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Continuous Low-thrust Time-optimal Orbital Maneuver

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Abstract— **Indirect optimization for the continuous low-thrust minimum time orbital maneuvers including the transfer, intercept and rendezvous are studied in this paper. The difference among the three maneuvers mainly lie in the terminal constraint conditions. By the Pontryagin Maximum Principle, the trajectory optimization problem is converted into the two-point boundary-value problem, and the terminal constraint conditions of the three orbital maneuver missions are also investigated respectively. The single shooting method is used to solve the corresponding boundary-value problem, and the simulations show that the product of the minimum flight time and the maximal thrust is near constant in the three maneuver problems.**

I. INTRODUCTION

An important problem in astronautics is to transfer a satellite between elliptic orbits, which have been widely studied by many researchers in the impulse case and the continuous low-thrust case [1], [2]. Recently, much attention has been focused on computing optimal trajectories of the low-thrust orbital transfer, which can be performed by minimizing the cost of the final time or the final mass [2]–[8].

The movement of the satellite is usually described by Keplerian equation using the position-speed variables in the so-called Cartesian coordinates or Gauss equation using the modified equinoctial elements in the Gauss coordinates. The research group of B. Bonnard and J. Gergaud studied the low-thrust time-optimal and minimum fuel-consumption orbital transfer problem using the Gauss equation, and the controllability properties of the system, the existence of the optimal control and the π-singularity observed in the problem were also proposed in the geometrical analysis viewpoint [3]–[5]. In numerical experiment of the minimum transfer time problem, some researchers found that the minimum time and the modulus of the maximal thrust have the relationship $t_{f\min} \times T_{\max} \approx c$ where c is a constant [5], [7], but whether there exists a positive constant c or not such that $t_{f\min} \times T_{\max}$ tends to c when $T_{\max}$ tends to zero is still an open problem [8].

The orbital maneuver also include the orbital rendezvous and intercept problems, which are different from the orbital transfer problem mainly in the terminal constraint conditions. As early as the 1950s-1960s, the rendezvous and intercept problems had been widely investigated in the impulse thrust case [9]. As for the rendezvous problem, the relative dynamics of the satellites (for example: Hill equation) is mainly used lately [10]. The relative dynamics using the position-speed variables has also been used to deal with the orbital intercept problem [11]. However, the research of these two maneuvers are not as wide as the transfer problem in the low thrust time optimal case using the Gauss coordinates.

So in this paper, we study the low-thrust time-optimal orbital maneuver such as transfer, intercept and rendezvous under the unified framework using the Gauss equation expressed by the modified equinoctial elements. The trajectory optimization problem is deduced into the two-point boundary-value (TPBV) problem by using the Pontryagin Maximum Principle [12], and the corresponding terminal conditions of the orbital transfer, intercept and rendezvous problems are studied respectively.

The main contributions of this paper are as follows. Firstly, the indirect optimization method is applied to the discussion of the continuous-thrust orbital intercept and rendezvous problems in the Gauss coordinates, which is quite different from the previous results on these problems obtained in the impulse thrust mode and/or by using the relative dynamics of the spacecrafts. To the best knowledge of the authors, this is new in dealing with these problems. Secondly, the results of the numerical simulations show that the phenomena of the product of the minimum flight time and the maximal thrust being nearly constant appears also in the orbital rendezvous and intercept problems. The simulation further shows that the orbital transfer and rendezvous problem share almost the same optimal trajectory with a fixed maximum magnitude of thrust.

In this paper, we use the following notations: Let $\langle , \rangle$ and $\bigwedge$ indicate the inner product and cross product of two vectors. We denote by $\frac{\partial}{\partial(\bullet)}$ the natural bases and by $|\bullet|$ the finite-dimensional Euclidean norm.

II. PROBLEM STATEMENT AND PRELIMINARIES

In order to give the mathematical formulation of the orbital maneuver problem, the satellite can be supposed to be modeled as a particle and the high-order terms of the earth gravitational field and perturbations are neglected. We denote by m the mass of the satellite and $\boldsymbol{T} = (T_1, T_2, T_3)$ the thrust of the engine which is bounded by $|\boldsymbol{T}| \leq T_{\max}$. In the inverse square law field, the equation describing the satellite dynamics in Cartesian coordinates is

$$\ddot{\boldsymbol{r}} = -\mu \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} + \frac{\boldsymbol{T}}{m} \tag{1}$$

Where $\boldsymbol{r} = (r_1, r_2, r_3)$ is the position vector of the satellite measured in a fixed inertial frame I, J, K whose origin is the Earth's center, and μ is the gravitation constant.

