_{a}&=\boldsymbol{u}\end{aligned} \tag{7}$$

by the input transformation

$$\boldsymbol{\tau}=(\boldsymbol{m}_{\mathrm{aa}}-\boldsymbol{m}_{\mathrm{ap}}\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}})\boldsymbol{u}+\boldsymbol{c}_2(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{m}_{\mathrm{ap}}\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \tag{8}$$

where $\boldsymbol{u}=\ddot{\boldsymbol{q}}_{\mathrm{a}}$ is the new input. Then, the system (7) can be written to a form in state space as

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u},\quad \boldsymbol{x}\in R^{2n},\boldsymbol{u}\in R^{m} \tag{9}$$

where

$$\boldsymbol{x}=\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\\ \boldsymbol{x}_4\end{bmatrix}=\begin{bmatrix}\boldsymbol{q}_{\mathrm{p}}\\ \dot{\boldsymbol{q}}_{\mathrm{p}}\\ \boldsymbol{q}_{\mathrm{a}}\\ \dot{\boldsymbol{q}}_{\mathrm{a}}\end{bmatrix},\ \boldsymbol{f}(\boldsymbol{x})=\begin{bmatrix}\dot{\boldsymbol{q}}_{\mathrm{p}}\\ -\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{c}_1(\boldsymbol{q},\dot{\boldsymbol{q}})\\ \dot{\boldsymbol{q}}_{\mathrm{a}}\\ 0\end{bmatrix},\ \boldsymbol{g}(\boldsymbol{x})=\begin{bmatrix}0\\ -\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}}\\ 0\\ \boldsymbol{I}_m\end{bmatrix}.$$

The main difficulty for motion planning and control design for the underactuated system in (9) is that after partial feedback linearization, the new control $\boldsymbol{u}$ appears in the dynamics of both subsystems $(\boldsymbol{x}_1,\boldsymbol{x}_2)$ and $(\boldsymbol{x}_3,\boldsymbol{x}_4)$. It will be shown in next section that the underactuated system (9) can be transformed to a linear system if it is differentially flat.

III. Differentially Flat Underactuated Mechanical Systems

Differential flatness was first defined by Fliess et al. [9] using the formalism of differential algebra. Roughly speaking, a system is differentially flat if one can find a set of outputs such that all states and inputs can be determined from these outputs without integration. More rigorously, if the system has states $\boldsymbol{x}\in R^n$, and inputs $\boldsymbol{u}\in R^m$ then the system is flat if we can find outputs $\boldsymbol{y}\in R^m$ (equal in number of inputs) with the form $\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}},\ldots,\boldsymbol{u}^{(\alpha)})$ such that the states and inputs can be expressed as

$$\begin{aligned}\boldsymbol{x}&=\varphi(\boldsymbol{y},\dot{\boldsymbol{y}},\ldots,\boldsymbol{y}^{(\beta)})\\ \boldsymbol{u}&=\gamma(\boldsymbol{y},\dot{\boldsymbol{y}},\ldots,\boldsymbol{y}^{(\beta)})\end{aligned}$$

Since the algebraic structure between the states and the flat outputs, the motion planning and control design for the differentially flat nonlinear systems become simple. Unfortunately, so far there is no systematic way to determine if a multi-inputs nonlinear system is differentially flat, or what the flat outputs for a multi-inputs nonlinear system are. For the single-input nonlinear system, the sufficient and necessary condition was presented by the following theorem [6]:

Theorem 1: A single input nonlinear system of the form $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},u),\quad \boldsymbol{x}\in R^n,\boldsymbol{u}\in R$ is differentially flat if and only if it is feedback linearizable.

