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On Eliminating the Drift from Affine Control Systems

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Abstract: In this paper the problem of eliminating the drift from affine control systems is discussed. We investigate the problem from two perspectives: the state extension system with particular form and the quotient driftless system. We give sufficient and necessary conditions on the affine control system such that it admits the above two structures respectively. Relations between the conditions are discussed. Some global results are given.

Key Words: Affine control system, drift, driftless, motion planning

1. Introduction

Motion planning problem (MPP) for a given control system is to design the control law such that it steers the control system from a given initiated state to a given final state in an assigned time. The existing results for MPP mainly focus on the driftless control systems [1-6]. For affine control systems, MPP appears more challenge. In the recent years some researchers have developed the optimal control methods to solve the MPP [7-9]. Another consideration is based on the existing results for driftless control systems. Some results have appeared for a special class of affine control systems-the simple mechanical control systems [10-13].

In this paper, we consider the methods of eliminating the drift from the affine control system such that we can apply the existing motion planning methods for driftless control systems. Two perspectives are investigated: we first study the state extension system for the affine control system such that MPP for the new system is easy to solve. More precisely, (Problem 1) given an affine control system Σ_Q

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u^i f_i(x) \tag{1}$$

on the n-dimensional manifold Q, we seek conditions on the vector fields $f_0, f_1, \cdots, f_m$ such that there exists a control system having Σ_Q as its subsystem and it is state feedback equivalent to the control system having the following form

$$\begin{aligned} \dot{\tilde{x}} &= v^i h_i(\tilde{x}) \\ \dot{\tilde{y}} &= h(\tilde{y}) \end{aligned} \tag{2}$$

A related paper is [14] where the authors focused on invariant control systems on matrix Lie group. We identify that the diffeomorphsim of the state space defined there is actually a diffeomorphsim of the extension state space. Motivated by this, we will work on state extension for general affine control system as described above.

From the perspective of quotients, we consider the following problem (Problem 2): given an affine control system Σ_Q as above, when does it admit a quotient driftless control system $\Sigma_{\tilde{Q}}$ with trajectory lift property? That is, if there exists a driftless control system $\Sigma_{\tilde{Q}}$ on the manifold $\tilde{Q}$ with $\phi: Q \to \tilde{Q}$ a submersion such that for every trajectory $\tilde{x}(t)$ of $\Sigma_{\tilde{Q}}$, there exists a trajectory $x(t)$ of Σ_Q, such that $\phi(x(t)) = \tilde{x}(t)$. In [15], the trajectory lift map between two general control systems was studied. Here we refine the map between affine and driftless control systems. We will investigate the conditions on Σ_Q such that it admits such a quotient driftless system. Then relations between Problem 1 and Problem 2 will be discussed since they seem to be independent.

The paper is organized as follows: Preliminaries are given in section 2. Problem 1 is investigated in section 3. In section 4 we study Problem 2 and discuss the relations between these two problems. Some global results are given in section 5. We make a conclusion in section 6. As a reference for notation employed and geometric concepts, see [16, 17].

2. State Feedback Equivalence

Definition 2.1: The affine control system

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} v^i g_i(z) \tag{3}$$

on the manifold M is said to be state feedback equivalent to the system (1) if and only if there exist a global diffeomorphism $\Phi: M \to N$ and a feedback

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transformation $v^i = \alpha^i_j(z)u^j + \beta^i(z)$ with (α^i_j) nonsingular such that

$$\Phi_*\left(g_0(z) + \beta^i(z)g_i(z)\right) = f_0(\Phi(z))$$
$$\Phi_*(\alpha^i_j(z)g_i(z)) = f_j(\Phi(z)) .$$

If the diffeomorphism above is locally around some $z_0 \in M$, then (3) is said to be locally state feedback equivalent to the system (1).