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If $\boldsymbol{r}\wedge\dot{\boldsymbol{r}}\neq 0$, the three-dimensional control, which is expressed in the mobile referential coordinate attached to the satellite, can be decomposed into the tangential-normal frame and radial-orthoradial frame [3].

Because of the large number of revolutions and strong oscillations of the variables in Cartesian coordinates described by 1, the Gauss coordinates system which describe the movement of the satellite in a more orbit-related point of view is preferred [4], [6]. There we use the first five components of the state vector to characterize the osculating orbit, while the sixth component indicates the current position of the satellite on the orbit. The state variables (P,e_x,e_y,h_x,h_y,L) are now defined by [2] : P is the semi-latus rectum; (e_x,e_y) is the eccentricity vector, in the orbit plane, oriented towards perigee; (h_x,h_y) is the rotation vector, in the equatorial plane, collinear to the intersection of orbit and equatorial plane; L is the cumulative longitude. The state variables are labeled as the modified equinoctial elements of Gauss coordinate, which are expressed by the classical orbital elements as

$$P=a(1-e^2), e_x=e\cos(\omega+\Omega), e_y=e\sin(\omega+\Omega)$$
$$h_x=\tan(i/2)\cos(\Omega), h_y=\tan(i/2)\sin(\Omega), L=\Omega+\omega+f$$

where in the classical orbital element, a is semi-major axis, e is eccentricity, i is inclination, Ω is longitude of the ascending node, ω is argument of perigee, and f is true anomaly. We set the coordinates $\boldsymbol{x}=[P\ e_x\ e_y\ h_x\ h_y\ L]^T\in\mathbb{R}^6$ and the thrust is decomposed into the radial-orthoradial frame [3]

$$\dot{\boldsymbol{x}}=\boldsymbol{f}_0(\boldsymbol{x})+\frac{1}{m}\sum_{i=1}^{3}T_i\boldsymbol{f}_i(\boldsymbol{x}) \tag{2}$$

In these coordinates,

$$\boldsymbol{f}_0=\sqrt{\frac{\mu}{P}}\frac{W^2}{P}\frac{\partial}{\partial L}$$
$$\boldsymbol{f}_1=\sqrt{\frac{P}{\mu}}(\sin L\frac{\partial}{\partial e_x}-\cos L\frac{\partial}{\partial e_y})$$
$$\boldsymbol{f}_2=\sqrt{\frac{P}{\mu}}\frac{1}{W}(2P\frac{\partial}{\partial P}+(W\cos L+\eta_x)\frac{\partial}{\partial e_x}+(W\sin L+\eta_y)\frac{\partial}{\partial e_y})$$
$$\boldsymbol{f}_3=\sqrt{\frac{P}{\mu}}\frac{1}{W}(-Ze_y\frac{\partial}{\partial e_x}+Ze_x\frac{\partial}{\partial e_y}+\frac{C}{2}\cos L\frac{\partial}{\partial h_x}+\frac{C}{2}\sin L\frac{\partial}{\partial h_y}+Z\frac{\partial}{\partial L})$$

with $W=1+e_x\cos L+e_y\sin L, \eta_x=e_x+\cos L, \eta_y=e_y+\sin L, Z=h_x\sin L-h_y\cos L$ and $C=1+h_x^2+h_y^2$.

We also take into account the fact that the mass flow is proportional to the modulus of the thrust [5]

$$\dot{m}=-\frac{1}{I_{sp}g}|\boldsymbol{T}|=-\beta|\boldsymbol{T}| \tag{3}$$

Where I_{sp} is the engine specific impulsion, g is the earth sea-level gravitational parameter. So that the state of the satellite is in fact $(\boldsymbol{x},m)\in M\times\mathbb{R}^+$, where M is the smooth submanifold of $\mathbb{R}^6$.

We restrict ourselves to elliptic trajectories, and define the path constraint $P\geq\Pi^0(\Pi^0>0)$ to prevent the satellite from colliding with the earth. Analogously, the mass of the satellite has to remain greater than the mass without fuel m^0. Accordingly, to ensure compactness of the set of admissible trajectory, the trajectory is assumed to stay in a secure zone

$$A=\{(\boldsymbol{x},m)|P\geq\Pi^0, e_x^2+e_y^2<1, m>m^0\}$$

But we shall assume in the sequel that the final mass is free and that the constraint $m>m^0$ is not active [3].