Exact linearization via nonlinear feedback is one of the most important fruits of the geometric control theory in nonlinear system during the past two decades [13]. The feedback linearization theorem for the single-input nonlinear system can be recited as follow [13]:

Theorem 2: The single-input nonlinear system of the form $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}$, $x\in R^n,\boldsymbol{u}\in R$, is feedback linearizable if and only if there exist a neighborhood U of a point $\boldsymbol{x}^0$ and a real-valued function $h(\boldsymbol{x})$, defined on U, such that the single input and single output (SISO) system

$$\begin{aligned}\dot{\boldsymbol{x}}&=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})u\\ y&=h(\boldsymbol{x})\end{aligned} \tag{10}$$

satisfies conditions:

(a). $L_gL_f^kh(\boldsymbol{x})=0,\qquad k<n-1$

(b). $L_gL_f^{n-1}h(\boldsymbol{x})\neq 0$.

The Theorem 2 indicates the single-input nonlinear system is feedback linearizable if and only if the real-valued function $h(\boldsymbol{x})$ defined at a point $\boldsymbol{x}^0$ can be found such that the relative degree of the SISO system satisfies $r=n$. Obviously, the relative degree of a nonlinear system depends on the selection of the output function, and if the point $\boldsymbol{x}^0$ can be given arbitrarily, the nonlinear system is globally linearizable by nonlinear feedback. If the output function $h(\boldsymbol{x})$ is available, then the Theorem 2 indicates the derivative of the output has relationships

$$\begin{aligned}\dot{y}&=L_fh(\boldsymbol{x})\\ &\vdots\\ y^{(n-1)}&=L_f^{n-1}h(\boldsymbol{x})\\ y^{(n)}&=L_f^nh(\boldsymbol{x})+L_gL_f^{n-1}h(\boldsymbol{x})u\end{aligned} \tag{11}$$

If one defines the coordinates transformation as

$$z_i=\phi_i(\boldsymbol{x})=L_f^{i-1}h(\boldsymbol{x}),\qquad 1\le i\le n \tag{12}$$

and defines the input change as

$$u = \frac{1}{L_g L_f^{n-1} h(\boldsymbol{x})} (-L_f^n h(\boldsymbol{x}) + v) \tag{13}$$

then the system (10) can be transformed to a linear system with form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned} \tag{14}$$

It is obviously that the linear system (14) is controllable. Therefore, the main task for linearizing the nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x} \in R^n, u \in R$ by nonlinear feedback (13) is to seek an output function $h(\boldsymbol{x})$ that makes the system has relative degree $r = n$. Thanks to the following theorem [13] the output function can be found for the SISO nonlinear system.

Theorem 3: There exists the output function $h(\boldsymbol{x})$ for which the single-input nonlinear system with form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}$, $\boldsymbol{x} \in R^n, u \in R$, has relative degree $r = n$ at a point $\boldsymbol{x}^0$ if and only if the following conditions are satisfied.

(a) The matrix $[\boldsymbol{g} \quad ad_f\boldsymbol{g} \quad \cdots \quad ad_f^{n-2}\boldsymbol{g} \quad ad_f^{n-1}\boldsymbol{g}]$ is full rank at $\boldsymbol{x}^0$;

(b) The distribution $\Delta = \mathrm{span}\{\boldsymbol{g}, ad_f\boldsymbol{g}, \ldots, ad_f^{n-2}\boldsymbol{g}\}$ is involutive near $\boldsymbol{x}^0$.

Note that if the point $\boldsymbol{x}^0$ can be specified arbitrarily in the state space, then the Theorem 3 is globally effective. Assuming this is the case we here consider, then the following Theorem is easy to be proven.

Theorem 4: The single nonlinear system with form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x} \in R^n, u \in R$ is differentially flat if and only if the following conditions are satisfied.

(a) The matrix $[\boldsymbol{g} \quad ad_f\boldsymbol{g} \quad \cdots \quad ad_f^{n-2}\boldsymbol{g} \quad ad_f^{n-1}\boldsymbol{g}]$ is full rank;

(b) The distribution $\Delta = \mathrm{span}\{\boldsymbol{g}, ad_f\boldsymbol{g}, \ldots, ad_f^{n-2}\boldsymbol{g}\}$ is involutive.

