Finite-time Optimal Formation Control on Lie Groups

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Abstract: This paper studies the problem of finite-time optimal formation control for agents evolving on Lie groups SE(2), for the situation when the formation time and/or the cost function need to be considered. The finite-time optimal formation control laws are proposed for the two-agent case. Considering that disturbances exist after the terminal time, the trajectory tracking control law is given to keep the formation. Finally, some numerical examples are given to illustrate the effectiveness of the theoretical results.

Key Words: Finite-time Formation, Optimal, Lie Group, Trajectory Tracking.

1 Introduction

Recently, the formation control of multiagent systems has attracted much attention for the potential applications, such as satellite attitude control, unmanned aircraft formation flying and sampling, distributed sensor networks, automated highway systems (AHS), etc [1–3]. Compared to the traditional monolithic systems, the formation control reduces the systems cost, breaches the size constraints, and prolongs the life span of the systems [4]. Furthermore, the robustness and flexibility are enhanced.

For systems evolving on Euclidean spaces, various consensus algorithms are designed to achieve the desired formation and many relevant engineering issues are considered, such as time-delays, switching topology, and finite-time formation [5-8]. However, in many applications, the agents evolve on nonlinear manifold such as satellite attitudes on SO(3) and vehicles move in SE(2) or SE(3); these particular manifolds share the geometric structure of a Lie group [9]. Formation control on nonlinear manifolds is inherently more difficult than on Euclidean spaces [10]. For the formation problem on nonlinear manifolds, the designing methods for control laws can be divided into two main categories. The traditional method is to locally convert the formation problem on nonlinear manifold into the consensus problem on Euclidean space by parametrization [11–15]. The second method is to directly design control laws on nonlinear manifolds [9, 10, 16–19]. The attitude control of multiple spacecraft is considered in [11–15]. In [11], the attitude of spacecraft is described in terms of the Modified Rodriguez Parameters, which have a geometric singularity. The unit quaternion is used to describe the attitude of spacecraft in [12-15], and the derived results are local, because the unit quaternion is not unique. By contrast, the methods in [9, 10, 16-19] can apply to arbitrary initial conditions. The authors took into account the geometry structure of the manifold, i.e. symmetries and directly design control laws on the nonlinear manifold. In [17], the authors presented a Lie group setting for the problem of control of formations, and the set of all possible relative equilibria for arbitrary G-invariant curvature control is described (where G = SE(2) is a symmetry group for the control law), and the proposed control

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law for the two-agent case is proved to stabilize the relative equilibria, which is determined by the model of agents. The coordination on Lie groups is considered in [9]. It gave a general problem formulation, analyzed ensuing conditions and proposed the control laws for the coordinated motion. However, the relative configuration between the agents are constants, which are determined by initial conditions, rather than task requirements. The authors in [18] studied the optimal control problems on some 6-D Lie groups for the case of one agent. The stable synchronization on Lie groups is considered in [19]. The asymptotical control laws are proposed to stabilize the desired relative equilibrium. In many practical applications, the formation algorithms, that obtain the formation in finite time, are more desirable, especially when the multi-maneuver is needed and a high precision control is required. However, it can be seen that for multiple agents evolving on Lie group the finite-time formation problem with the general formation conditions is unsolved. Furthermore, for the formation systems it is significant to guarantee that some performance index is optimal in practical applications. For example, the minimal fuel consumption for the satellites formation with limited fuel is crucial. Therefore, studies of finite-time optimal formation control on Lie groups make good sense in practical cases.

Motivated by the above analysis, the finite-time optimal formation problem for agents evolving on Lie groups SE(2) is considered in the present paper. For the given formation time and the cost function, the finite-time optimal formation control laws are proposed for the two-agent case to achieve the formation in the terminal time and optimize the cost function during the formation. Then, considering that disturbances exist after the terminal time, the trajectory tracking control law is given to keep the formation. Compared to the asymptotical formation control, this control laws derive the formation in finite time, which is specified according to the task requirements in advance, and the cost function is guaranteed to be optimal.

The remainder of this paper is organized as follows. The preliminaries and problem formulation are given in section 2. In section 3, the finite-time optimal formation control laws are proposed for the two-agent systems evolving on Lie group SE(2), and considering that disturbances exist after the terminal time, the trajectory tracking control law is given

to keep the formation. The numerical examples are given in section 4 to illustrate the theoretical results. Concluding remarks are made in section 5.