For the orbital maneuver mission, the initial and terminal orbits are prescribed, so we have the boundary constraints

$$\Phi(t_0,\boldsymbol{x}(t_0),m(t_0))=0,\quad \Phi(t_f,\boldsymbol{x}(t_f),m(t_f))=0 \tag{4}$$

We also suppose that the maximum thrust magnitude $T_{\max}$ is fixed during the orbital maneuver, so the control is parameterized by [2]

$$\boldsymbol{T}=T_{\max}\boldsymbol{u},\quad |\boldsymbol{u}|\leq 1 \tag{5}$$

In our continuous low-thrust time-optimal orbital maneuver problem, the performance index to be minimized is the flight time $t_f(t_0=0)$. For a given maximum thrust $T_{\max}$, the minimum time orbital maneuver problem will be referred to as $(TP)_t$ as follows:

$$(TP)_t\begin{cases} min\quad J=\int_0^{t_f}dt\\ \dot{\boldsymbol{x}}=\boldsymbol{f}_0(\boldsymbol{x})+\frac{T_{\max}}{m}\sum_{i=1}^{3}u_i\boldsymbol{f}_i(\boldsymbol{x})\\ \dot{m}=-\beta T_{\max}|\boldsymbol{u}|\\ \Phi(0,\boldsymbol{x}(0),m(0))=0,\Phi(t_f,\boldsymbol{x}(t_f))=0\\ |\boldsymbol{u}|\leq 1\end{cases} \tag{6}$$

Since the drift $\boldsymbol{f}_0$ is periodic and the tangent space at any point is spanned by the brackets of $\boldsymbol{f}_0,\dots,\boldsymbol{f}_3$, $lie_{\boldsymbol{x}}(\boldsymbol{f}_0,\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3)=T_{\boldsymbol{x}}M$, so that no matter how low the thrust might be, the system remains controllable if the mass of the satellite without fuel m^0 is small enough. Hence, the set of admissible trajectories and control is nonempty and the existence of optimal control proceeds from the Filippov theorem [13].

III. TIME OPTIMAL CONTROL

In this section, we study the optimal control for the minimum time orbit maneuver problem $(TP)_t$.The maximum principle is applies and the associated Hamiltonian is

$$H=p_0+H_0+\frac{T_{\max}}{m}\sum_{i=1}^{3}u_iH_i-\beta T_{\max}p_m|\boldsymbol{u}| \tag{7}$$

In 7, p_0 is a non-positive constant, the H_i is the Hamiltonian lift $\langle\boldsymbol{p},\boldsymbol{f}_i\rangle$, $\boldsymbol{p}$ being the dual to $\boldsymbol{x}$, p_m the dual to m. In the normal case, p_0 is negative and normalized to -1. Defining $\Psi=[H_1\ H_2\ H_3]$, we have the following proposition.

Proposition 1: [3] In the secure zone, along an optimal solution, whenever $\Psi=[H_1\ H_2\ H_3]$ is not zero, the optimal control of $(TP)_t$ is given by

$$\boldsymbol{u}=\frac{\Psi}{|\Psi|} \tag{8}$$

Remark 1: In this paper, the Pontryagin Maximum Principle is applied. If we use the Minimum Principle, p_0 should be normalized to 1, and the optimal control should be given by $\boldsymbol{u} = -\Psi/|\Psi|$. But the two cases have the same optimal trajectory.

Let $(\boldsymbol{x}, m, \boldsymbol{p}, p_m, \boldsymbol{u})$ be an extremal, the classification of regular extremals is based upon the contact order of the trajectory with the switching surface $\{\Psi = 0\}$, the extremal is said to be of order zero if $\boldsymbol{u}$ is smooth and given by (8) whenever $\Psi \neq 0$, and to be singular if $\{\Psi \equiv 0\}$, which is a vector valued function of time corresponds to switching points in the time domain [3]. It has been proved in [4] that Ψ is continuously differentiable, and the geometric analysis showed that there is only a finite number of switching points. So we have the following result.

Proposition 2: [4] For a fixed magnitude of thrust $T_{\max}$, if there exists an optimal trajectory $(\boldsymbol{x}, m)$ that keep staying in the interior of the secure zone A, then the corresponding optimal control of $\boldsymbol{u}$ of $(TP)_t$ will be such that $|\boldsymbol{u}| = 1$ almost everywhere.

Remark 2: In [3] and [4], the authors investigate the time optimal orbital transfer problem, and obtain the results described by proposition 1 and 2. By the Pontryagin Maximum Principle, the difference between orbital transfer problem and orbital intercept or rendezvous problem lies only in the terminal constraint conditions. Hence, the proposition 1 and 2 can still be applicable to the orbital intercept or rendezvous problem.