3. State extension control Systems

The state extension systems are very important in the proof of Pontryagain's Maximum Principle for optimal control [18]. Here we consider whether the drift can be eliminated by state extension. We have:

Theorem 3.1: Let Σ_Q be an affine control system having the form (1). Locally, there exists a control system having Σ_Q as its subsystem and is state feedback equivalent to the system with the form (2) if and only if $\exists$ m^2 functions α_i^j and m functions β^j on Q such that $[f_0, f_i] = \alpha_i^j f_j + \beta^j[f_i, f_j], \forall i \in \{1, \cdots, m\}$

Proof: (Necessity)

Let
$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u^i f_i(x)$$
$$\dot{y} = g_0(x,y) + \sum_{j=1}^{k} u^j g_j(x,y)$$
be the control system state feedback equivalent to the system (2).

Then there exists feedback transformation

$$u^i = \alpha^i_j(x,y)v^j + \beta^i(x,y)$$

where the matrix (α^i_j) is nonsingular and a local diffeomorphism

$$\Phi : U \subseteq IR^k \to \tilde{U} \subseteq IR^k$$
$$(x,y) \to (\tilde{x}, \tilde{y})$$

such that

$$\Phi_*\begin{pmatrix} f_0(x) + f_i(x)\beta^i(x,y) \\ g_0(x,y) + g_j(x,y)\beta^j(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ h(\tilde{y}) \end{pmatrix}$$
$$\Phi_*\begin{pmatrix} f_i(x)\alpha^i_j(x,y) \\ g_i(x,y)\alpha^i_j(x,y) \end{pmatrix} = \begin{pmatrix} h_j(\tilde{x}) \\ 0 \end{pmatrix}$$

So

$$\left[\begin{pmatrix} f_0(x) + f_i(x)\beta^i(x,y) \\ g_0(x,y) + g_j(x,y)\beta^j(x,y) \end{pmatrix}, \begin{pmatrix} f_i(x)\alpha^i_j(x,y) \\ g_i(x,y)\alpha^i_j(x,y) \end{pmatrix}\right] = 0$$

We have

$$\alpha^i_j[f_0, f_i] + f_0(\alpha^i_j)f_i + \alpha^i_j\beta^l[f_i, f_l] + \alpha^i_j f_i(\beta^l)f_l$$
$$-\beta^l f_l(\alpha^i_j)f_i - \alpha^i_j g_i(\beta^l)f_l + (g_0 + g_k\beta^k)(\alpha^i_j)f_i = 0$$

Since (α^i_j) is nonsingular,

We have

$$[f_0, f_i] = \gamma_i^j f_j + \beta^l[f_i, f_l], \forall i \in \{1, \cdots, m\}$$

For $[f_0, f_i], f_j, [f_i, f_l]$ are vector fields on the manifold Q, there exist m^2 functions α_i^j and m functions β^j on Q such that

$$[f_0, f_i] = \alpha_i^j f_j + \beta^j[f_i, f_j], \forall i \in \{1, \cdots, m\}$$

(Sufficiency)

If $[f_0, f_i] = \alpha_i^j f_j + \beta^j[f_i, f_j], \forall i \in \{1, \cdots, m\}$, then

$$[f_0 + \beta^j f_j, f_i] = \alpha_i^j f_j + \beta^j[f_i, f_j] + \beta^j[f_j, f_i]$$
$$-f_i(\beta^j)f_j \subseteq span\{f_j\}$$

Let $\tilde{f}_0 = (f_0 + \beta^j f_j)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $\tilde{f}_j = f_j\frac{\partial}{\partial x}$.

There must exist m^2 functions $\alpha_i^j(x,y)$ such that

$$[\tilde{f}_0, \alpha_i^j\tilde{f}_j] = \alpha_i^j[f_0 + \beta^k f_k, f_j]\frac{\partial}{\partial x} + (f_0 + \beta^k f_k)(\alpha_i^j)$$
$$f_j\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(\alpha_i^j)f_j\frac{\partial}{\partial x} = (\alpha_i^j\gamma_j^k(x) + (f_0 + \beta^j f_j)(\alpha_i^k)$$
$$+\frac{\partial}{\partial y}(\alpha_i^k))f_k\frac{\partial}{\partial x} = 0 \qquad (4)$$