2 Preliminaries and Problem Formulation

2.1 Model of the Agent

In this paper, the agents are considered to evolve on Lie group SE(2), i.e. the special Euclidean group in the plane. SE(2) describes the configuration of the planar rigid body (translations and rotations) and the element of SE(2) is denoted by

$$g = \left[\begin{array}{cc} R & d \\ 0 & 1 \end{array} \right],$$

where $d \in \mathbb{R}^2$ is the position vector in the plane and $R \in SO(2)$ is an orthogonal matrix with positive determinant. Let g^{-1} denote the group inverse of $g \in SE(2)$. $T_gSE(2)$ is the tangent space to SE(2) at the base element g, and for g = e (identity element), $T_eSE(2)$ is denoted by se(2) and is called Lie algebra of the group SE(2).

Suppose that the kinematics model of the agent is given by

$$\dot{g} = g\hat{\xi} = g\hat{e}_1 u + g\hat{e}_2 v + g\hat{e}_3 w,$$

where the matrix $\hat{\xi} \in se(2)$ is called a twist which can be written as a linear combination of the basis of se(2), i.e. $\hat{e}_i(i = 1, 2, 3)$ defined as follows,

$$\hat{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \hat{e}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\hat{e}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $[u, v, w]^T \in \mathbb{R}^3$ is considered as the control input.

Let $\xi = [u, v, w]^T$. There exists an invertible mapping \wedge : $\mathbb{R}^3 \to se(2)$.

$$\wedge : \left(\begin{array}{c} u \\ v \\ w \end{array}\right) \longrightarrow \left(\begin{array}{cc} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{array}\right)$$

It is easy to prove that se(2) is isomorphic to \mathbb{R}^3 . The inverse mapping is denoted by $\vee : se(2) \to \mathbb{R}^3$.

2.2 Linear Functionals on $T_gSE(2)$ and se(2)

Let $P_g \in T_g^*SE(2)$ denote the linear functional on $T_gSE(2)$, where $T_g^*SE(2)$ is the cotangent space at g, which is dual to $T_gSE(2)$. Using the left-invariant property, it is easy to obtain that

$$P_g(\dot{g}) = g^* P_g(g^{-1}\dot{g}) = \hat{p}^*(\hat{\xi}),$$

where $\hat{p}^* \in se(2)^*$ is the linear functional on se(2). For the matrix Lie group concerned, we have

$$P_g = g^{-T} \hat{p}^*. \tag{1}$$

For the linear functional on se(2), the following definition is given.

Definition 1. $\hat{p}^*(\hat{\xi}) = tr(diag(\frac{1}{2}, \frac{1}{2}, 1)\hat{p}^T\hat{\xi}).$

Define the dual basis of $se(2)^*$ as $\hat{e}_i^* = \hat{e}_i^T$ (i = 1, 2, 3), such that

$$\hat{e}_i^*(\hat{e}_j) = \delta_{ij} \ (i, j = 1, 2, 3),$$

where δ_{ij} is the Kronecker delta. Let $p_i = \hat{p}^*(\hat{e}_i)$ (i = 1, 2, 3). Thus, \hat{p}^* can be expressed as

$$\hat{p}^* = p_1 \hat{e}_1^* + p_2 \hat{e}_2^* + p_3 \hat{e}_3^*,$$

and $[p_1, p_2, p_3]^T \in \mathbb{R}^3$ is the coordinates of \hat{p}^* with respect to the dual basis \hat{e}_i^* (i = 1, 2, 3). Then we have

$$P_g(\dot{g}) = \hat{p}^*(\hat{\xi}) = \langle p, \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ presents the inner product on \mathbb{R}^3 .

Next, in order to describe the formation condition in the following, the definition of relative configuration is introduced.

Definition 2. [9] The relative configuration on SE(2) of agent j with respect to agent k is $g_{jk} = g_k^{-1}g_j$.

Remark 1. In the present paper, the left-invariant relative configuration $g_{jk} = g_k^{-1}g_j$ is considered. However, the proposed designing methods can also been applied for the case of the right-invariant relative configuration $\rho_{jk} = g_j g_k^{-1}$.