And the flat output can be solved by

$$\frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}} = [0 \quad 0 \quad \cdots \quad 0 \quad \delta(\boldsymbol{x})] \, \boldsymbol{C}^{-1}(\boldsymbol{x}) \tag{15}$$

where

$$\boldsymbol{C}(\boldsymbol{x}) = [\boldsymbol{g} \quad ad_f\boldsymbol{g} \quad \cdots \quad ad_f^{n-2}\boldsymbol{g} \quad ad_f^{n-1}\boldsymbol{g}] \tag{16}$$

is addressed as the ***controllability matrix*** of the system, and $\delta(\boldsymbol{x}) \neq 0$.

The Theorem 4 provides a systemic way to find the flat output for the single-input underactuated mechanical system. Assume the flat output could be obtained by equation (15), the coordinate transformation (12) gives

$$\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x}) \tag{17}$$

where

$$\boldsymbol{z} = [z_1 \quad z_2 \quad \ldots \quad z_n]^{\mathrm{T}} = [y \quad \dot{y} \quad \cdots \quad y^{(n-1)}]^{\mathrm{T}}$$
$$\boldsymbol{\Phi}(\boldsymbol{x}) = [h(x) \quad L_f h(x) \quad \cdots \quad L_f^{n-1} h(x)]^{\mathrm{T}}$$

For $\forall \boldsymbol{x} \in R^n$ satisfying $L_g L_f^{n-1} h(\boldsymbol{x})) \neq 0$, the matrix $\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{\Phi}(\boldsymbol{x})$ has full rank [13], i.e. there exist the inverse coordinate transformation

$$\boldsymbol{x} = \boldsymbol{\Phi}^{-1}(\boldsymbol{z}) = \boldsymbol{\Phi}^{-1}(y, \dot{y}, \ldots, y^{(n-1)}) \tag{18}$$

and the input change

$$u = \frac{1}{L_g L_f^{n-1} h(\boldsymbol{\Phi}^{-1}(\boldsymbol{z}))} (-L_f^n h(\boldsymbol{\Phi}^{-1}(\boldsymbol{z})) + v) \tag{19}$$

therefore both the states and input can be expressed as the functions of flat output and its finite order derivatives.

IV. Optimal Motion Planning for the Flat Underactuated Mechanical System

For the flat underactuated mechanical system, the problem of finding a feasible trajectory $(x(t), u(t))$ between the initial state $\boldsymbol{x}^0 = \boldsymbol{x}(t_0)$ and the final state $\boldsymbol{x}^1 = \boldsymbol{x}(t_1)$ for the underactuated system is changed to the problem of finding a flat output curve $y(t)$ satisfying boundary conditions $\boldsymbol{z}^0 = [y \quad \dot{y} \quad \cdots \quad y^{(n-1)}]^{\mathrm{T}}(t_o)$ and $\boldsymbol{z}^1 = [y \quad \dot{y} \quad \cdots \quad y^{(n-1)}]^{\mathrm{T}}(t_1)$ specified by $\boldsymbol{x}^0 = \boldsymbol{x}(t_0)$ and $\boldsymbol{x}^1 = \boldsymbol{x}(t_1)$ respectively. Therefore, the problem finding a trajectory satisfying the underactuated constraints becomes the relatively simple algebraic problem of finding a curve to fit initial and final conditions on $y(t)$. By the inverse coordinate transformation (18), any curve $y(t)$ maps directly to a consistent pair of state and control histories $\boldsymbol{x}(t)$ and $u(t)$. For instance, the flat output can be parameterized by the polynomial

$$y = \sum_{s=1}^{k} a_{s-1} \left(\frac{t}{\mathrm{T}} \right)^{s-1} \tag{20}$$

where $t \in [t_0, t_1]$, $\mathrm{T} = t_1 - t_0$, and $a_{s-1}, s = 1, 2, \ldots, k$ are the design parameters. Define $\tau_{\mathrm{T}} = t/\mathrm{T} \in [0,1]$ to be a new time variable, and let $k > 2n$, where n is the dimension of the state space. Substitute the boundary conditions into (20), one has

$$\begin{bmatrix} \boldsymbol{z}^0 \\ \boldsymbol{z}^1 \end{bmatrix} = \boldsymbol{A}\boldsymbol{a} \tag{21}$$

