Formation Control for Multiple Vehicles Based on Motion Planning Approach

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Abstract: The paper studies the problem of finite time formation control for multiple vehicles based on the motion planning approach. The vehicle is modeled as the full actuated rigid body with the dynamics evolving on the tangent bundle of Euclidean group, both the formation time which is finite and the geometric structure of the formation are assumed to be specified by the task. For the required formation, an open optimal control law is derived by using Pontryagins minimum principle. In order to overcome the deficiency that the open control law is sensitive to disturbances, the open control law is converted to the closed form by feeding the current state back and initializing the control law at the current time, under the assumption that the mode of communication between the vehicles is all-to-all. For the demonstration of the result, some numerical examples of formation for planar vehicles are included.

Key Words: Formation Control, Consensus, Multiple Vehicles, Motion Planning

1 Introduction

Formation control for multiple vehicles has attracted enormous research efforts during the past decade. Motivations for such research stem from the inherent strength and robustness in a coordinated group when dealing with tasks such as searching, surveying and mapping. For example, a fleet of underwater vehicles is used collectively for adaptive ocean sampling in [1]. From a networked control point of view, the formation control for multiple vehicles is a cooperative control of a networked system with its nodes to be vehicles under some given communication topology. The dynamics of the networked system depends both on the dynamics of the nodes and on the communication topology. The research on this first originate from the study of multiple integrators, which propose mainly a philosophy to reveal a certain logic between the behavior of the group as a whole and the behaviors of the individuals connected by communication [2]. For the model of network consisting of multiple integrators, the researchers [3–6, 8, 9, 17] put a little more emphasis upon the communication topology between individuals and the coordinated behaviors of the group, and designed various consensus based algorithm in order to coordinate the group behaviors such as aggregation, formation and leader-follower coordination etc. The results on the networks of integrators are rich and diversified, and the theories are relative mature, we are not going to elaborate on them here. The integrators in these works would at most represent the dynamics of particles, even though they are trying to be explained under the meaning of dynamics of vehicles. It would be too much for the model of integrator to describe the vehicles when considered as rigid bodies. Then, linear systems, as the model of dynamics of the nodes, are considered [10–12], the problem of consensus and formation of multiple linear systems are studied, and a series of fruitful results are obtained. Compared with integrator, linear system has more complex dynamical behaviors and can locally represent the model of the vehicle as a rigid body by linearization near some equilibrium. Nevertheless, For the problem of coordination undergoing a large overall motion, especially when the initial state of the group is far from the coordinated state, the nonlinearity will play the role and the linear model maybe rough and preliminary even failure in representing the real dynamics of the vehicle. To take a step further, some researchers considered controlled Euler-Lagrange (EL) systems as the dynamics of the nodes in the networks, and made an important progress in the study of consensus and cooperative tracking [13–15]. However, when the vehicle is considered as a rigid body, its configuration space ia a Euclidean group $\mathrm{SE}(3)$, which ia a nonlinear manifold. Thus, from the theoretic point of view, the EL systems are the only local Euclidean space representation of the dynamics of the vehicle. When the large angle maneuver is unavoidable in the attitude coordination, it is difficult to say how valid the EL systems will be.

The research works [16, 17] considered the formation of vehicles under the setting of Lie group, in which, the kinematics of the vehicle is molded by Frenet-Serret equations of motion, the speed of motion is supposed to be unit and the rotation velocity is taken as the control input, and presented the control law that leads to vehicle swarm. From the formation control point of view, these are the relatively early works considering the configuration of the vehicle globally on Euclidean group instead of locally homeomorphic to some open set of Euclidean space. In order to keep the formation while moving in space, some researchers [18] studied the problem of coordinated motion on Lie groups by converting the problem of coordination to that of consensus on Lie algebra which is a linear space, and proposed the control law for the coordination. Compared with works in which the dynamics of the vehicle evolves on Euclidean space, the works [16–18] are more realistic in describing the dynamics of vehicle when the attitude is taken into account in the formation control. But in their works, they do not deal with the problem of

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achieving a specified formation which is quite important for some real applications. Recently, we studied the problem for multiple vehicles to achieve an arbitrary specified formation under the framework of Euclidean group, and proposed the asymptotic convergent formation control laws for both kinematic model [19] and dynamic model [20]. We also considered the problem of finite time optimal formation control of multiple vehicles under the framework of Euclidean group, in which both the formation time and the geometric structure of the formation are specified by the task, and we proposed an optimal formation control law for the the kinematic model[21].