2.3 Problem Formulation

Consider two identical agents evolving on Lie group SE(2). The kinematics model of the agent *i* is described by

$$\dot{g}_i = g_i \hat{\xi}_i = g_i \hat{e}_1 u_i + g_i \hat{e}_2 v_i + g_i \hat{e}_3 w_i, \ i = 1, 2.$$
 (2)

Suppose the cost function is given as

$$J = \frac{1}{2} \int_{t_0}^{t_f} \left(\xi_1^T(t) \xi_1(t) + \xi_2^T(t) \xi_2(t) \right) dt,$$
(3)

where t_0 and t_f are the initial time and terminal time, respectively. In general, the minimization of the above cost function is to minimize the length of geodesics or the control energy.

The agents are said to achieve a formation defined by a given relative configuration $g_{21} \in SE(2)$, if their configurations satisfy the following condition

$$g_1^{-1}(t_f)g_2(t_f) - g_{21} = 0, (4)$$

where g_{21} is given according to the formation requirements. Let g_{12} denote the inverse of g_{21} .

In this paper, the objective is to design ξ_i (i = 1, 2) for system (2) such that the specified formation condition (4) is achieved at the given terminal time t_f , and the cost function (3) is minimized.

3 Main Results

3.1 Finite-time Formation

In the above section, the problem of finite-time optimal formation control is converted into the optimal control problem. Therefore, the next task is to solve the optimal control problem. Considering the system (2), the Hamiltonian is written as

$$H = -\frac{1}{2} \left(\xi_1^T \xi_1 + \xi_2^T \xi_2 \right) + P_{g_1} \left(g_1 \hat{\xi}_1 \right) + P_{g_2} \left(g_2 \hat{\xi}_2 \right), \quad (5)$$

where $P_{g_i} \in T_{g_i}^*SE(2)$ (i = 1, 2) represents the costate. For convenience, P_{g_i} (i = 1, 2) is denoted by P_i . Then, it follows from Pontryagin's maximum principle (PMP) that the necessary conditions of optimality are written as [20]

$$\dot{g}_i = \frac{\partial H}{\partial P_i} = g_i \hat{\xi}_i,$$
 (6)

$$\dot{P}_i = -\frac{\partial H}{\partial g_i} = -P_i \hat{\xi}_i^T, \quad i = 1, 2.$$
(7)

Lemma 1. For the given formation condition (4), the transversality condition is given by

$$P_1(t_f) = -P_2(t_f)g_{21}^T.$$

Proof. For the given formation condition (4), let

$$f = g_1^{-1}(t_f)g_2(t_f) - g_{21},$$

and f^{ij} is the (i, j)th entry of f. According to PMP [20], we have

$$P_i^{pq}(t_f) = \sum_{j,k=1}^3 \lambda_{jk} \frac{\partial f^{jk}}{\partial g_i^{pq}}(t_f)$$

where $P_i^{pq}(t_f)$ and g_i^{pq} are the (p,q)th entry of $P_i(t_f)$ and $g_i(t_f)$, respectively, and λ_{jk} (j, k = 1, 2, 3) is the undetermined parameter.

Let $\Lambda \in \mathbb{R}^{3 \times 3}$ denote the undetermined matrix with λ_{jk} being the (j, k)th entry. Thus

$$P_i^{pq}(t_f) = \operatorname{tr}\left(\Lambda^T \frac{\partial f}{\partial g_i^{pq}}(t_f)\right)$$
$$= \operatorname{tr}\left(\Lambda^T \frac{\partial (g_1^{-1}g_2)}{\partial g_i^{pq}}(t_f)\right).$$

