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Stabilization of Relative Equilibria for Coordinated Underwater Vehicles*

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Abstract: This paper considers the problem of relative equilibria stabilization for two underwater vehicles coupled by a coordinating law that depends only on their relative configurations. Both the coordinating and stabilizing control laws are derived using energy shaping on a reduced phase space after symmetry reduction. The proposed method is physically motivated and avoids linearization or cancellation of nonlinearities.

Key Words: Underwater Vehicles, Relative Equilibria, Coordination, Stabilization

1 INTRODUCTION

Underwater vehicles (UVs) are enjoying more and more attention from ocean scientists as well as control theorists. A fleet of multiple UVs moving together in a prescribed pattern (formation) can form an efficient mobile sensor network for environment monitoring, aquatic community surveys, and oil exploration. UVs also provide a rich test bed for control techniques developed for multiple mechanical systems such as geometric control and coordination control. Representative work on UVs' dynamics and control includes [1, 2]. Recent work on control of multiple mechanical systems includes [3, 4].

This paper investigates the stabilization technique for steady motions, called relative equilibria, of two UVs which are coupled by control inputs that depend only on the relative configuration (position and attitude). Most man-built UVs, under fairly assumptions (an isotropic surrounding fluid and bottom heavy structure), enjoy symmetries in their dynamics, i.e., the dynamics is invariant to the translations in any direction and rotations about the axis parallel to the local gravity. For a UV group, the most obvious symmetry is associated with the invariance of the dynamics to the absolute position and attitude of the group. The dynamics of a mechanical system with symmetry can be reduced to simpler dynamics evolving on a smaller phase space using mechanical reduction [5]. The equilibria of the reduced dynamics are called the original system's relative equilibria. In this paper, the relative equilibria correspond to UVs' steady motions with constant body-fixed velocity while maintaining a relative configuration within the group. Parameterized by different kinds of steady motions and relative configurations, the relative equilibria build a family of coordinated trajectories that can be used in motion planning and formation maneuver.

Stabilization of relative equilibria for a single mechanical system with application to underactuated UVs has been investigated in [6], a potential shaping with damping control method is proposed there to exponentially stabilize the relative equilibria of a mechanical system with symmetry. The proposed method assumes the external force inputs and natural stability of the unforced system in certain motion directions. Paper [7] employs kinetic energy shaping to stabilize the relative equilibria of an UV using only internal rotors. The asymptotical stability is achieved by adding feedback dissipations. Paper [8] studies the relative equilibria and their stability in a coordinated mechanical system group with applications to UV and satellite groups. The artificial potential based control developed there allows for achievement of any prescribed relative configuration across the group. However, the stabilization of group's relative equilibria involving unstable node dynamics is not considered. Paper [9] studies the stability of the relative equilibria in a coordinated network of rotating rigid bodies in the case that individual node has unstable dynamics. The developed stabilization method uses kinetic energy shaping.

The aforementioned papers share a common idea, energy shaping, which means modifying the energy of the original mechanical system to make the desired state a stable equilibrium. Control torques/forces implement the shaped energy. The well known controlled lagrangian method [10], a generalization of energy shaping, is developed for stabilizing a class of underactuated mechanical systems with symmetry. We adopt this idea to stabilize relative equilibria of a group of two UVs, which involve unstable dynamics. To coordinate, we introduce the control inputs that artificially couple the individuals such that the two-UV group acts as one multibody mechanical system. To stabilize the relative equilibria, an energy shaping term is added to the Hamiltonian of the UV group such that the desired coordinated motion is a stable relative equilibrium for the controlled dynamics.

The paper is organized as follows. In Section 2, the dynamics of a two-UV group is described. In Section 3, the coordinating and stabilizing control laws are derived using energy shaping. In Section 4, a numerical simulation is given to demonstrate the results of the control method. In Section 5, we conclude the paper and give research problems motivated by the derived method for future research.

2 UNDERWATER VEHICLE DYNAMICS

The UV is modeled as a neutrally buoyant (the gravity is balanced by buoyancy) rigid body in an ideal fluid. The dynamics is described in a Hamiltonian setting which helps exploiting the symmetry structure of the system. The notation in this section mostly follows [1].