According to the proposition 2, $|\boldsymbol{u}| = 1$ almost everywhere, so the mass is known explictly as a function of the time,

$$m(t) = m^0 - \beta T_{\max} t \tag{9}$$

and $(TP)_t$ can be given by an equivalent nonautonomous formulation. Further, for the simplicity, we recast the problem by scaling the flight time on $[0,1]$, treating the final time as an additional constant state variable by letting $\tau = t/t_f$, so $(TP)_t$ can be deduced into the model in the Mayer form as follows:

$$(TP)_\tau \begin{cases} \min \quad J = t_f(1) \\ \dot{\boldsymbol{x}} = t_f\left(\boldsymbol{f}_0(\boldsymbol{x}) + \frac{T_{\max}}{m(t_f\tau)} \sum_{i=1}^{3} u_i \boldsymbol{f}_i(\boldsymbol{x})\right), \tau \in [0,1] \\ \dot{t_f} = 0 \\ \Phi(0, \boldsymbol{x}(0), t_f(0)) = 0, \Phi(1, \boldsymbol{x}(1), t_f(1)) = 0 \\ |\boldsymbol{u}| \leq 1 \end{cases} \tag{10}$$

If $(\boldsymbol{x}, t_f, u)$ is the solution of $(TP)_\tau$, there will be absolutely continuous costates $\boldsymbol{p} = [p_P \; p_{e_x} \; p_{e_y} \; p_{h_x} \; p_{h_y} \; p_L]^T$ and p_{t_f} associated to $\boldsymbol{x}$ and t_f, respectively, such that $(\boldsymbol{x}, t_f, \boldsymbol{p}, p_{t_f})$ is a solution of the two-point boundary-value problem obtained from first-order necessary condition of the Pontryagin Maximum Principle [12].

$$\dot{\boldsymbol{x}} = \frac{\partial H^*}{\partial \boldsymbol{p}} \tag{11a}$$

$$\dot{t_f} = 0 \tag{11b}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H^*}{\partial \boldsymbol{x}} \tag{11c}$$

$$\dot{p_{t_f}} = -\frac{\partial H^*}{\partial t_f} \tag{11d}$$

with initial boundary condition

$$\begin{aligned}\Phi(0) = (&P(0) - P^0, e_x(0) - e_x^0, e_y(0) - e_y^0, h_x(0) - h_x^0, \\ &h_y(0) - h_y^0, L(0) - L^0, p_{t_f}(0) - p_{t_f}^0) = 0\end{aligned} \tag{12}$$

and final boundary condition

$$\Phi(1) = \Phi(1, \boldsymbol{x}(1), t_f(1), \boldsymbol{p}(1), p_{t_f}(1)) = 0 \tag{13}$$

which is determined by the different orbital maneuver mission. And, in (11),

$$H^* = \left\langle \boldsymbol{p}, t_f \left(\boldsymbol{f}_0(\boldsymbol{x}) + \frac{T_{\max}}{m(t_f\tau)} \sum_{i=1}^{3} u_i^* \boldsymbol{f}_i(\boldsymbol{x})\right)\right\rangle \tag{14}$$

is the Hamiltonian of the $(TP)_\tau$, where the optimal control $\boldsymbol{u}^* = [u_1^* \; u_2^* \; u_3^*]$ is still defined as a smooth function in (8).

Now, the continuous-thrust minimum time orbital maneuver problem is deduced into the two-point boundary value problem. In the next section, we shall discuss the terminal boundary conditions for different orbital maneuver mission.

IV. BOUNDARY VALUE AND TRANSVERSALITY CONDITIONS

The main difference in the orbital maneuver missions such as transfer, intercept and rendezvous is the terminal constraint conditions. In this paper, we suppose that the three orbital maneuver missions have common initial boundary conditions (12), where $p_{t_f}^0 = 0$ by transversality. For showing the terminal condition, we will consider the position and velocity. In Gauss coordinate, the cartesian position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ of the satellite (in an inertial geocentric reference frame) are given by [2]

$$r_1 = \frac{P}{CW}\left((1 + h_x^2 - h_y^2)\cos L + 2h_x h_y \sin L\right) \tag{15a}$$

$$r_2 = \frac{P}{CW}\left((1 - h_x^2 + h_y^2)\sin L + 2h_x h_y \cos L\right) \tag{15b}$$