The work of this paper studies the formation control problem based on the motion planning approach and can be seen as the continuation of the paper [21], but the dynamic model of the vehicle is used, in which the control input can be explained physically as generalized force/moment. This is quite different from the kinematic model in which the control input is the generalized velocity. In the work of this paper, the formation time and the geometric structure of the formation are also arbitrarily specified by the task. For this formation control problem, we proposed a finite time optimal formation control law under the assumptions that the vehicles are full actuated with aero initial velocities and that the communications between the vehicles are all-to-all. As we know, this is new in that it first propose an explicit optimal formation control law for the dynamic model of vehicle evolving on Euclidean group, and that both formation time and the geometric structure of the formation are arbitrarily specified by the task.

The paper is organized as follows. In section two, we introduce some notions and preliminary results which will be used in this paper. In section three, we focus mainly on the formulation of the problem. Section four is devoted to develop the main results of the paper. Section five includes some examples and their numerical simulation results. To conclude the paper, we summarize the main results and gives the further problems that are worth being pursued in section six.

2 Preliminaries

In this section, we introduce some notions and preliminary results which will be used in the paper.

Consider a rigid body moving in space. In order to describe its position and attitude in a given spacial coordinate, we use the Euclidian group

$$\mathrm{SE}(N)=\left\{\begin{bmatrix} R & p \\ 0_{1\times 3} & 1\end{bmatrix} : R\in \mathrm{SO(N)}, p\in\mathbb{R}^N\right\}, N=2,3,$$

to denote the configuration manifold of a rigid body. Where $\mathrm{SO}(N)$ is a $N\times N$ orthogonal matrix group with its element $R\in\mathrm{SO}(N)$ satisfies $\det R=1$. Each column of R is the base vector of the coordinate system fixed on the rigid body, thus R can be used to represent the attitude of the body, and $p\in\mathbb{R}^N$ is the position vector of the body.

Let $T\mathrm{SE}(N)$ denote the tangent bundle of $\mathrm{SE}(N)$ and $T_g\mathrm{SE}(N)$ be the tangent space of $\mathrm{SE}(N)$ at $g\in\mathrm{SE}(N)$. For the special case when g is the identity I, the tangent space $T_I\mathrm{SE}(N)$ that is denoted by $\mathfrak{se}(N)$ has the following structure,

$$\mathfrak{se}(N)=\left\{\begin{bmatrix} \hat\omega & v \\ 0_{1\times N} & 0\end{bmatrix} : \hat\omega\in\mathfrak{so}(N), v\in\mathbb{R}^N\right\},$$

where $\mathfrak{so}(N)\subset\mathbb{R}^{N\times N}$ is the set of anti-symmetric matrices that represent the velocities of rotation of the body, and $v\in\mathbb{R}^N$ is a real vector that represents the transition velocity of the body. It is obvious that $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$, and the outer product defined on $\mathbb{R}^3$, e.g., $\omega_1\times\omega_2$ can be written as $\hat\omega_1\omega_2$ for $\omega_1,\omega_2\in\mathbb{R}^3$, and $\hat\omega_1\in\mathfrak{so}(3)$ is the isomorphic of $\omega_1\in\mathbb{R}^3$. It is obvious that $\mathfrak{se}(3)$, as a linear space, is isomorphic to $\mathbb{R}^6$. For this fact, we define an isomorphic map $\wedge:\mathbb{R}^6\to\mathfrak{se}(3)$, such that

$$\wedge:\eta=\begin{bmatrix}\omega\\ v\end{bmatrix}\to\begin{bmatrix}\hat\omega & v\\ 0_{1\times 3} & 0\end{bmatrix}=\hat\eta,\ \omega,v\in\mathbb{R}^3$$