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2.1 Single UV

Fig. 1 illustrates the notation. Let $\{I_x, I_y, I_z\}$ denote the inertial coordinate frame, and $\{b_x, b_y, b_z\}$ denote the body coordinate frame with origin at the center of buoyancy (CB). Let $\{e_1, e_2, e_3\}$ be the standard Euclidean basis for $\mathbb{R}^3$. The vector from CB to the center of gravity (CG) in the body frame is denoted as le_3. The distance $l > 0$ means that the UV is bottom heavy. Lengths of three semi axes are assumed $l_x > l_y$, $l_x > l_z$. Let $\Upsilon = R^T e_3$ be the direction of the gravity in the body frame, where $R \in SO(3)$ is the rotational matrix describing the attitude of the UV in the inertial frame. The group $SO(3)$ is known as the special orthogonal group in 3 dimensions, and is the set of all 3×3 matrices with determinant equal to $+1$.

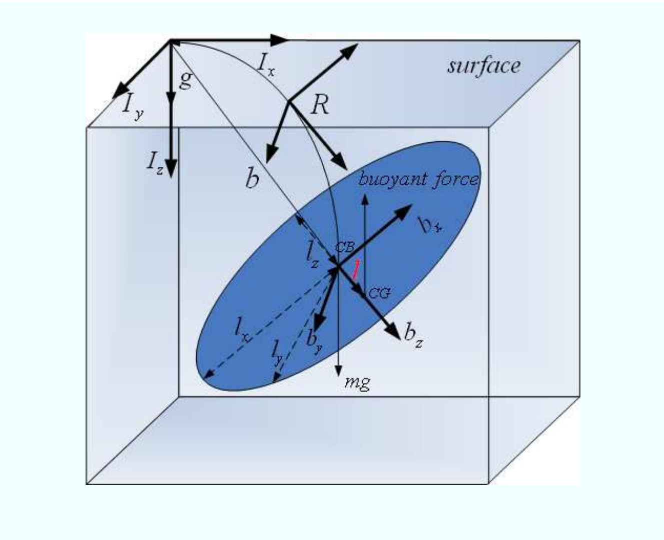

Fig. 1 Notation for single UV model

The kinematic equations for the UV are

$$\dot{R} = R\hat{\Omega}, \dot{b} = Rv, \tag{1}$$

where $R \in SO(3)$, $b \in \mathbb{R}^3$ denote the UV's configuration, $\Omega \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ denote the UV's velocity expressed in the body frame, and the operator $\hat{\cdot}$ is defined as $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$.

Let m be the mass of the UV, J_b be the moment of inertial matrix of the UV. Let m_a and J_a be the added mass matrix and the added moment of inertial matrix due to the influence of the fluid surrounding the UV. Let

$$\begin{aligned} &M = mI_3 + m_a = \text{diag}\left(m_x, m_y, m_z\right), \\ &J = J_b + J_a = \text{diag}\left(J_x, J_y, J_z\right), \\ &D = ml\hat{e}_3, \end{aligned}$$

where I_3 is the 3×3 identity matrix. By definition, the following matrix

$$\bar{M} := \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} J & D \\ D^T & M \end{pmatrix}^{-1}$$

is positive definite.

The linear and angular momentum of the UV in the body frame are given as

$$P = Mv + D^T\Omega, \ \Pi = J\Omega + Dv.$$

In the Hamiltonian setting, the state of UV can be written as (R, b, Π, P) and the Hamiltonian is the sum of the kinetic energy K and the potential energy V,

$$\begin{aligned} H\left(\Pi, P, \Upsilon\right) &= K + V \\ &= \tfrac{1}{2}\left(\Pi^T A \Pi + 2\Pi^T B^T P + P^T C P\right) \\ &\quad -mgl\left(\Upsilon \cdot e_3\right). \end{aligned}$$

Define a Lie group

$$G_S := \left\{(R, b) \in SE\left(3\right) \middle| R^T e_3 = e_3\right\},$$

where $SE(3) = SO(3) \times \mathbb{R}^3$ is the three-dimensional Special Euclidean group. The (left) action of an element $(\bar{R}, \bar{b}) \in G_S$ on the UV's state is given as

$$(\bar{R}, \bar{b})(R, b, \Pi, P) = (\bar{R}R, \bar{R}b + \bar{b}, \Pi, P).$$

Hence under the action of G_S, the Hamiltonian (and likewise the dynamics of the UV) holds

$$\begin{aligned} (R, b)H\left(\Pi, P, \Upsilon\right) &= H\left(R\Pi, RP, R\Upsilon\right) \\ &= H\left(\Pi, P, \Upsilon\right), \end{aligned} \tag{2}$$

for all $(R, b) \in G_S$.