where $\boldsymbol{a} = [a_0 \quad a_1 \quad \cdots \quad a_k]^{\mathrm{T}}$, and the matrix $\boldsymbol{A} \in R^{2n \times k}$ has form

$$\boldsymbol{A} = \begin{bmatrix} 1 & \tau_{\mathrm{T}}(0) & \tau_{\mathrm{T}}^2(0) & \cdots & \tau_{\mathrm{T}}^{k-1}(0) \\ 0 & 1/\mathrm{T} & 2\tau_{\mathrm{T}}(0)/\mathrm{T} & \cdots & (\mathrm{k}-1)\tau_{\mathrm{T}}^{k-2}(0)/\mathrm{T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)!\tau_{\mathrm{T}}^{k-n}(0)/((\mathrm{k-n})!\mathrm{T}^{\mathrm{n-1}}) \\ 1 & \tau_{\mathrm{T}}(1) & \tau_{\mathrm{T}}^2(1) & \cdots & \tau_{\mathrm{T}}^{k-1}(1) \\ 0 & 1/\mathrm{T} & 2\tau_{\mathrm{T}}(1)/\mathrm{T} & \cdots & (\mathrm{k}-1)\tau_{\mathrm{T}}^{k-2}(1)/\mathrm{T} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)!\tau_{\mathrm{T}}^{k-n}(1)/((\mathrm{k-n})!\mathrm{T}^{\mathrm{n-1}}) \end{bmatrix}$$

where $\tau_{\mathrm{T}}(0) = 0, \tau_{\mathrm{T}}(1) = 1$. Obviously, the matrix $\boldsymbol{A}$ has full rank with $\mathrm{rank}(\boldsymbol{A}) = 2n$. Since $k > 2n$, the general solution of algebraic equation (21) can be written as

$$\boldsymbol{a}=\boldsymbol{A}^{+}\begin{bmatrix}\boldsymbol{z}^{0}\\ \boldsymbol{z}^{1}\end{bmatrix}+\mu(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{\varepsilon} \tag{22}$$

where $\boldsymbol{A}^{+}$ denote the Moore-Penrose generalized inverse and is given by $\boldsymbol{A}^{+}=\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}$, $\mu>0$ is a scale multiplier, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\varepsilon}$ is an arbitrary vector. The term $(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})$ span the null space of $\boldsymbol{A}$. This formulation provides a decoupled solution. The first term is the special solution that makes $\|a\|_2$ be minimized, and it gives a set of design parameters $\boldsymbol{a}^{*}$ such that the polynomial $y(t)$ satisfies the boundary conditions. The second term $\mu(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{\varepsilon}$ can change the shape of the curve $y(t)$ while not violating the boundary conditions $\boldsymbol{z}^{0}$ and $\boldsymbol{z}^{1}$ since $\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{\varepsilon}\equiv 0$. Therefore, the equation (22) provides an approach to optimizing the motion of the flat mechanical system for specific sake.

To improve the energy efficiency is a broadly interested task for designing the control for the mechanical system. As an example, considering the optimization problem that minimizes a measure about the kinetic energy. The measure can be defined as

$$K=\frac{1}{2}\int_{t_0}^{t_1}\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{x}\right)\mathrm{d}t \tag{23}$$

where $\boldsymbol{G}$ is a weighted matrix . With considering the equation (16), the measure (23) can be rewritten as

$$K(\boldsymbol{a})=\frac{1}{2}\int_{t_0}^{t_1}\left[\left(\boldsymbol{\Phi}^{-1}(z)\right)^{\mathrm{T}}\boldsymbol{G}\left(\boldsymbol{\Phi}^{-1}(z)\right)\right]\mathrm{d}t \tag{24}$$