$$r_3 = \frac{P}{CW}(2Z) \tag{15c}$$

$$v_1 = \frac{1}{C}\sqrt{\frac{\mu}{P}}\left(2h_x h_y(e_x + \cos L) - (1 + h_x^2 - h_y^2)(e_y + \sin L)\right) \tag{15d}$$

$$v_2 = \frac{1}{C}\sqrt{\frac{\mu}{P}}\left((1 - h_x^2 + h_y^2)(e_x + \cos L) - 2h_x h_y(e_y + \sin L)\right) \tag{15e}$$

$$v_3 = \frac{1}{C}\sqrt{\frac{\mu}{P}}\left(2h_x(e_x + \cos L) + 2h_y(e_y + \sin L)\right) \tag{15f}$$

In the sequel, we use $(P^f, e_x^f, e_y^f, h_x^f, h_y^f, L^f)$ to denote the final orbit of the satellite or the virtual object.

A. Orbital Transfer Problem

The orbital transfer problem has been widely studied, the terminal boundary constraints meet [3], [4]

$$\left(P(1) - P^f, e_x(1) - e_x^f, e_y(1) - e_y^f, h_x(1) - h_x^f, h_y(1) - h_y^f\right) = 0$$

and $(p_L(1) - p_L^f, p_{t_f}(1) - p_{t_f}^f) = 0$. Where $p_L^f = 0, p_{t_f}^f = -1$, which is determined by transversality condition applying the Pontryagin Maximum Principle, because of the free final longitude and the Mayer form performance index, t_f. So for the orbital transfer problem, the terminal boundary vector function is defined by

$$\begin{aligned}\Phi(1) = (P(1) - P^f, e_x(1) - e_x^f, e_y(1) - e_y^f, h_x(1) - h_x^f, \\ h_y(1) - h_y^f, p_L(1), p_{t_f}(1) + 1) = 0\end{aligned} \tag{16}$$

Remark 3: In fact, the fixed final longitude L_f means that both the position of the satellite on the final orbit and the number of revolutions are fixed. This case can be considered as the orbital rendezvous problem provided that the flight time t_f is also fixed [14]. However, in the minimum time orbital maneuver problem, the longitude L is the same fast variable as the time t, so the final longitude can not be chosen optionally because that is related to the maximum thrust $T_{\max}$, and the product of the minimum longitude and the maximal thrust is near constant [2], [6]. Therefore, the problem of minimum orbital transfer time with fixed final longitude can not be simply seen as rendezvous.

B. Orbital Intercept Problem

The orbital intercept problem is that the relative position of two satellites is zero but relative velocity is free in the terminal. We call the two satellites as interceptor and target respectively, and suppose that the target moves on a fixed orbit without thrust. For a given target satellite, the terminal constraint of the interceptor is not a fixed orbit but a sub-manifold of the bundle TM satisfying the following constraints

$$\phi_1 \doteq r_1(1) - r_1^f = 0 \tag{17a}$$
$$\phi_2 \doteq r_2(1) - r_2^f = 0 \tag{17b}$$
$$\phi_3 \doteq r_3(1) - r_3^f = 0 \tag{17c}$$

where (r_1^f, r_2^f, r_3^f) is the position of the target which can be obtained by substituting $(P^f, e_x^f, e_y^f, h_x^f, h_y^f, L^f)$ into (15a-15c). However, for a given initial position, the final position on the orbit of the target is not fixed if the flight time is not known prior, because the longitude L^f varies with time without thrust

$$\frac{dL^f}{d\tau} = t_f\sqrt{\frac{\mu}{P^f}}\frac{(1 + e_x^f\cos L^f + e_y^f\sin L^f)^2}{P^f} \tag{18}$$

For convenience, we introduce a new variable L^f described by (18) into $(TP)_\tau$, so the state variables $(\boldsymbol{x}, t_f, L^f) \in \mathbb{R}^8 \times \mathbb{R} \times \mathbb{R}$, and $L^f(0) - L^{f^0} = 0$ is added into (12) as another initial condition constraint. By virtue of the Maximum Principle, the conditions of transversality are determined from the terminal constraints (17):