Equality (2) means that the Hamiltonian is unchanged if we translate the inertial frame in any direction and rotate it about the local gravity direction. This invariance property is called *symmetry* and G_S is called the symmetry group. We can factor out the absolute position and gravity-directional orientation in the UV's equations of motion to obtain the reduced dynamics (symmetry reduction in [1]) as

$$\begin{aligned} \begin{pmatrix} \dot{\Pi} \\ \dot{P} \\ \dot{\Upsilon} \end{pmatrix} &= \Lambda\left(\Pi, P, \Upsilon\right) \nabla H \\ &= \begin{pmatrix} \Pi \times \Omega + P \times v - mgl\left(\Upsilon \times e_3\right) \\ P \times \Omega \\ \Upsilon \times \Omega \end{pmatrix}, \end{aligned} \tag{3}$$

where ∇ is the gradient operator, and Λ is the Poisson tensor given by

$$\Lambda\left(\Pi, P, \Upsilon\right) = \begin{pmatrix} \hat{\Pi} & \hat{P} & \hat{\Upsilon} \\ \hat{P} & 0 & 0 \\ \hat{\Upsilon} & 0 & 0 \end{pmatrix}. \tag{4}$$

An *orbit* of the action of G_S through a point $(R_x, b_x) \in SE(3)$ is the set of configurations which can be reached from (R_x, b_x) under the G_S action, i.e.,

$$\text{orb}\left(R_x, b_x\right) := \left\{RR_x, Rb_x + b \,|(R, b) \in G_S\right\}.$$

Definition 1 A relative equilibrium for the UV is a symmetry group orbit that corresponds to an equilibrium for the reduced dynamics (3).

In Fig. 2, a UV, with its b_z axis aligned with the direction of gravity, constantly translates along its longest axis b_x without spinning ($\Omega = 0$) is a relative equilibrium. This relative equilibrium takes the form

$$\left[\Pi_e, P_e, \Upsilon_e\right] = \left[DM^{-1}P_e, P_e, \Upsilon_e\right],$$

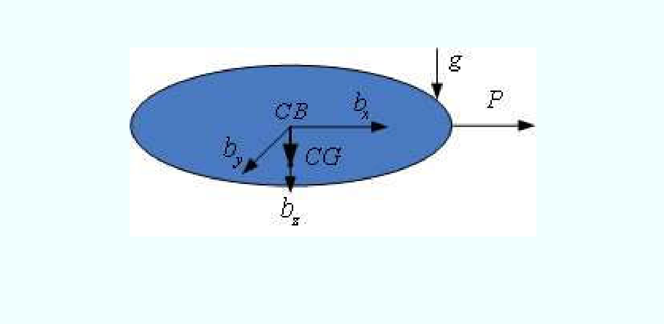

Fig. 2 A relative equilibrium of single UV

where

$$\Pi_e = \begin{bmatrix} 0 \\ mlP_e^x/m_x \\ 0 \end{bmatrix}, P_e = \begin{bmatrix} P_e^x \\ 0 \\ 0 \end{bmatrix}, \Upsilon_e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

Unfortunately, a bottom heavy UV's translation along any but its shortest axis is unstable [1]. Therefore, the relative equilibrium (5) is unstable.

2.2 Two-UV Group

In this paper, we assume that two UVs are identical. Fig. 3 illustrates the notation.

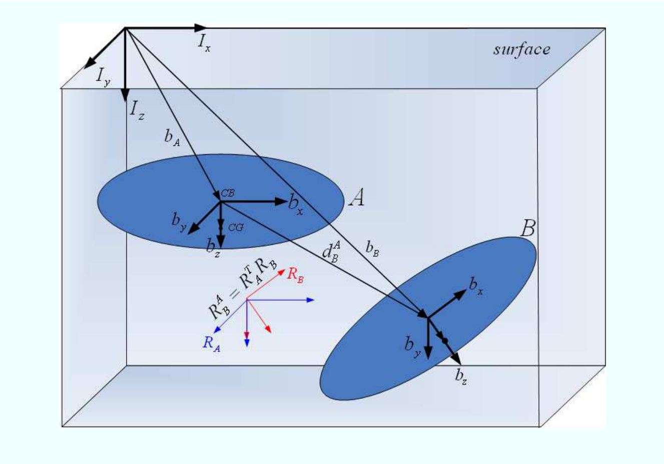

Fig. 3 Notation for 2-UV group

Let (R_A, b_A, R_B, b_B) describe the configurations of two UVs in the inertial frame. The relative attitude (B with respect to A) is defined as

$$R_B^A := R_A^T R_B = [\Sigma_1, \Sigma_2, \Sigma_3],$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are three columns of the matrix. The relative position (B with respect to A) is defined as

$$d_B^A := R_A^T (b_B - b_A).$$

As mentioned in the introduction, we admit the invariance of the UV-group dynamics to the absolute position and attitude of the UV group. Therefore, the symmetry group for the UV group is

$$\begin{aligned} G_{(K,r)} \quad := \quad & \{(R_1, b_1, R_2, b_2) \in G_S \times G_S | \\ & R_1^T K R_2 = K, R_1 r + b_1 - K b_2 = r\}, \end{aligned}$$

where $K \in SO(3)$ and $r \in \mathbb{R}^3$ describe the relative configuration between two UVs.