Therefore, the kinetic energy is a function of the design parameters of the polynomial (20). Let

$$\boldsymbol{\varepsilon}=-\frac{\partial}{\partial \boldsymbol{a}}K(\boldsymbol{a}) \tag{25}$$

with $\mu>0$, then the equation (22) gives a locally optimal solution for minimizing the measure (23). If the arbitrary vector $\boldsymbol{\varepsilon}$ can be expressed explicitly, then by adjusting the scale μ, a better suboptimal solution can be obtained generally. If the arbitrary vector $\boldsymbol{\varepsilon}$ cannot be obtained explicitly, a global optimization algorithm such as the evolution methods (for instance, genetic algorithm) or the randomized methods (for instance, probabilistic roadmap method) can be used to solve the problem, and the formulation (22) is also useable as long as one regards the arbitrary vector $\boldsymbol{\varepsilon}$ and the scale multiplier μ as new design variables. Some global optimization methods permit determining the arbitrary vector $\boldsymbol{\varepsilon}$ without using the gradient of the measure. Therefore, the energy efficiency measure can be defined by a function without depending on the design parameters $\boldsymbol{a}$ explicitly. For instance, the measure can also be defined as

$$K=\int_{t_0}^{t_1}\left|\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{q}}_{\mathrm{a}}\right|\mathrm{d}t \tag{26}$$

The measure (26) calculates the total works of the actuators on the actuation duration. This is a more direct way to evaluate the energy efficiency of the specific motion. Since the generalized actuation force τ_i is not an algebraic function of state variables, the gradient of the measure (26) cannot be obtained by (25), the optimization problem can but be solved by a optimization method without using the gradient of the measure.

V. MOTION PLANNING AND CONTROL OF THE INERTIA WHEEL PENDULUM

The inertia wheel pendulum (IWP) is a planar inverted pendulum with a revolving wheel at the end, as shown in Figure 1. The wheel is actuated and the joint of the pendulum at the base is passive. The IWP was first introduced by Spong et al. [14]. The task is to stabilize the pendulum at its upright equilibrium point while the wheel stops to a given position. The IWP is the first example a flat underactuated of mechanical system with two DOF and single actuator. This is due to the constant inertia matrix of the system [15]. In this section, the IWP is considered to verify the feasibility of the optimization method.

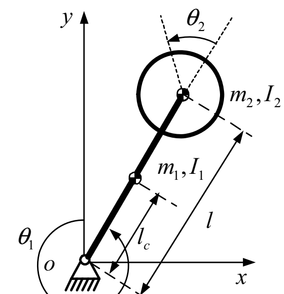

Fig.1. The inertia wheel pendulum

A. The flat output of the IWP

Given the length of the link is l, the distance between the center of mass (CM) of the link and the passive joint is l_c, the mass of the link and the wheel are m_1 and m_2 respectively, I_1 and I_2 are the inertias of the link and wheel respectively. Denote the generalized coordinates of the system to be $\boldsymbol{q}=[\theta_1 \quad \theta_2]^{\mathrm{T}}$. The dynamics of the IWP is given by

$$\begin{aligned} m_{11}\ddot{\theta}_1+m_{12}\ddot{\theta}_2+c_1(\theta_1)&=0\\ m_{21}\ddot{\theta}_1+m_{22}\ddot{\theta}_2+c_2&=\tau \end{aligned} \tag{27}$$

where

$$m_{11}=m_1l_c^2+m_2l^2+I_1+I_2$$
$$m_{12}=m_{21}=m_{22}=I_2$$
$$c_1(\theta_1)=-(m_1l_c+m_2l)g\sin\theta_1$$
$$c_2=0$$

Let $\boldsymbol{x}=[\theta_1 \quad \dot{\theta}_1 \quad \theta_2 \quad \dot{\theta}_2]^{\mathrm{T}}$, $h_0=\dfrac{(m_1l_c+m_2l)g}{m_1l_c^2+m_2l^2+I_1+I_2}$, $h_1=\dfrac{-I_2}{m_1l_c^2+m_2l^2+I_1+I_2}$, the state space equation of the system (27) can be written as