which is denoted by $\wedge(\eta)=\hat\eta$. The inverse of the map $\wedge$ is denote by $\vee:\mathfrak{se}(3)\to\mathbb{R}^6$. For $\mathfrak{se}(2)$, it is isomorphic to $\mathbb{R}^3$, the similar isomorphic map and its inverse can be defined, it is unnecessary to go into details. The meaning of the notion " ˆ " used for vectors in $\mathfrak{so}(3)$, $\mathfrak{se}(3)$ and $\mathfrak{se}(2)$ can be distinguished from the context.

Once defining Lie bracket on $\mathfrak{se}(N)$ by

$$[\hat x,\hat y]\triangleq\hat x\hat y-\hat y\hat x,\ \hat x,\hat y\in\mathfrak{se}(N),$$

$\mathfrak{se}(N)$ is a Lie algebra corresponding to $\mathrm{SE}(N)$. For a given $\hat x\in\mathfrak{se}(N)$, it defines a linear map $\mathrm{ad}_{\hat x}:\mathfrak{se}(N)\to\mathfrak{se}(N)$, such that $\mathrm{ad}_{\hat x}(\hat y)=[\hat x,\hat y],\hat y\in\mathfrak{se}(N)$. In the following, we shall only discuss for the case of $N=3$, but the results will also hold for the case of $N=2$.

For the Lie bracket defined on $\mathfrak{se}(3)$, we have the following lemma.

Lemma 1. *Let $\hat\eta,\hat\xi_1,\hat\xi_2\in\mathfrak{se}(3)$, if $[\hat\eta,\hat\xi_i]=0, i=1,2$, then $[\hat\xi_1,\hat\xi_2]=0$.*

Proof. Suppose that

$$\hat\eta=\begin{bmatrix}\hat\omega & v\\ 0_{1\times 3} & 0\end{bmatrix}\neq 0,$$

we show that $[\hat\eta,\hat\xi]=0$ for some $\hat\xi\in\mathfrak{se}(3)$, if and only if $\hat\xi$ has the structure of

$$\hat\xi=\begin{bmatrix}\hat\xi_\omega & \xi_v\\ 0_{1\times 3} & 0\end{bmatrix}=\begin{bmatrix}\alpha\hat\omega & \alpha v+\beta\omega\\ 0_{1\times 3} & 0\end{bmatrix},\ \alpha,\beta\in\mathbb{R}.$$

Since

$$\vee\left([\hat\eta,\hat\xi]\right)=\begin{bmatrix}\hat\omega & 0\\ \hat v & \hat\omega\end{bmatrix}\begin{bmatrix}\xi_\omega\\ \xi_v\end{bmatrix}=\begin{bmatrix}\hat\omega\xi_\omega\\ \hat v\xi_\omega+\hat\omega\xi_v\end{bmatrix},$$

then $\hat\omega\xi_\omega=\omega\times\xi_\omega=0$ if and only if $\xi_\omega=\alpha\omega$, for some scalar $\alpha\in\mathbb{R}$, and $\hat v\xi_\omega+\hat\omega\xi_v=(\alpha v-\xi_v)\times\omega=0$, if and only if $(\alpha v-\xi_v)=-\beta\omega$ for some scalar $\beta\in\mathbb{R}$, and thus $\xi_v=\alpha v+\beta\omega$. Now suppose that

$$\hat\xi_i=\begin{bmatrix}\alpha_i\hat\omega & \alpha_i v+\beta_i\omega\\ 0_{1\times 3} & 0\end{bmatrix},\ i=1,2,$$

then, it is easy to check that

$$\vee\left([\hat\xi_1,\hat\xi_2]\right)=\begin{bmatrix}\alpha_1\hat\omega & 0\\ \alpha_1\hat v+\beta_1\hat\omega & \alpha_1\hat\omega\end{bmatrix}\begin{bmatrix}\alpha_2\omega\\ \alpha_2 v+\beta_2\omega\end{bmatrix}=0,$$

which leads to that $[\hat\xi_1,\hat\xi_2]=0$. $\square$

The dual of TSE(3), T_gSE(3), and $\mathfrak{se}(3)$ will be denoted by T^*SE(3), T_g^*SE(3) and $\mathfrak{se}^*(3)$ respectively.