Similarly, we can factor out the absolute configuration of each UV in space and only retain information on the relative configuration in the group. The reduced dynamics is

$$\begin{aligned} & \left(\dot{\Pi}_A, \dot{P}_A, \dot{\Upsilon}_A, \dot{\Pi}_B, \dot{P}_B, \dot{\Upsilon}_B, \dot{\Sigma}_1, \dot{\Sigma}_2, \dot{\Sigma}_3, \dot{d}_B^A\right)^T \\ & = \Lambda \nabla H_{AB} (\Pi_A, P_A, \Upsilon_A, \Pi_B, P_B, \Upsilon_B), \end{aligned} \tag{6}$$

where $H_{AB} = H_A + H_B$ is the UV group's Hamiltonian. The Poisson tensor Λ is given by

$$\Lambda = \begin{bmatrix} \Lambda_A & 0 & \tilde{\Sigma} & -\tilde{\Theta}^T \\ 0 & \Lambda_B & \Psi & -\Re^T \\ -\tilde{\Sigma}^T & \Psi & 0 & 0 \\ \tilde{\Theta} & \Re & 0 & 0 \end{bmatrix}, \tag{7}$$

where Λ_A, Λ_B have the same form of (4), $\tilde{\Sigma}$ stands for $\begin{bmatrix} \hat{\Sigma}_1 & \hat{\Sigma}_2 & \hat{\Sigma}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Ψ stands for $\begin{bmatrix} 0 & -\Sigma_3 & \Sigma_2 \\ \Sigma_3 & 0 & -\Sigma_1 \\ -\Sigma_2 & \Sigma_1 & 0 \end{bmatrix}$, $\tilde{\Theta}$ has the form $\left[\hat{d}_B^A, -I_3, 0\right]$, and $\Re$ has the form $\left[R_B^A, 0, 0\right]$. Alternatively, the reduced equations of motion are

$$\begin{aligned} \dot{\Pi}_i &= \Pi_i \times \Omega_i + P_i \times v_i - mgl(\Upsilon_i \times e_3), \\ \dot{P}_i &= P_i \times \Omega_i, \\ \dot{\Upsilon}_i &= \Upsilon_i \times \Omega_i, i = A, B, \\ \dot{\Sigma}_1 &= \Omega_{Bz}\Sigma_2 - \Omega_{By}\Sigma_3 + \Sigma_1 \times \Omega_A, \\ \dot{\Sigma}_2 &= \Omega_{Bx}\Sigma_3 - \Omega_{Bz}\Sigma_1 + \Sigma_2 \times \Omega_A, \\ \dot{\Sigma}_3 &= \Omega_{By}\Sigma_1 - \Omega_{Bx}\Sigma_2 + \Sigma_3 \times \Omega_A, \\ \dot{d}_B^A &= d_B^A \times \Omega_A - v_A + R_B^A v_B, \end{aligned} \tag{8}$$

where $\Omega_{Bx}, \Omega_{By}, \Omega_{Bz}$ are three components of Ω_B. We write the relative configuration evolution part, i.e. the last four equations, in a collective form

$$\dot{X} = \Xi(X, \Omega_A, \Omega_B, v_A, v_B),$$

where $X = \left[\Sigma_1, \Sigma_2, \Sigma_3, d_B^A\right]^T$.

By Definition 1, for the UV group, a class of relative equilibria (also an equilibrium of (8)), without spinning, has the following form

$$\begin{aligned} & (DM^{-1}P_{Ae}, P_{Ae}, \Upsilon_{Ae}, DM^{-1}P_{Be}, P_{Be}, \Upsilon_{Be}, \\ & \Sigma_{1e}, \Sigma_{2e}, \Sigma_{3e}, d_{Be}^A). \end{aligned}$$

3 COORDINATION AND STABILIZATION

To make the equilibrium of (8) possess a desired relative configuration $X = X_d$, coordinating controls should be applied. The following coordinating design is based on the introduction of a coupling artificial potential, while preserving the system (6)'s Hamiltonian structure (7).

Assumption 1 Each UV is fully actuated.

Assumption 2 There is a fixed, undirected, connected communication topology across the two-UV group, which means that two UVs can sense and communicate each other about the relative configuration information.