For i = 1,

$$P_{1}^{pq}(t_{f}) = \operatorname{tr}\left(\Lambda^{T}\frac{\partial(g_{1}^{-1}g_{2})}{\partial g_{1}^{pq}}\right)(t_{f})$$

$$= \operatorname{tr}\left(-\Lambda^{T}g_{1}^{-1}\frac{\partial g_{1}}{\partial g_{1}^{pq}}g_{1}^{-1}g_{2}\right)(t_{f})$$

$$= \operatorname{tr}\left(-\Lambda^{T}g_{1}^{-1}E_{pq}g_{1}^{-1}g_{2}\right)(t_{f})$$

$$= -\operatorname{tr}\left(E_{pq}g_{1}^{-1}g_{2}\Lambda^{T}g_{1}^{-1}\right)(t_{f})$$

$$= -\left(g_{1}^{-1}g_{2}\Lambda^{T}g_{1}^{-1}\right)^{qp}(t_{f}),$$

where $E_{pq} \in \mathbb{R}^{3\times 3}$ is a constant matrix with only (p,q)thentry being 1 and the others being 0. For i = 2,

$$P_2^{pq}(t_f) = \operatorname{tr}\left(\Lambda^T \frac{\partial (g_1^{-1}g_2)}{\partial g_2^{pq}}\right)(t_f)$$
$$= \operatorname{tr}\left(\Lambda^T g_1^{-1} \frac{\partial g_2}{\partial g_2^{pq}}\right)(t_f)$$
$$= \operatorname{tr}\left(\Lambda^T g_1^{-1} E_{pq}\right)(t_f)$$
$$= (g_1^{-T} \Lambda)^{pq}(t_f).$$

In the matrix form,

$$P_{1}(t_{f}) = -\left(g_{1}^{-1}g_{2}\Lambda^{T}g_{1}^{-1}\right)^{T}(t_{f}) = -(g_{1}^{-T}\Lambda g_{21}^{T})(t_{f})$$
$$P_{2}(t_{f}) = \left(g_{1}^{-T}\Lambda\right)(t_{f}).$$

Thus

$$P_1(t_f) = -P_2(t_f)g_{21}^T.$$

Lemma 2. $P_i(t)g_i^T(t)$ is a constant.

Proof. From (6), the control input is written as

$$\hat{\xi}_i = g_i^{-1} \dot{g}_i$$

Substituting the above equality into (7), we obtain

$$\dot{P}_i = -P_i (g_i^{-1} \dot{g}_i)^T = -P_i \dot{g}_i^T (g_i^T)^{-1}$$

Thus

$$\dot{P}_i g_i^T + P_i \dot{g}_i^T = 0.$$

From the equality

$$\frac{d}{dt}(P_i g_i^T) = \dot{P}_i g_i^T + P_i \dot{g}_i^T,$$

it is easy to see that $P_i g_i^T$ is invariant with respect to time t, i.e.

$$\frac{d}{dt}(P_i g_i^T) = 0.$$

Consequently, $P_i g_i^T$ is a constant. From Lemma 2, it can be assumed that

$$P_i(t)g_i^T(t) = \hat{c}_i^*, (i = 1, 2), \tag{8}$$

where \hat{c}_i^* is an unknown constant. From (1), it is obtained that

$$g_i^{-T}(t)\hat{p}_i^*(t)g_i^T(t) = \hat{c}_i^*$$

Using the coadjoint operator, we have

$$\operatorname{Ad}_{g_i^{-1}(t)}^* \hat{p}_i^*(t) = \hat{c}_i^*.$$

Thus

$$\hat{p}_{i}^{*}(t) = g_{i}^{T}(t)\hat{c}_{i}^{*}g_{i}^{-T}(t) = \mathrm{Ad}_{g_{i}(t)}^{*}\hat{c}_{i}^{*},$$
$$\hat{p}_{i}(t) = g_{i}^{-1}(t)\hat{c}_{i}g_{i}(t) = \mathrm{Ad}_{g_{i}^{-1}(t)}\hat{c}_{i}.$$
(9)

In a similar manner to [18], the Hamiltonian (5) is function on the cotangent bundle $T^*SE(2)$ which can be trivialized such that $T^*SE(2) = SE(2) \times se(2)^*$. Therefore the appropriate Hamiltonian is a function on $se(2)^*$, the dual of the Lie algebra se(2) of SE(2). The Hamiltonian (5) can be pulled back by the left transformation and then is written as

$$H = -\frac{1}{2} \left(\xi_1^T \xi_1 + \xi_2^T \xi_2 \right) + \hat{p}_1^* \left(\hat{\xi}_1 \right) + \hat{p}_2^* \left(\hat{\xi}_2 \right)$$
$$= \sum_{i=1}^2 -\frac{1}{2} \langle \xi_i, \xi_i \rangle + \langle p_1, \xi_1 \rangle + \langle p_2, \xi_2 \rangle$$