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})u \tag{28}$$

where

$$\boldsymbol{f}(\boldsymbol{x})=[x_2 \quad h_0\sin x_1 \quad x_4 \quad 0]^{\mathrm{T}}$$
$$\boldsymbol{g}(\boldsymbol{x})=[0 \quad h_1 \quad 0 \quad 1]^{\mathrm{T}}$$

$$u = \dot{x}_4$$

The controllability matrix of the system (28) is given by

$$\boldsymbol{C}(\boldsymbol{x}) = \left[\boldsymbol{g}, ad_f \boldsymbol{g}, ad_f^2 \boldsymbol{g}, ad_f^3 \boldsymbol{g}\right] \tag{29}$$

Let $\delta(\boldsymbol{x}) = -h_0 h_1 \cos x_1$, the flat output for the IWP can be obtained by the equation (15), and gives

$$y = h(x) = x_1 - h_1 x_3 \tag{30}$$

The derivatives of the flat output can be obtained as

$$\dot{y} = x_2 - h_1 x_4 \text{ , } \ddot{y} = h_0 \sin x_1 \text{ , } \dddot{y} = h_0 x_2 \cos x_1 \tag{31}$$

then, the inverse coordinate transformation can be given by

$$\begin{aligned} x_1 &= \arcsin(\ddot{y}/h_0) \\ x_2 &= \dddot{y}\Big/\left(h_0\sqrt{1-(\ddot{y}/h_0)^2}\right) \\ x_3 &= \frac{1}{h_1}\left(\arcsin(\ddot{y}/h_0) - y\right) \\ x_4 &= \frac{1}{h_1}\left(\dddot{y}\Big/\left(h_0\sqrt{1-(\ddot{y}/h_0)^2}\right) - \dot{y}\right) \end{aligned} \tag{32}$$

Substituted (32) into (19), the input transformation has form

$$u = -\frac{1}{h_1}\ddot{y} + \frac{1}{h_0 h_1 \sqrt{1-(\ddot{y}/h_0)^2}}\left[y^{(4)} + \frac{(\dddot{y})^2 \ddot{y}}{1-(\ddot{y}/h_0)^2}\right] \tag{33}$$

Since both the states and the input can be expressed as the function of the output (30), as shown by equation (32) and (33), this confirms the output (30) is indeed a flat output for the IWP system.

Refer to the equations (32) and (33), one can find that there is a singularity at the point $x_1 = \theta_1 = \dfrac{\pi}{2} \pm k\pi, k = 0,1,2,\ldots$. Outside the singularity region, by the nonlinear feedback

$$u = -\frac{1}{h_1}\ddot{y} + \frac{1}{h_0 h_1 \sqrt{1-(\ddot{y}/h_0)^2}}\left[v + \frac{(\dddot{y})^2 \ddot{y}}{1-(\ddot{y}/h_0)^2}\right] \tag{34}$$

the system is equivalent to a fourth order linear system

$$y^{(4)} = v \tag{35}$$

where v is the auxiliary control input defined in (34).

B. Optimal motion planning for the IWP system

The underactuated mechanical systems are appealing for the application fields where the weight of the system has rigorous limit. The energy efficiency of the underactuated mechanical must be considered even if the system is carefully designed to be flat, such as the flat biped robots [16] and the flat space manipulators [17]. For the IWP system (27), the benchmark single input flat underactuated mechanical system with flat output (30), the trajectory in the flat output space can be parameterized as a eighth-order polynomial

$$y(t) = \sum_{s=1}^{9} a_{s-1}\left(\frac{t}{\mathrm{T}}\right)^{s-1} \tag{36}$$

where a redundant design parameter is included. Assume the initial state and the final state are $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ respectively, and the corresponding positions in flat output space are $\boldsymbol{z}^0$ and $\boldsymbol{z}^1$ respectively. The matrix $\boldsymbol{A}$ of the equation (21) is given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mathrm{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/\mathrm{T}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6/\mathrm{T}^3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1/\mathrm{T} & 2/\mathrm{T} & 3/\mathrm{T} & 4/\mathrm{T} & 5/\mathrm{T} & 6/\mathrm{T} & 7/\mathrm{T} & 8/\mathrm{T} \\ 0 & 0 & 2/\mathrm{T}^2 & 6/\mathrm{T}^2 & 12/\mathrm{T}^2 & 20/\mathrm{T}^2 & 30/\mathrm{T}^2 & 42/\mathrm{T}^2 & 56/\mathrm{T}^2 \\ 0 & 0 & 0 & 6/\mathrm{T}^3 & 24/\mathrm{T}^3 & 60/\mathrm{T}^3 & 120/\mathrm{T}^3 & 210/\mathrm{T}^3 & 336/\mathrm{T}^3 \end{bmatrix} \tag{37}$$