Without loss of generality, we consider the case when two UVs are to be aligned and to achieve a relative position vector d directed from $\mathrm{UV_A}$ to $\mathrm{UV_B}$. We choose the artificial potential (used in [8]) to couple the UVs as

$$\begin{array}{rcl} V & = & \frac{1}{2}\left(d_B^A - d\right)^T G_d \left(d_B^A - d\right) \\ & + & \lambda\left(\Sigma_1 \cdot e_1 + \Sigma_2 \cdot e_2 + \Sigma_3 \cdot e_3\right), \end{array}$$

where $G_d > 0, \lambda < 0$.

Lemma 1 The following controlled equations

$$\begin{array}{rcl} \dot{\Pi}_A & = & L_A - mgl\left(\Upsilon_A \times e_3\right) + \tau_A, \\ \dot{P}_A & = & P_A \times \Omega_A + f_A, \\ \dot{\Upsilon}_A & = & \Upsilon_A \times \Omega_A, \\ \dot{\Pi}_B & = & L_B - mgl\left(\Upsilon_B \times e_3\right) + \tau_B, \\ \dot{P}_B & = & P_B \times \Omega_B + f_B, \\ \dot{\Upsilon}_B & = & \Upsilon_B \times \Omega_B, \\ \dot{X} & = & \Xi\left(X, \Omega_A, \Omega_B, v_A, v_B\right), \end{array} \tag{9}$$

where $L_i = \Pi_i \times \Omega_i + P_i \times v_i, i = A, B$, with the coordinating forces and torques

$$\begin{array}{rcl} \tau_A & = & \lambda\left(\Sigma_1 \times e_1 + \Sigma_2 \times e_2 + \Sigma_3 \times e_3\right) \\ & & +d_B^A \times G_d\left(d_B^A - d\right), \\ \tau_B & = & -\lambda\left(\Sigma_1 \times e_1 + \Sigma_2 \times e_2 + \Sigma_3 \times e_3\right), \\ f_A & = & G_d\left(d_B^A - d\right), \\ f_B & = & -\left[\Sigma_1, \Sigma_2, \Sigma_3\right]^T G_d\left(d_B^A - d\right), \end{array} \tag{10}$$

possess a relative equilibrium of interest as

$$\begin{array}{l} \left(\Pi_A, P_A, \Upsilon_A, \Pi_B, P_B, \Upsilon_B, \Sigma_1, \Sigma_2, \Sigma_3, d_B^A\right) = \\ \left(DM^{-1}P_e, P_e, e_3, DM^{-1}P_e, P_e, e_3, e_1, e_2, e_3, d\right), \end{array} \tag{11}$$

where $P_e = \left[P_e^x, P_e^y, P_e^z\right]^T = \left[P_e^x, 0, 0\right]^T$.

Proof It can be seen that the aligned attitudes $R_A = R_B$ along with the separation $d_B^A = d$ minimize the artificial potential V. We modify the original Hamiltonian as

$$H_V = H_{AB} + V.$$

Substitute H_V for H_{AB} in (6), new terms appearing at the right side of equations are identified with control inputs (10), and new equations (9) are interpreted as controlled equations. Under Assumption 1, the derived control can be applied. Direct substitution shows that (9) varnishes at (11), thus proves the relative equilibrium.

Remark 1 The relative equilibrium (11) corresponds to both UVs moving along their longest axes (perpendicular to the gravity) constantly, aligned without spinning, and maintaining their relative position. However, this relative equilibrium is unstable since each UV translates along its longest (unstable) axis. Additional stabilizing control need to be applied to each UV.

3.1 Stabilization

We take the potential shaping idea in paper [11] as a reference for the stabilizing design. To preserve the symmetry in the UV group, we derive the stabilizing control that only depends on relative configurations.

The relative attitude at equilibrium (11) is $R_{Be}^A = R_{Ae}^B = I_3$. Let $v_e = M^{-1}P_e$, then the unit vector $k = R_{Ae}^B v_e / \| v_e \| = v_e / \| v_e \|$ describes the direction of $\mathrm{UV_A}$'s linear velocity in $\mathrm{UV_B}$'s body frame when at equilibrium. Define $\tilde{k}_A = R_B^A k$ as k in $\mathrm{UV_A}$'s body frame. Consider the energy shaping term for $\mathrm{UV_A}$ as

$$E_{SA} = \tilde{k}_A \cdot \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e,$$

where α is the control gain. The modified Hamiltonian is

$$\begin{array}{rcl} H_{SA} & = & H_A + E_{SA} \\ & = & K - mgl\left(\Upsilon \cdot e_3\right) \\ & & +\tilde{k}_A \cdot \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e. \end{array}$$

The control inputs can be obtained by substituting H_{SA} into (3), and reading the new terms appearing at the right side of the controlled equations. Therefore we get the stabilizing torque

$$\tau_{stA} = \tilde{k}_A \times \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e. \tag{12}$$

Remark 2 The above stabilizing design is inspired by the naturally stabilizing effect of gravity and buoyancy in the case of a rising or falling UV [1]. The control (12) introduces a new restoring torque relative to the direction of translation (mimics the structure of the gravity-buoyancy induced restoring torque).