For each $\hat{x} \in \mathfrak{se}(3)$, it defines a left invariant vector field $\hat{x}_L : \text{SE}(3) \to T\text{SE}(3)$, such that $\hat{x}_L(q) = q\hat{x} \in T_q\text{SE}(3)$, for $q \in \text{SE}(3)$. We shall denote $\hat{x}_L(q)$ by $\hat{x}_q$ in the sequel. From the definition of the left invariant vector field, for any given $q \in \text{SE}(3)$, it defines a left action map denoted simply by $q : \mathfrak{se}(3) \to T_q\text{SE}(3)$ with abuse of notion.

For linear spaces T_qSE(3) and $\mathfrak{se}(3)$, defining inner products on the spaces,

$$G_q(\hat{x}_q, \hat{y}_q) \triangleq G_I(\hat{x}, \hat{y}) \triangleq x^T y, \tag{1}$$

then T_qSE(3) and $\mathfrak{se}(3)$ are inner product spaces, and SE(3) will be a Riemannian manifold with a Riemannian metric induced by the inner product.

Definition 1. *Let V be linear space, $B : V \times V \to \mathbb{R}$ be a bilinear map. The linear map $B^\flat : V \to V^*$ is called the flat map of B, if $B^\flat(u), u \in V$ satisfies*

$$(B^\flat(u))(v) = B(v, u), \forall v \in V,$$

and denote the image of the flat map $B^\flat(u)$ by $u^* \in V^*$. And if the flat map $B^\flat$ is invertible, then the inverse is denoted by $B^\sharp : V^* \to V$ which is called the sharp map of B, such that

$$B(v, B^\sharp(u^*)) = u^*(v), \forall v \in V.$$

By this definition and the above notions, we denote $\hat{x}_q^* = G_q^\flat(\hat{x}_q)$ and $\hat{x}^* = G_I^\flat(\hat{x})$, and it is ready to show that

$$\hat{x}_q^* = (q^{-1})^*(\hat{x}^*),\ \hat{x}^* = \vee^*(x), \tag{2}$$

and

$$\hat{x}_q^*(\hat{y}_q) = \hat{x}^*(\hat{y}) = x^T y. \tag{3}$$

By using the inner product of matrices, we have

$$\hat{x}^*(\hat{y}) = \left\langle [\hat{x}^*]^T, \hat{y} \right\rangle_{\mathbb{R}^{4\times 4}} = \text{tr}(\hat{x}^*\hat{y}), \tag{4}$$

and

$$\hat{x}_q^*(\hat{y}_q) = \left\langle \left[\hat{x}_q^*\right]^T, \hat{y}_q \right\rangle_{\mathbb{R}^{4\times 4}} = \text{tr}(\hat{x}_q^*\hat{y}_q), \tag{5}$$

where $\hat{x}^* = \text{diag}\left(\frac{1}{2}I_3, 1\right)\hat{x}^T$, $\hat{x}_q^* = \text{diag}\left(\frac{1}{2}I_3, 1\right)\hat{x}^T q^{-1}$.

Let $\{e_i\}$, $\{\hat{e}_i\}$, and $\{\hat{e}_{i,q}\}$ denote the orthonormal basis of $\mathbb{R}^6$, $\mathfrak{se}(3)$ and T_qSE(3) respectively, which satisfy

$$\hat{e}_{i,q} = q(\hat{e}_i) = q(\wedge(e_i)), i = 1, \cdots, 6,$$

where $e_i \in \mathbb{R}^6$ is a column vector with its i-th component being 1 and others being 0. And the dual basis of $(\mathbb{R}^6)^*$, $\mathfrak{se}^*(3)$ and T_q^*SE(3) will be

$$e_i^T = G_{\mathbb{R}^6}^\flat(e_i),\ \hat{e}_i^* = G_I^\flat(\hat{e}_i),\ \hat{e}_{i,q}^* = G_q^\flat(\hat{e}_{i,q}).$$