As in section 4, the measure of the IWP system can be defined to be $K = \frac{1}{2}\int_{t_0}^{t_1}[x_2 \quad x_4]\boldsymbol{M}[x_2 \quad x_4]^{\mathrm{T}}\mathrm{d}t$, where $\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. Despite the arbitrary vector in the equation (22) can be calculated as

$$\boldsymbol{\varepsilon} = -\frac{\partial}{\partial \boldsymbol{a}}K(\boldsymbol{a}) = -\int_{t_0}^{t_1}\left(\left[\frac{\partial x_2}{\partial \boldsymbol{a}} \quad \frac{\partial x_4}{\partial \boldsymbol{a}}\right]\boldsymbol{M}\left[x_2 \quad x_4\right]^{\mathrm{T}}\right)\mathrm{d}t \tag{38}$$

where $\dfrac{\partial x_2}{\partial \boldsymbol{a}} \in R^{8\times 1}$ and $\dfrac{\partial x_4}{\partial \boldsymbol{a}} \in R^{8\times 1}$ are vectors, the negative gradient of the measure only gives the best direction for reducing the energy dissipation. We adopt the global optimization method to optimize the motion for the IWP system. In the simulations section, the vector $\boldsymbol{\varepsilon}$ and the scale multiplier μ are determined by the standard genetic algorithm, which can be found in many textbooks.

C. Simulation results

The physical parameters of the IWP are listed in the Appendix. For the motion planning task, we assume the initial state is $\boldsymbol{x}^0\big|_{t_0=0s} = \left[30^{\circ} \quad 0 \quad 0 \quad 0\right]^{\mathrm{T}}$ and the final state is $\boldsymbol{x}^1\big|_{t_1=10s} = \left[0 \quad 0 \quad 0 \quad 0\right]^{\mathrm{T}}$, the duration of the task is $\mathrm{T} = t_1 - t_0 = 10s$. By the genetic algorithm, the approximately optimal values for the arbitrary vector and the scale multiplier are obtained to be $\varepsilon = -\left[0 \ 0 \ 0 \ 0 \ 5 \ 1 \ 1 \ 0 \ -1\right]^T$ and $\mu = 6\times 10^4$ respectively. Fig. 2 and Fig. 3 show the motion planning (dashed line) and trajectories tracking (solid line) simulation results, of which the former corresponds to $\mu = 0$ and the later corresponds to $\mu = 6\times 10^4$. The energy efficiency is evaluated by $E = \int_0^{\mathrm{T}}\left|\tau\dot{\theta}_2\right|\mathrm{d}t$. For the fourth-order linear system (35), it is easy to design a closed-loop controller

$$v = y^{(4)d} + k_3(\dddot{y}^d - \dddot{y}) + k_2(\ddot{y}^d - \ddot{y}) + k_2(\dot{y}^d - \dot{y}) + k_0(y^d - y) \tag{39}$$

where y^d is the flat trajectory given by (36). The controller (39) is stabilizable by choosing the parameters $k_i, i = 1,2,3,4$ such that the closed loop characteristic polynomial $\lambda^4 + k_3\lambda^3 + k_2\lambda^2 + k_1\lambda + k_0$ has all its roots in the left half of the complex plane. For testing the performance of the closed loop controller, the state of the system is started from a new initial state $\boldsymbol{x}^0\big|_{t_0=0s} = \left[70^{\circ} \quad 0 \quad -\left(3\times 10^3\right)^{\circ} \quad 0\right]^{\mathrm{T}}$, which significantly deviates from the target trajectory. Refer to the Fig.2 and 3, one can find that not only the controller (39) is exponentially stable but also the actual energy dissipation is reduced correspondingly.