$$p_P(1) = -\left(\lambda_1\frac{\partial r_1}{\partial P} + \lambda_2\frac{\partial r_2}{\partial P} + \lambda_3\frac{\partial r_3}{\partial P}\right)_{(1)} \tag{19a}$$
$$p_{e_x}(1) = -\left(\lambda_1\frac{\partial r_1}{\partial e_x} + \lambda_2\frac{\partial r_2}{\partial e_x} + \lambda_3\frac{\partial r_3}{\partial e_x}\right)_{(1)} \tag{19b}$$
$$p_{e_y}(1) = -\left(\lambda_1\frac{\partial r_1}{\partial e_y} + \lambda_2\frac{\partial r_2}{\partial e_y} + \lambda_3\frac{\partial r_3}{\partial e_y}\right)_{(1)} \tag{19c}$$
$$p_{h_x}(1) = -\left(\lambda_1\frac{\partial r_1}{\partial h_x} + \lambda_2\frac{\partial r_2}{\partial h_x} + \lambda_3\frac{\partial r_3}{\partial h_x}\right)_{(1)} \tag{19d}$$
$$p_{h_y}(1) = -\left(\lambda_1\frac{\partial r_1}{\partial h_y} + \lambda_2\frac{\partial r_2}{\partial h_y} + \lambda_3\frac{\partial r_3}{\partial h_y}\right)_{(1)} \tag{19e}$$
$$p_L(1) = -\left(\lambda_1\frac{\partial r_1}{\partial L} + \lambda_2\frac{\partial r_2}{\partial L} + \lambda_3\frac{\partial r_3}{\partial L}\right)_{(1)} \tag{19f}$$
$$p_{L^f}(1) = \left(\lambda_1\frac{\partial r_1^f}{\partial L^f} + \lambda_2\frac{\partial r_2^f}{\partial L^f} + \lambda_3\frac{\partial r_3^f}{\partial L^f}\right)_{(1)} \tag{19g}$$

where $\lambda_i (i = 1, 2, 3)$ is the Lagrange multiplier.

We can get λ_i by solving any three equations choosing from (19), then obtain four terminal constraints by substituting λ_i into the remaining four equations of (19). So the four terminal constraints, (17) and $p_{t_f}(1) = -1$ form eight terminal constraint conditions of the orbital intercept problem.

C. Orbital Rendezvous Problem

Different from the orbital intercept problem, for the orbital rendezvous problem, both relative position and relative velocity are required to be zero in the terminal, that is

$$\phi_1 \doteq r_1(1) - r_1^f = 0 \tag{20a}$$
$$\phi_2 \doteq r_2(1) - r_2^f = 0 \tag{20b}$$
$$\phi_3 \doteq r_3(1) - r_3^f = 0 \tag{20c}$$
$$\phi_4 \doteq v_1(1) - v_1^f = 0 \tag{20d}$$
$$\phi_5 \doteq v_2(1) - v_2^f = 0 \tag{20e}$$
$$\phi_6 \doteq v_3(1) - v_3^f = 0 \tag{20f}$$

Similarly, L^f described by (18) is introduced into $(TP)_\tau$ as a new state variable, and by transversality, the costate $\boldsymbol{p} = [p_p\ p_{e_x}\ p_{e_y}\ p_{h_x}\ p_{h_y}\ p_L]^T$ dual to the state x meets terminal constraint as follows

$$\boldsymbol{p}(1) = -\sum_{i=1}^{6}\lambda_i\frac{\partial\phi_i}{\partial\boldsymbol{x}}_{(1)} \tag{21}$$

Solving the algebraic equations (21), we can get the Lagrange multipliers $\lambda_i, i = 1, \cdots, 6$, then substituting into the next terminal constraint equation on L^f

$$\begin{aligned}p_{L^f}(1) = (\lambda_1\frac{\partial r_1^f}{\partial L^f} + \lambda_2\frac{\partial r_2^f}{\partial L^f} + \lambda_3\frac{\partial r_3^f}{\partial L^f} \\ + \lambda_4\frac{\partial v_1^f}{\partial L^f} + \lambda_5\frac{\partial v_2^f}{\partial L^f} + \lambda_6\frac{\partial v_3^f}{\partial L^f})_{(1)}\end{aligned} \tag{22}$$

So that, (20), (22) and $p_{t_f}(1) = -1$ also form eight terminal constraint conditions of the orbital rendezvous problem.

TABLE I

BOUNDARY CONDITIONS

Variables	Initial Condition	Terminal Condition
P	15.6Mm	32.1Mm
e_x	0.75	0.16
e_y	0.0	0.30
h_x	0.612	0.0
h_y	0.0	0.1
L	π	free
L^f	1.507	free

V. NUMERICAL SIMULATION

We examine now the numerical results of the orbital maneuvers including transfer, intercept and rendezvous. In section 3 and 4, the continuous-thrust time-optimal orbital maneuver problem is deduced to the TPBV problem by applying the Pontryagin Maximum Principle. In this section, we use the single shooting method to solve the corresponding TPBV problem. Taking the orbital transfer problem as example, the boundary-value problem is equivalent to the so-called shooting function: find $(\boldsymbol{p}^0, t_f^0) \in \mathbb{R}^6 \times \mathbb{R}$ such that

$$S(\boldsymbol{p}^0, t_f^0) = b(\Phi^0(\boldsymbol{x}^0, t_f^0, \boldsymbol{p}^0, p_{t_f}^0)) = 0 \tag{23}$$

where the boundary function b is defined by (16).