Since $(f_0 + \beta^j f_j)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is nondegenerate, we have local coordinates $(x^1, \cdots, x^n, x^{n+1})$ on $Q \times R$ such that

$$f_0 + \beta^j f_j + \frac{\partial}{\partial y} = \frac{\partial}{\partial x^1}.$$

Then (4) becomes

$$\alpha_i^j\gamma_j^k + \frac{\partial}{\partial x^1}(\alpha_i^k) = 0 \qquad (5)$$

Let $(\alpha_i^j)(0) = I$, then (5) has solutions α_i^j such that the matrix (α^i_j) is nonsingular in sufficient small neighborhood of zero point according to the theory of ordinary differential equation.

Denote the involutive distribution $D = Lie^\infty(f_1, \cdots, f_m)$, we have $[\tilde{f}_0, D] \subseteq D$ because

$$[\tilde{f}_0, f_i] = [f_0 + \beta^j f_j, f_i] \in D.$$

Then there exists local coordinates $(\tilde{x},\tilde{y})$ on $Q\times R$ such that $D=span\left\{\frac{\partial}{\partial\tilde{x}^1},\cdots,\frac{\partial}{\partial\tilde{x}^k}\right\}$ with $\dim D=k$, and $\tilde{f}_0=\frac{\partial}{\partial\tilde{y}}$ according to Proposition 3.50 in [19].

Since $[\tilde{f}_0,\alpha_i^j\tilde{f}_j]=0$,

we have $\alpha_i^j\tilde{f}_j=h_i^1(\tilde{x})\frac{\partial}{\partial\tilde{x}^1}+\cdots+h_i^k(\tilde{x})\frac{\partial}{\partial\tilde{x}^k}$.

That is the control system $\begin{array}{l}\dot{x}=f_0(x)+\sum_{i=1}^m u^i f_i(x)\\ \dot{y}=1\end{array}$ is locally state feedback equivalent to the system with the form (2).

Remark 3.1: From the proof of the above theorem, we know when Σ_Q has a state extension system which is state feedback equivalent to the system with the form (2), we can always choose the state extension system as $\begin{array}{l}\dot{x}=f_0(x)+\sum_{i=1}^m u^i f_i(x)\\ \dot{y}=1\end{array}$, then (2) becomes $\begin{array}{l}\dot{\tilde{x}}=v^i h_i(\tilde{x})\\ \dot{\tilde{y}}=1\end{array}$. That is, a state extension with time variable is always enough to eliminate the drift.

4. Quotient Driftless Control Systems

Theorem 4.1 [15]: If $f:M\times\Omega\to TM$ and $h:N\times\Theta\to TN$ be C^1 control systems. If $\Phi:M\to N$ is a C^1 mapping, and if every compactly defined trajectory $\psi:[a,b]\to N$ of h satisfying $\psi(a)\in\Phi(M)$ can be locally lifted from h to f via Φ. then for every $x\in M$ we have

$$\left\{d\Phi_x f(x,\omega)\middle|\omega\in R^p\right\}\supseteq\left\{h(\Phi(x),\theta)\middle|\theta\in\Theta\right\}$$

Theorem 4.2: Let Σ_Q be an affine control system having the form (1). Then locally it admits a quotient driftless control system $\Sigma_{\tilde{Q}}$ with trajectory lift property if and only if there exists an involutive distribution $\tilde{D}$ such that $f_0\in\tilde{D}+F$ and $[\tilde{D},\tilde{F}]\subseteq\tilde{D}+\tilde{F}$ where $F=span\{f_1,\cdots,f_m\}$ and $\tilde{F}$ is a distribution contained in F.