Definition 2. *Let V be a finite dimensional linear space with a basis $\{\varepsilon_i\}$, $f \in C^1(V, \mathbb{R})$ be a function defined on V, then the derivative of f with respect $x = \sum_i \varepsilon_i x^i \in V$ is defined by*

$$\frac{\partial f}{\partial x} = \sum_i \varepsilon_i \frac{\partial f}{\partial x^i} \in V.$$

Remark 1. *It follows from this definition that*

$$\begin{aligned} \frac{\partial \hat{x}^*(\hat{y})}{\partial \hat{x}} &= \frac{\partial \hat{y}^*(\hat{x})}{\partial \hat{x}} = \sum_i \hat{e}_i \frac{\partial x^T y}{\partial x^i} = \hat{y}, \\ \frac{\partial \hat{x}^*(\hat{y})}{\partial \hat{x}^*} &= \frac{\partial \hat{y}^*(\hat{x})}{\partial \hat{x}^*} = \sum_i \hat{e}_i^* \frac{\partial x^T y}{\partial x^i} = \hat{y}^*. \end{aligned} \tag{6}$$

Lemma 2. *Let $(q, \hat{x}_q^*) \in T^*SE(3)$, and $\hat{y} \in \mathfrak{se}(3)$, then for $\hat{x}_q^*(q\hat{y})$ as a function of $q \in SE(3)$,*

$$\frac{\partial \hat{x}_q^*(q\hat{y})}{\partial q^T} = \hat{y}\hat{x}_q^*.$$

Proof. By imbedding SE(3) into $\mathbb{R}^{4\times 4}$ and writing $q = \sum_{ij} E_{ij} q_{ij}$, where E_{ij} is the base matrix with its (i, j) component being 1 and others being 0, then the inner product of matrix space (see (5)) can be used, and by definition 2, we have

$$\begin{aligned} \frac{\partial \hat{x}_q^*(q\hat{y})}{\partial q^T} &= \sum_{ij} E_{ji} \frac{\partial \hat{x}_q^*(q\hat{y})}{\partial q_{ij}} = \sum_{ij} E_{ji} \left\langle [\hat{x}_q^*]^T, E_{ij}\hat{y} \right\rangle_{\mathbb{R}^{4\times 4}} \\ &= \sum_{ij} E_{ji} \left\langle E_{ji}, \hat{y}\hat{x}_q^* \right\rangle_{\mathbb{R}^{4\times 4}} = \hat{y}\hat{x}_q^*. \end{aligned}$$

□

For a given $q \in \text{SE}(3)$, an adjoint map $\text{Ad}_q : \mathfrak{se}(3) \to \mathfrak{se}(3)$ is defined as $\text{Ad}_q(\hat{x}) \triangleq q\hat{x}q^{-1}$, for $\hat{x} \in \mathfrak{se}(3)$, and $\text{Ad}_q(\hat{x})$ is denote by $\hat{x}^s$. Although both $\hat{x}$ and $\text{Ad}_q(\hat{x})$ are in $\mathfrak{se}(3)$, we shall still distinguish the image space $\text{Ad}_q(\mathfrak{se}(3))$ of the adjoint map from its domain $\mathfrak{se}(3)$. By the similar way, we can define inner product on $\text{Ad}_q(\mathfrak{se}(3))$,

$$G_{\text{Ad}_q}(\hat{x}^s, \hat{y}^s) \triangleq x^T y,\ \hat{x}^s, \hat{y}^s \in \text{Ad}_q(\mathfrak{se}(3)),$$

and denote the image of $\hat{x}^s$ under the flat map of G_{Ad_q} by $(\hat{x}^s)^* = G_{\text{Ad}_q}^\flat(\hat{x}^s)$, such that

$$(\hat{x}^s)^*(\hat{y}^s) = G_{\text{Ad}_q}(\hat{x}^s, \hat{y}^s),\ \forall \hat{y}^s \in \text{Ad}_q(\mathfrak{se}(3)).$$