In numerical experiment example, the physical constants in the system $(TP)_\tau$ are listed in the following [3]

$$\begin{aligned} \mu &= 5165.8620912 \quad && \mathrm{Mm^3/h^2} \\ \beta &= 1.42 \times 10^{-2} && \mathrm{h/Mm} \\ m^0 &= 1500 && \mathrm{kg} \end{aligned}$$

And the boundary conditions are summarized in table I.

To begin with, we show in Fig.1 the solution of the state for the orbital transfer, intercept and rendezvous problem in the same figure with the maximum thrust of 4N. In Fig.1, the graphs includes orbital elements P, e_x, e_y, h_x, h_y, L and the mass m of the satellite with the time as the abscissa, and the solid line, dashdot line and dashed line represent the orbital transfer, rendezvous and intercept respectively. The desired value of the final orbital elements are also showed by the level solid line in this figure. For the maximum thrust of 4N, the minimum flight time is 322.428h for the transfer problem, 318.986h for the rendezvous problem and 169.0513h for the intercept problem respectively.

It can be seen from Fig.1 that the evolution of the state variables is quite smooth, due to the use of the Modified equinoctial orbital elements, and also to the low continuous thrust of the propulsion. It also can be seen that the states of the transfer problem and rendezvous problem almost share the same trajectory, and the final values of the elements in the orbital rendezvous also reach the desired orbital elements same as that of the orbital transfer. However, the final value of the elements in the orbital intercept problem are much different from those of the elements of the object orbit, which can be seen from Fig.2 that the final orbit of the interceptor is nearly perpendicular to the orbit of the target.

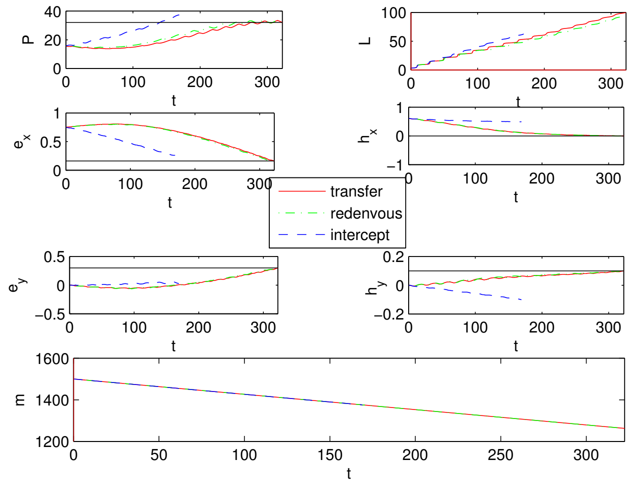

Fig. 1. Optimal solution of the state,thrust of $4N$

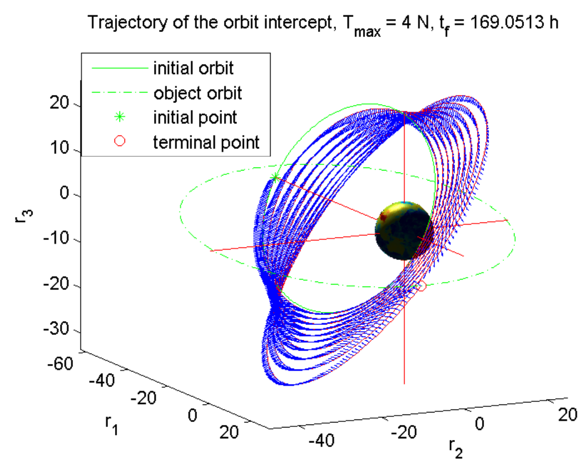

Fig. 2. Optimal 3D trajectory of the orbital intercept,thrust of 4N

Fig.2 shows the 3D optimal trajectory in $(r_1, r_2, r_3)$of the orbital intercept problem, and the arrows picture the action of the control. The solid line expresses the orbit of the intercept, and dashdot line expresses the orbit of the target. While the intercept point is labeled as a circle. From Fig.2, we can see that the change in the inclination is lower than that in orbital transfer problem shown in Fig.3. Because the orbital rendezvous problem share almost the same trajectory as that of the orbital transfer problem, so the trajectory of the rendezvous problem is not shown here.