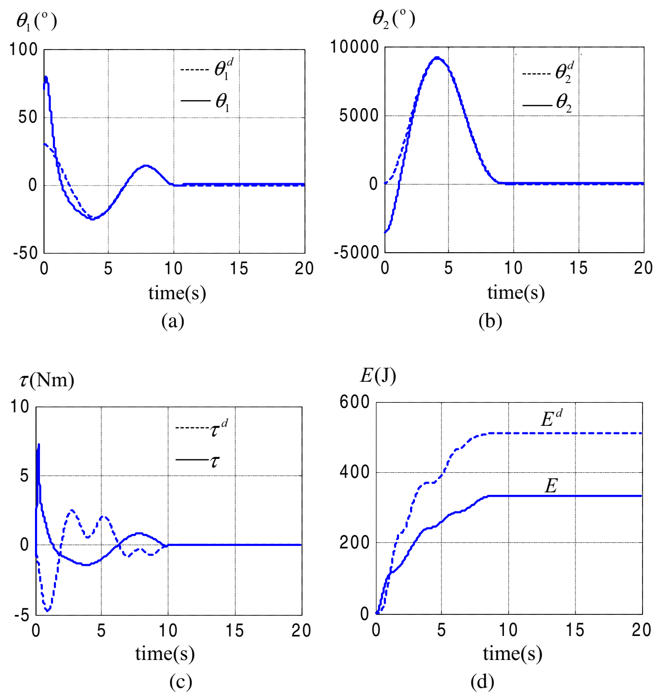

Fig.2. The trajectory tracking control results with $\mu = 0$

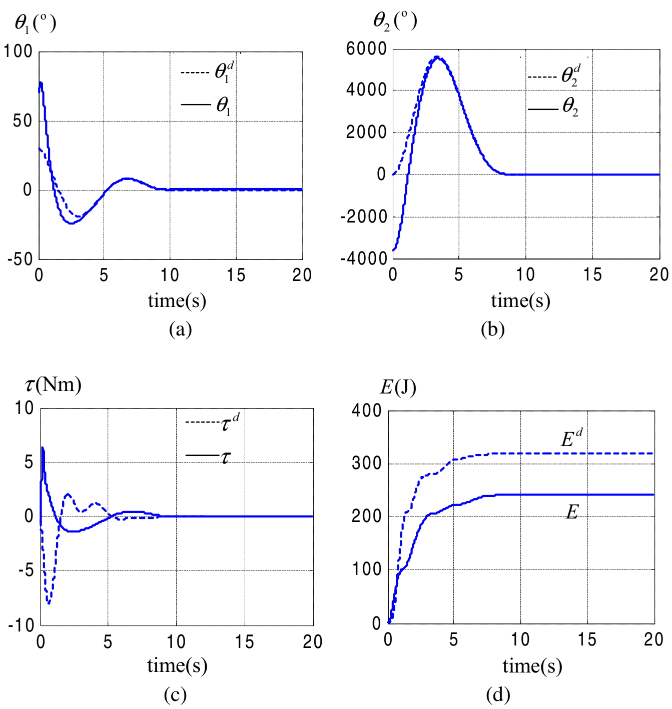

Fig. 3. The trajectory tracking control results with $\mu = 6\times10^4$

VI. CONCLUSIONS

For a flat underactuated mechanical system, it is shown that the motion of the system can be optimized by constructing a shape adjustable curve in the flat output space. By the benchmark singe-input underactuated mechanical system, the inertia wheel pendulum system, it is verified that the proposed optimal motion planning method can effectively improve the energy efficiency of the IWP system for a given position control task. Since there is no systemic way to determine if a multi-inputs nonlinear system is flat or what the flat outputs for a multi-inputs nonlinear system are, the motion planning and control for general multi-inputs underactuated mechanical systems are still open problems, say nothing of the motion optimization.

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