The closed loop system for $\mathrm{UV_A}$ is

$$\begin{pmatrix} \dot{\Pi}_A \\ \dot{P}_A \\ \dot{\Upsilon}_A \end{pmatrix} = \begin{pmatrix} L_A + \tilde{k}_A \times \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e - mgl\left(\Upsilon \times e_3\right) \\ P_A \times \Omega_A \\ \Upsilon_A \times \Omega_A \end{pmatrix}, \tag{13}$$

where L_A is defined as in (9). Note that

$$\left(\Pi, P, \Upsilon\right) = \left(DM^{-1}P_e, P_e, e_3\right) \tag{14}$$

is an equilibrium for the closed loop system (13) as desired. For the coordinated UV group, we have

$$\begin{array}{rcl} \dot{\Pi}_A & = & L_A + \tilde{k}_A \times \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e \\ & & -mgl\left(\Upsilon_A \times e_3\right) + \tau_A, \\ \dot{P}_A & = & P_A \times \Omega_A + f_A, \\ \dot{\Upsilon}_A & = & \Upsilon_A \times \Omega_A, \\ \dot{\Pi}_B & = & L_B + \tilde{k}_B \times \left(\|v_e\| I_3 - \alpha M^{-1}\right) P_e \\ & & -mgl\left(\Upsilon_B \times e_3\right) + \tau_B, \\ \dot{P}_B & = & P_B \times \Omega_B + f_B, \\ \dot{\Upsilon}_B & = & \Upsilon_B \times \Omega_B, \\ \dot{X} & = & \Xi\left(X, \Omega_A, \Omega_B, v_A, v_B\right), \end{array} \tag{15}$$

where $\tilde{k}_B = R_A^B R_{Be}^A v_e / \| v_e \| = R_A^B k$. The stability result is given in the following theorem.

Theorem 1 The relative equilibrium (11) is also an equilibrium for the closed loop system (15) and (10), and it is Lyapunov stable if $\alpha I_3 - \left(\|v_e\| + \lambda\right) M > 0$.

Proof By Lemma 1 and comparing the form of (14) with (11), there comes the relative equilibrium. The Lyapunov stability is proved using the energy Casimir method [12] in three steps. First, we need to find several independent Casimir functions (also see [12])

$$C\left(\Pi_i, P_i, \Upsilon_i, \Sigma_1, \Sigma_2, \Sigma_3, d_B^A\right), i = A, B$$

which satisfy

$$\nabla C \subset Null\left(\Lambda^T\right),$$

where Λ is the Poisson tensor (7). This means that Casimir functions are conserved quantities along the solution of the system. Second, a Lyapunov function is constructed as

$$H_\Phi = H_V + E_{SAB} + \Phi\left(C\right),$$

where E_{SAB} is the stabilizing energy shaping term for the UV group, Φ is chosen to be a smooth function such that the desired relative equilibrium is a critical point for H_Φ. Third, the condition that the second derivative of H_Φ be definite at the relative equilibrium is sufficient for Lyapunov stability of the relative equilibrium.

In our case, there are six Casimir functions,

$$\begin{aligned}
C_1 &= \tilde{k}_A^T \tilde{k}_A, \\
C_2 &= \tilde{k}_B^T \tilde{k}_B, \\
C_3 &= \|\Sigma_1\|^2 + \|\Sigma_2\|^2 + \|\Sigma_3\|^2, \\
C_4 &= \tilde{k}_A^T P_e + \tilde{k}_B^T P_e, \\
C_5 &= \tilde{k}_A^T v_e + \tilde{k}_B^T v_e, \\
C_6 &= P_A^T P_A + 2P_B^T R_A^B P_A + P_B^T P_B.
\end{aligned}$$

By choosing Φ_C to satisfy the first order condition, we have

$$\begin{aligned}
\Phi\left(C\right) = &-\tfrac{\lambda}{2}\left(\|\Sigma_1\|^2 + \|\Sigma_2\|^2 + \|\Sigma_3\|^2\right) \\
&-\tfrac{1}{2}\left(\left(M^{-1}P_A\right)^T P_A + \left(M^{-1}P_B\right)^T P_B\right) \\
&-\left(M^{-1}P_B\right)^T R_A^B P_A \\
&+\alpha\left(\tilde{k}_A^T \tilde{k}_A + \tilde{k}_A^T v_e + \tilde{k}_B^T \tilde{k}_B + \tilde{k}_B^T v_e\right) \\
&-\|v_e\|\left(\tilde{k}_A^T P_e + \tilde{k}_B^T P_e\right).
\end{aligned}$$