We also adjust the magnitude of the maximal thrust and repeat the numerical experiment, and the results about the relationship between the minimum flight time and the maximal thrust are pictured in Fig.4. From the left graph of this figure, one can see that the minimum flight time of the orbital transfer and rendezvous problem are almost equal for the same fixed maximum thrust $T_{\max}$, and the minimum time

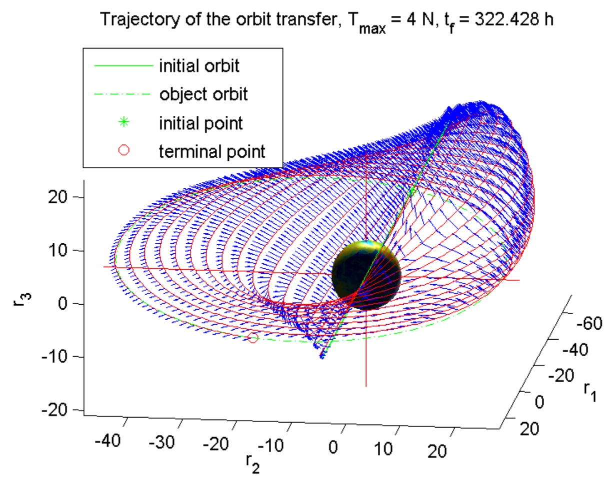

Fig. 3. Optimal 3D trajectory of the orbital transfer,thrust of 4N

of the orbital intercept is lower than that of the orbital transfer or rendezvous. Though some researchers found that the minimum time and the modulus of the maximal thrust have the near relationship $t_{f\min} \times T_{\max} \approx c$ [5], [7], the result seems to be also suitable for the orbital intercept and rendezvous problem. However, we also can see from the local of the left graph and the right graph that the relationship is not fit for the case with higher thrust, the minimum flight time does not decrease significantly with the increasing of the magnitude of the maximal thrust when the thrust is greater than a magnitude(for example:60N). Then, it becomes natural to suppose the impulse thrust as the limit of the continuous low-thrust orbital maneuver when the maximum thrust magnitude grows, but which is not proven in theory.

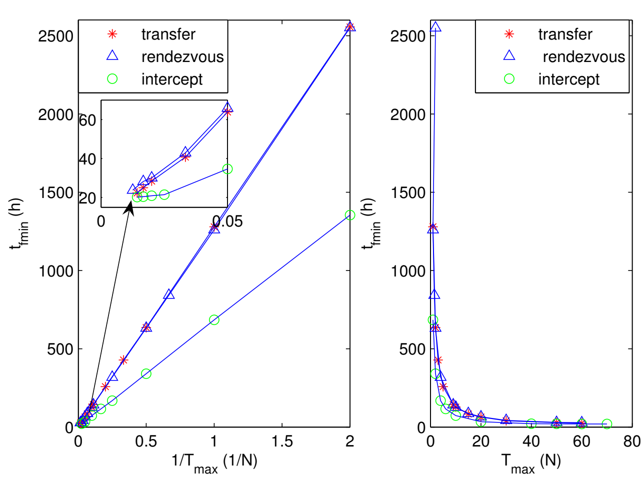

Fig. 4. Near constancy of the product $t_{f\min} \times T_{\max}$

VI. CONCLUSIONS

We have considered in this paper the continuous low-thrust minimum time orbital maneuver which include the transfer, intercept and rendezvous problem. The modified equinoctial elements in the Gauss coordinates are used to describe the movement of the satellite, and the terminal constraints of the three maneuver missions are studied respectively by transversality conditions. Under the unified framework, the time-optimal maneuver problem is deduced into the two-point boundary-value problem by the Maximum Principle.

Numerically, we have used the single shooting method which has been proved to be very efficient. The simulation results demonstrate that: 1) for a fixed maximum thrust, the orbital transfer and rendezvous problem share almost the same optimal trajectory; 2)for the same maximum thrust, the minimum flight time of the intercept mission is lower than that of the transfer or rendezvous problem, which is obvious; 3)the conjecture that the product of the minimum time and the maximal thrust is nearly constant is also fit for the intercept and rendezvous problem, but one also can see that the relationship is not tenable in the high thrust case. However, the above conclusions are just obtained from the numerical experiment, we will try to make it clear theoretically in the succeeded research.

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