Then, it follows from the PMP [20] that the optimal control laws are determined from the following condition:

$$\frac{\partial H}{\partial \xi_i} = -\xi_i + p_i = 0$$

Thus

$$\xi_i^{op} = p_i, \ i = 1, 2.$$

Using (9), we have

$$\hat{\xi}_{i}^{\text{op}}(t) = \hat{p}_{i}(t) = \operatorname{Ad}_{q_{i}^{-1}(t)}\hat{c}_{i}.$$
 (10)

For the problem of finite-time optimal formation control on Lie group SE(2), the following theorem is proposed.

Theorem 1. For system (2), the specified formation (4) is achieved at the given terminal time t_f , and the corresponding cost function (3) is optimal with the following control laws

$$\hat{\xi}_{1}^{\text{op}}(t) = \frac{1}{2(t_{f}-t_{0})} \operatorname{Ad}_{g_{1}^{-1}(t)} \log \left(g_{2}(t_{0}) g_{12} g_{1}^{-1}(t_{0}) \right),
\hat{\xi}_{2}^{\text{op}}(t) = \frac{1}{2(t_{f}-t_{0})} \operatorname{Ad}_{g_{2}^{-1}(t)} \log \left(g_{1}(t_{0}) g_{21} g_{2}^{-1}(t_{0}) \right).$$
(11)

Proof. From Lemma 1, the transversality condition is given by

$$P_1(t_f) = -P_2(t_f)g_{21}^T.$$

Thus

$$P_1(t_f)g_1^T(t_f) = -P_2(t_f)g_2^T(t_f).$$

Using Lemma 2 and (8), it is obtained that

$$P_1(t)g_1^T(t) = -P_2(t)g_2^T(t), \quad \forall t \in [t_0, t_f].$$

Therefore,

$$\hat{c}_1^* = -\hat{c}_2^*, \ \hat{c}_1 = -\hat{c}_2.$$

Substituting the above equality into (10) gives

$$\hat{\xi}_1^{\text{op}}(t) = \operatorname{Ad}_{g_1^{-1}(t)}\hat{c}_1, \hat{\xi}_2^{\text{op}}(t) = -\operatorname{Ad}_{g_2^{-1}(t)}\hat{c}_1.$$

For the system (2), we have

$$\dot{g}_i = g_i \operatorname{Ad}_{g_i^{-1}(t)} \hat{c}_i = \hat{c}_i g_i.$$

Solving the above equation gives

$$g_i(t) = e^{\hat{c}_i(t-t_0)}g_i(t_0), \quad i = 1, 2.$$
 (12)

Substituting (12) into the formation condition (4) gives

$$g_1^{-1}(t_0)e^{-2\hat{c}_1(t_f-t_0)}g_2(t_0) = g_{21}.$$

Thus

$$\hat{c}_1 = -\frac{1}{2(t_f - t_0)} \log \left(g_1(t_0) g_{21} g_2^{-1}(t_0) \right) = \frac{1}{2(t_f - t_0)} \log \left(g_2(t_0) g_{12} g_1^{-1}(t_0) \right).$$

Then the optimal control laws are given by

$$\hat{\xi}_{1}^{\text{op}}(t) = \frac{1}{2(t_f - t_0)} \operatorname{Ad}_{g_1^{-1}(t)} \log \left(g_2(t_0) g_{12} g_1^{-1}(t_0) \right),$$
$$\hat{\xi}_{2}^{\text{op}}(t) = \frac{1}{2(t_f - t_0)} \operatorname{Ad}_{g_2^{-1}(t)} \log \left(g_1(t_0) g_{21} g_2^{-1}(t_0) \right).$$

3.2 Trajectory Tracking

For the above finite-time optimal formation problem, the proposed control laws are imposed on agents until the desired formation is achieved. However, it is common that disturbances exist after the terminal time. For this reason, the following trajectory tracking algorithm on SE(2) is given to keep the formation.