Since

$$(\hat{x}^s)^*(\hat{y}^s) = \hat{x}^*(\hat{y}) = \hat{x}^*(\text{Ad}_{q^{-1}}((\hat{y}^s)) = \text{Ad}_{q^{-1}}^*\hat{x}^*(\hat{y}^s)$$

holds for all $\hat{y}^s \in \text{Ad}_q(\mathfrak{se}(3))$, this leads to

$$G_{\text{Ad}_q}^\flat(\text{Ad}_q\hat{x}) = (\hat{x}^s)^* = \text{Ad}_{q^{-1}}^*\hat{x}^*. \tag{7}$$

And it is easy to show that

$$\text{Ad}_{q^{-1}}^*\hat{x}^* = q\hat{x}^*q^{-1}. \tag{8}$$

Let $\exp : \mathfrak{se}(3) \longrightarrow \text{SE}(3)$ be the exponential map which can be considered as the map defined on operator space $\mathbb{C}^{n\times n}$ with the image in the same space. For this exponential map, we have the following lemma.

Lemma 3. *(Baker Campbell Hausdorff [22]) For the two non-commutative operator X and Y, if Z is defined as $\exp Z = \exp X \cdot \exp Y$, then Z can be rewritten as*

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots, \tag{9}$$

where $[\cdot, \cdot]$ is the Lie bracket defined by the commutator of operators.

We also can define the inverse of the exponential map. When the operator $A \in \mathbb{C}^{n\times n}$ has no eigenvalues on $\mathbb{R}^{-1}$, there is a unique $X \in \mathbb{C}^{n\times n}$ with its eigenvalues in the strip $\{z : -\pi < \mathrm{Im}(z) < \pi\}$, such that $A = \exp X$. We denote this fact by $X = \log(A)$, and refer the X as to the logarithm of operator A (see [23]). For the logarithm map, we have the following lemmas.

Lemma 4. *[19] Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^{-1}$. If $B \in \mathbb{C}^{n\times n}$ is invertible such that BAB^{-1} has no eigenvalues on $\mathbb{R}^{-1}$, then,*

$$B(\log(A))B^{-1} = \log(BAB^{-1}).$$

Lemma 5. *[24] (Differential of Exponential) Let $g(t)$ be a smooth curve on SE(3), $x(t) = \log(g(t))$ be the exponential coordinates of $g(t)$, $\hat{\xi}^b = g^{-1}\dot{g}$ the body velocity and $\hat{\xi}^s = \dot{g}g^{-1}$ the spatial velocity. Then it holds*

$$\dot{x} = \sum_{k=0}^{\infty} \frac{B_k}{k!} ad_{-x}^k(\hat{\xi}^b) = \sum_{k=0}^{\infty} \frac{B_k}{k!} ad_x^k(\hat{\xi}^s), \tag{10}$$

where $\{B_k\}$ are Bernoulli numbers.

3 Problem Formulation

Consider n vehicles with the dynamics given by

$$\begin{cases} \dot{g}_i = g_i\hat{\xi}_i^b, & g_i(t_0) = g_i^0 \\ \dot{\hat{\xi}}_i^b = \hat{u}_i, & \hat{\xi}_i^b(t_0) = \hat{\xi}_i^{b,0} \end{cases}, \ i = 1, \cdots, n, \tag{11}$$

where $g_i \in$SE(3) is the configuration of the i-th vehicle, $\hat{\xi}_i^b \in \mathfrak{se}(3)$ is the velocity seen form the body fixed corroborate of the i-th vehicle, and $\hat{u}_i \in \mathfrak{se}(3)$ is the control input of the i-th vehicle. The kinematics $\dot{g}_i = g_i\hat{\xi}_i^b$ can also be written as $\dot{g}_i = \hat{\xi}_i^s g_i$, where $\hat{\xi}_i^s = \mathrm{Ad}_g\hat{\xi}_i^b$ is the velocity of the i-th vehicle seen from the fixed spacial coordinate.