Recall that

$$\bar{M} := \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} J & D \\ D^T & M \end{pmatrix}^{-1} > 0,$$

the second derivative of H_Φ at the relative equilibrium has the following form

$$\begin{bmatrix} I_2 \otimes \bar{M} & 0 & 0 \\ 0 & I_3 \otimes \left(\alpha M^{-1} - \left(\|v_e\| + \lambda\right) I_3\right) & 0 \\ 0 & 0 & G_d \end{bmatrix}, \tag{16}$$

where the operator $\otimes$ stands for the Kronecker product. The condition $\alpha I_3 - \left(\|v_e\| + \lambda\right) M > 0$ ensures the positive definiteness of the matrix (16), thus is a sufficient condition for the Lyapunov stability.

4 SIMULATION EXAMPLE

We illustrate the coordinating and stabilizing laws derived in the Section 3 with a numerical simulation example in which two UVs are required to align each other, move along their longest axes b_x without spinning, and maintain a prescribed relative position.

The desired relative equilibrium for the UV group is

$$\left(\Pi_e, P_e, e_3, \Pi_e, P_e, e_3, e_1, e_2, e_3, d\right),$$

where $P_e = \left[m_x v_{ex}, 0, 0\right]^T$, $\Pi_e = \left[0, mlv_{ex}, 0\right]^T$, and the data are as follows

$$v_{ex} = 0.5 \text{ m/s}, d = \left[5, 9, 0\right]^T \text{m}, v_e = \left[v_{ex}, 0, 0\right]^T.$$

The physical parameters of the UV are in Tab. 1.

Tab. 1 Physical Parameters of the UV

principal axis	total mass	total inertial	length
b_x	2 kg	1 kg · m^2	4 m
b_y	5 kg	4 kg · m^2	2 m
b_z	5 kg	4 kg · m^2	2 m

Besides, the mass of UV is $m = 1$kg, and the distance from CB to CG is $l = 1$m.

The torques and forces for coordinating are

$$\begin{aligned}
\tau_A &= \lambda\left(\Sigma_1 \times e_1 + \Sigma_2 \times e_2 + \Sigma_3 \times e_3\right) \\
&\quad + d_B^A \times G_d\left(d_B^A - d\right), \\
\tau_B &= -\lambda\left(\Sigma_1 \times e_1 + \Sigma_2 \times e_2 + \Sigma_3 \times e_3\right), \\
f_A &= G_d\left(d_B^A - d\right), \\
f_B &= -\left[\Sigma_1, \Sigma_2, \Sigma_3\right]^T G_d\left(d_B^A - d\right).
\end{aligned}$$

The torques for stabilizing are

$$\begin{aligned}
\tau_{stA} &= \left(R_B^A e_1\right) \times \left(P_e^x \|v_e\| - \alpha P_e^x / m_x\right) e_1, \\
\tau_{stB} &= \left(R_A^B e_1\right) \times \left(P_e^x \|v_e\| - \alpha P_e^x / m_x\right) e_1.
\end{aligned}$$

The control parameters are chosen as $\lambda = -1$, $G_d = 2I_3$, $\alpha = 9$. The initial conditions of the two-UV group are

$$R_B^A\left(0\right) = \begin{bmatrix} 0.8201 & -0.5942 & 0.0606 \\ 0.2445 & 0.2341 & -0.9410 \\ 0.5449 & 0.7695 & 0.3330 \end{bmatrix},$$

$$d_B^A\left(0\right) = \left[4, -7, 25\right]^T,$$

$$\left(\Omega_A, v_A, \Omega_B, v_B\right)\left(0\right) = \left(0.2, 0.2, 0.2, 0, 0, 0, 0.3, 0.3, 0.3, 0, 0, 0\right).$$

Since the above control inputs only achieve Lyapunov stability for the relative equilibrium according to Theorem 1, dissipation controls can be introduced to achieve asymptotical stability. Therefore, we add (15) with linear dissipations in the $\dot{\Pi}_A, \dot{P}_A, \dot{\Pi}_B, \dot{P}_B$ equations. The damping coefficients are set to 2 respectively.