Proof: According to Theorem 4.1, if $\Sigma_{\tilde{Q}}:\dot{\tilde{x}}=\sum_{i=1}^k v^i g_i(\tilde{x})$ satisfies the trajectory lift property, then for every $x\in Q$

$$\left\{d\phi_x(f_0(x)+\sum_{i=1}^m u^i f_i(x))\middle|u\in R^m\right\}\supseteq$$
$$\left\{\sum_{i=1}^k v^i g_i(\phi(x))\middle|v\in R^k\right\}$$

Since $0\in\left\{\sum_{i=1}^k v^i g_i(\phi(x))\middle|v\in R^k\right\}$,

We know there exist m functions α^i on Q such that

$$d\phi_x(f_0(x)+\sum_{i=1}^m\alpha^i f_i(x))=0\text{, for every }x\in Q \tag{6}$$

And there exists a k-dimensional distribution $\tilde{F}$ contained in the distribution spanned by $\{f_1,\cdots,f_m\}$ such that

$$d\phi_x(\tilde{F}(x))=span\{g_i(\phi(x))\} \tag{7}$$

For $\phi:Q\to\tilde{Q}$ is a submersion, we know $\ker(d\phi)$ is an involutive distribution. We denote it by $\tilde{D}$.
So (6) is equivalent to $f_0\in\tilde{D}+F$,
and (7) is equivalent to $[\tilde{D},\tilde{F}]\subseteq\tilde{D}+\tilde{F}$.
On the other hand, if the conditions in Theorem 4.2 hold, then $\tilde{D}$ is a controlled invariant distribution for the subbundle $f_0+\tilde{F}$. Locally, the existence of the quotient driftless system with trajectory lift property is ensured by the local form of Σ_Q induced by $\tilde{D}$.

Remark 4.1: On considering the trajectory preserving property, we should let $\tilde{F}=F$ in the above theorem. The existence of quotient driftless system makes it possible to solve the MPP first for driftless system in the base space.
Remark 4.2: A special case that links Theorem 3.1 with Theorem 4.2 is when $[f_0,f_i]=span\{f_1,\cdots,f_m\},\forall i\in\{1,\cdots,m\}$.
With local feedback transformation $u^i=\alpha_j^i v^j$, Σ_Q can be transformed into the form $\begin{array}{l}\dot{x}=1\\ \dot{y}=v^i g_i(y)\end{array}$. MPP for Σ_Q is locally equivalent to MPP for the driftless system $\dot{y}=v^i g_i(y)$.

5. Corollaries

In this section, we give some examples of control systems where they admit global results for eliminating the drift.

Corollary 5.1: Consider the left-invariant control system on $GL(n)$: $\dot{g}=gX_0+g\sum_{i=1}^m u^i X_i$.

With the conditions that

$$ad_{X_0}X_j\in span\{X_1,\cdots,X_m,ad_{X_1}X_j,\cdots,ad_{X_m}X_j\}\quad\forall j\in\{1,\cdots,m\}$$

a global diffeomorphsim defined by

$$\tilde{g}=ge^{-X_0t}$$

together with a feedback transformation with the form

$$u = B(t)v + U$$

where $U \in R^m$ is a constant vector field transform the control system into a driftless system $\dot{\tilde{g}} = \tilde{g}\sum_{i=1}^{m} v^i X_i$.

Remark 5.1: Here we correct the conditions given by Theorem 1 in [14].

Corollary 5.2: Consider the left-invariant control system on $GL(n)$: $\dot{g} = gX_0 + g\sum_{i=1}^{m} u^i X_i$.

If there exists a Lie subalgebra $A \subseteq gl(n)$ such that $X_0 \in A + span\{X_1, \cdots, X_m\}$ and $[A, X_j] \subseteq A + span\{X_1, \cdots, X_m\}$, then there exists a driftless control system on the homogenous manifold $GL(n)/H$ where H is the Lie subgroup of $GL(n)$ with Lie algebra $A \subseteq gl(n)$, such that the trajectory preserving and lifting property are satisfied under the map $\pi : GL(n) \to GL(n)/H$.

6. Conclusion

In this paper we considered eliminating the drift from an affine control system. Two perspectives have been shown: from state extension and quotient systems. The sufficient and necessary conditions under which the methods are locally applicable have been given. The relations between these two methods were discussed. Some global results were given for invariant control systems on Lie groups.

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