Considering the following systems

$$\dot{g}_0 = g_0 \hat{\xi}_0,$$
 (13)

$$\dot{g}_1 = g_1 \xi_1, \tag{14}$$

where $\hat{\xi}_0 \in se(2)$ is the external input signal. Our goal is to design $\hat{\xi}_1 \in se(2)$ for system (14) to track the trajectory of system (13).

For arbitrary elements $g_0, g_1 \in SE(2)$, it is not ensured that $g_1 - g_0$ also belongs to SE(2). So, the state difference which is used in the Euclidean space can not be used to the tracking problem on SE(2). For the general linear group GL(n), a sufficiently small open neighborhood U of identity element I in GL(n) is diffeomorphic to an open neighborhood of zero element in matrix group M(n) (see the theorem 3.2.4 in [21]). It is assumed that $g_i(t) \in SE(2)$ (i = 0, 1)belongs to an open neighborhood U_I of identity element I. Thus, there exists the unique $X_i(t) \in \log(U_I) \bigcap se(2)$ (i =0, 1) such that $g_i(t) = e^{X_i(t)}$ or $\log(g_i(t)) = X_i(t)$. Let

$$\frac{d}{dt}\log(g_i(t)) = \hat{\xi}_i(t), (i=0,1).$$

Using Baker-Campbell-Hausdorff formula (see theorem 10.4 in [22]),

$$\log(\exp(-X_0)\exp X_1) = -X_0 + X_1 + \frac{1}{2}[-X_0, X_1] \\ + \frac{1}{12}[-X_0, [-X_0, X_1]] - \frac{1}{12}[X_1, [-X_0, X_1]] \\ - \frac{1}{24}[X_1, [-X_0, [-X_0, X_1]]] + \cdots.$$

Thus,

$$\log(g_0^{-1}g_1) = \log(g_1) - \log(g_0) + [\log(g_1), \log(g_0)]$$

+higher order term,

where $[\cdot, \cdot]$ is the Lie bracket defined on se(2). Let

$$\delta \hat{\xi} = \hat{\xi}_1 - \hat{\xi}_0 + \frac{d}{dt} [\log(g_1), \log(g_0)] + \text{higher order term.}$$

When $\log(g_0)$ and $\log(g_1)$ are sufficiently close to each other, it follows that

$$\delta \hat{\xi} \approx \hat{\xi}_1 - \hat{\xi}_0. \tag{15}$$

Therefore, considering the following system

$$\frac{d}{dt}(\log(g_0^{-1}g_1)) = \delta\hat{\xi},\tag{16}$$

the problem of trajectory tracking is converted into the stabilization of system (16). As mentioned above, se(2) is isomorphic to \mathbb{R}^3 . For convenience, the system (16) is written as

$$\frac{d}{dt}([\log(g_0^{-1}g_1)]^{\vee}) = \delta\xi.$$
(17)

This problem is solved by the classical Euclidean space state feedback control. For control input $\delta \xi$, the state feedback matrix $K \in \mathbb{R}^{n \times n}$ is designed such that

$$\delta\xi = K[\log(g_0^{-1}g_1)]^{\vee}, \tag{18}$$

and system (17) is asymptotically stable. Solving (17) gives

$$[\log(g_0^{-1}(t)g_1(t))]^{\vee} = e^{K(t-t_0)}[\log(g_0^{-1}(t_0)g_1(t_0))]^{\vee}.$$

Obviously, $e^{K(t-t_0)}[\log(g_0^{-1}(t_0)g_1(t_0))]^{\vee} \to 0$ implies th

Obviously, $e^{K(t-t_0)}[\log(g_0^{-1}(t_0)g_1(t_0))]^{\vee} \to 0$ implies that $g_1(t) \to g_0(t)$. Following from (15,18), the desired tracking control law for system (14) is given by

$$\hat{\xi}_1(t) = \hat{\xi}_0(t) + [K[\log(g_0^{-1}(t)g_1(t))]^{\vee}]^{\wedge}.$$
(19)

For the problem of trajectory tracking, the following theorem is proposed.

Theorem 2. Considering the systems (13,14) and assuming that the initial conditions for systems (13,14) are sufficiently close, the trajectory tracking control law is given by

$$\hat{\xi}_1(t) = \hat{\xi}_0(t) + [K[\log(g_0^{-1}(t)g_1(t))]^{\vee}]^{\wedge}, \qquad (20)$$

where $K \in \mathbb{R}^{n \times n}$ is chosen such that all the eigenvalues have negative real parts.