Figure 4 shows the movement of the UV group in the inertial frame for a simulation time of 80 seconds. Two UVs are separated by a distance more than 30 m, and not aligned at the initial time. During the motion, they align their attitudes, adjust their relative position to achieve a prescribed relative configuration. Finally, they all move along their longest body axes stably.

Figure 5 shows the dynamics of the relative attitude, i.e. the entries of the matrix $R_B^A = \left[\Sigma_1, \Sigma_2, \Sigma_3\right]$. It can be observed that three diagonal entries $\Sigma_1^1, \Sigma_2^2, \Sigma_3^3$ all tend to 1 while the remaining entries tend to zero.

Figure 6 shows the dynamics of the relative position expressed in the UV$_A$'s body frame. The three components of the relative position evolve from $\left[4, -7, 25\right]$ to $\left[5, 9, 0\right]$ as expected.

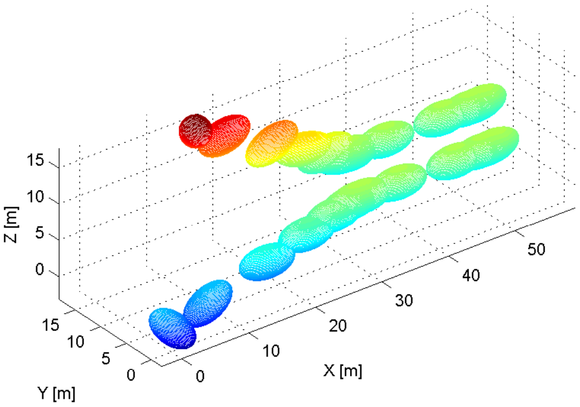

Fig. 4 Coordination of two UVs in 3D space

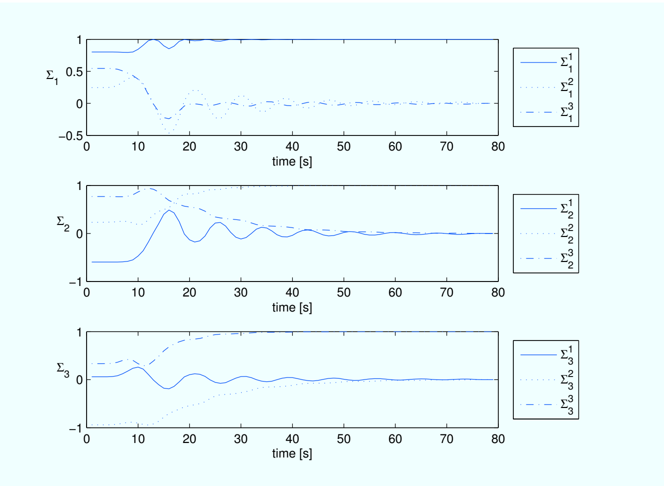

Fig. 5 Dynamics of relative attitude

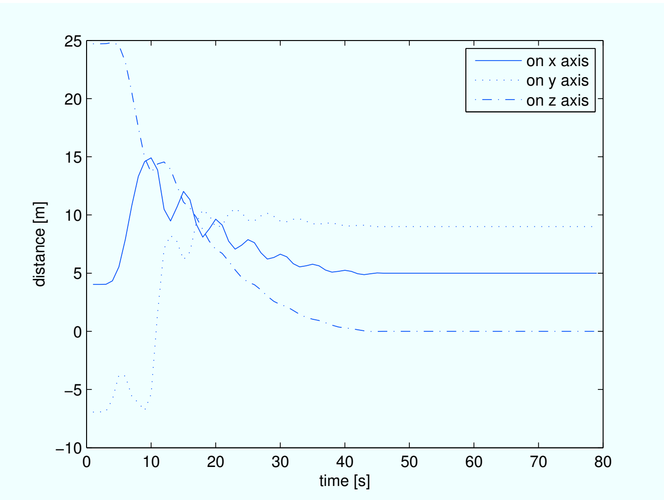

Fig. 6 Dynamics of relative position in UV_A's body frame

5 CONCLUSIONS

In this paper, we have considered the problem of coordinated motion stabilization for two UVs coupled by a coordinating law that depends only on their relative configuration. The symmetry reduction produces a reduced dynamics from which we can efficiently formulate the control law. An artificial potential is introduced to coordinate two UVs and energy shaping is used to stabilize the relative equilibria. The derived control law is physically motivated and do not rely on the linearization or cancellation of the nonlinearities.

However, throughout this paper, each UV is assumed to be fully actuated which is a strict condition. Another assumption is the complete communication topology across the group. Ongoing work includes energy shaping design in the relative equilibria stabilization for underactuated UVs or UV groups. It is also of interest to consider coordination problems in a $N > 2$ group with limited communications, e.g. unidirectional communication.

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