For system (2), the desired formation is achieved at the given terminal time. Then, for each agent, the control law switches to the tracking control law to keep the formation. The tracking control laws are given by

$$\hat{\xi}_1(t) = [K[\log(g_1(t_f)^{-1}(t)g_1(t))]^{\vee}]^{\wedge}, \\ \hat{\xi}_2(t) = [K[\log(g_2(t_f)^{-1}(t)g_1(t))]^{\vee}]^{\wedge}.$$

Remark 2. The above theorem requires that the initial conditions are sufficiently close. It restricts the application range. However, for the problem concerned, the initial conditions are justified when the desired formation is achieved at the terminal time. Besides, the simulation indicates a large basin of attraction.

4 Numerical Examples

In this section, some numerical examples are given to illustrate the effectiveness of the control laws proposed in the above sections. For simplicity, the initial time is given by $t_0 = 0$.

Example 1. Consider the system with two agents described by (2). The desired formation is that the configurations of these two agents achieve consensus, so the formation condition is given by $g_{21} = [1, 0, 0; 0, 1, 0; 0, 0, 1]$. And the desired formation time is 3. Select the initial configurations as follows

$$g_1(t_0) = \begin{bmatrix} \cos \alpha_1 & -\sin \alpha_1 & 3\\ \sin \alpha_1 & \cos \alpha_1 & 2\\ 0 & 0 & 1 \end{bmatrix},$$
$$g_2(t_0) = \begin{bmatrix} \cos \alpha_2 & -\sin \alpha_2 & 0\\ \sin \alpha_2 & \cos \alpha_2 & 4.5\\ 0 & 0 & 1 \end{bmatrix},$$



Fig. 1: Formation of two agents and $t_f = 3$, (a) Finite-time formation in the plane, (b) the position curves for two agents.

where $\alpha_1 = -10, \alpha_2 = 1.2$. The simulation time is 4.

Fig.1 (a) shows the formation for system (2) using the finite-time optimal control laws (11). The dark-colored agent and the light-colored agent represent Agent 1 and Agent 2, respectively. The agents without color are the initial configurations. The position curves for the agents are presented in Fig.1 (b), which indicate that the formation is achieved at terminal time 3. Besides, the performance index is 3251.9.

Example 2. Consider the systems (13,14). The external input is given by

$$\hat{\xi}_0 = \begin{bmatrix} 0 & -0.2t & 1\\ 0.2t & 0 & 2\\ 0 & 0 & 0 \end{bmatrix} \in se(2).$$

Select the initial configurations as follows

$$g_0(t_0) = \begin{bmatrix} \cos \alpha_0 & -\sin \alpha_0 & -2\\ \sin \alpha_0 & \cos \alpha_0 & 4.5\\ 0 & 0 & 1 \end{bmatrix},$$
$$g_1(t_0) = \begin{bmatrix} \cos \alpha_1 & -\sin \alpha_1 & 1\\ \sin \alpha_1 & \cos \alpha_1 & 2\\ 0 & 0 & 1 \end{bmatrix},$$

where $\alpha_0 = -1.2, \alpha_1 = 1$. The state feedback matrix is -E, where E is the unite matrix. The simulation time is 8.

Fig.2 (a) shows the trajectory tracking result for system (14) using the trajectory tracking control law (20). The dark-



(b)

Fig. 2: Trajectory tracking for two agents (a) Tracking result in the plane, (b) the position curves.

colored agent and the light-colored agent represent Agent 1 and Agent 0, respectively. The position curves for the agents are presented in Fig.2 (b), which show that the Agent 1 asymptotically tracks the trajectory of Agent 0.

5 Conclusions

In this paper, the finite-time optimal formation control laws are proposed for the two-agent system evolving on SE(2). The formation time is given in advance according to the task requirements and during the formation, the given performance index is optimal. Additionally, for the case when disburses exist after the terminal time, the trajectory tracking control law is designed to keep the formation.

Nevertheless, there are still some problems remained to be solved, such as the finite-time optimal formation control for the case of multiple agents and considering the dynamics model of the agents. The solving of these problems could be important both for theoretical research and practical applications.

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