Now the problem is for given relative configurations $\bar{g}_{ij} \triangleq g_i^{-1}g_j, i, j = 1, \cdots, n$, determined by the formation task, and initial states $(g_i^0, \hat{\xi}_i^{b,0}), i = 1, \cdots, n$, to find control input $\hat{u}_i, i = 1, \cdots, n$, such that

$$\begin{gathered} g_i^{-1}(t_f)g_j(t_f) = \bar{g}_{ij},\ \hat{\xi}_j^b(t_f) - \mathrm{Ad}_{\bar{g}_{ji}}\hat{\xi}_i^b(t_f) = 0, \\ i, j = 1, \cdots, n, \end{gathered} \tag{12}$$

where $t_f > t_0$ is the final time which is given, meanwhile, to minimize the following cost function,

$$J(\hat{u}_1, \cdots, \hat{u}_n) = \int_{t_0}^{t_f} \frac{1}{2}\sum_{k=1}^{n} G_I(\hat{u}_k, \hat{u}_k)dt. \tag{13}$$

The above problem is referred to as finite time optimal formation control. When $\bar{g}_{ij} = I, i, j = 1, \cdots, n$, then $\mathrm{Ad}_{\bar{g}_{ji}}$ will become an identity map, this implies that $\hat{\xi}_j^b(t_f) = \hat{\xi}_i^b(t_f), i, j = 1, \cdots, n$, then the corresponding problem is referred to as finite time optimal consensus control. For the relation between the problems of formation and consensus, we have the following lemma.

Lemma 6. *Let $\tilde{g}_i = g_i\bar{g}_{i1}$, $\tilde{\xi}_i^b = Ad_{\bar{g}_{1i}}\hat{\xi}_i^b$, and $\tilde{u}_i = Ad_{\bar{g}_{1i}}\hat{u}_i$, for $i = 1, \cdots, n$, then the systems (11) achieve formation under the control $\hat{u}_i, i = 1, \cdots, n$, if and only the following systems achieve consensus under the control $\tilde{u}_i, i = 1, \cdots, n$.*

$$\begin{cases} \dot{\tilde{g}}_i = \tilde{g}_i\tilde{\xi}_i^b, & \tilde{g}_i(t_0) = \tilde{g}_i^0 \\ \dot{\tilde{\xi}}_i^b = \tilde{u}_i, & \tilde{\xi}_i^b(t_0) = \tilde{\xi}_i^{b,0} \end{cases}, \ i = 1, \cdots, n. \tag{14}$$

Proof. By noting that

$$\dot{\tilde{g}}_i = \dot{g}_i\bar{g}_{i1} = g_i\hat{\xi}_i^b\bar{g}_{i1} = g_i\bar{g}_{i1}\bar{g}_{i1}^{-1}\hat{\xi}_i^b\bar{g}_{i1} = \tilde{g}_i\mathrm{Ad}_{\bar{g}_{1i}}\hat{\xi}_i^b.$$

it follows that $\dot{\tilde{g}}_i = \tilde{g}_i\tilde{\xi}_i^b$ and $\dot{\tilde{\xi}}_i^b = \mathrm{Ad}_{\bar{g}_{1i}}\hat{u}_i = \tilde{u}_i$. Besides, by utilizing $\bar{g}_{ij} = \bar{g}_{i1}\bar{g}_{1j}$

$$g_i^{-1}(t_f)g_j(t_f) = \bar{g}_{ij} = \bar{g}_{i1}\bar{g}_{1j}$$

and

$$\tilde{\xi}_j^b - \tilde{\xi}_i^b = \mathrm{Ad}_{\bar{g}_{1j}}\hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{1i}}\hat{\xi}_i^b = \mathrm{Ad}_{\bar{g}_{1j}}(\hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{ji}}\hat{\xi}_i^b)$$

it is easy to obtain that the consensus conditions

$$\tilde{g}_i^{-1}(t_f)\tilde{g}_j(t_f) = I,\ \tilde{\xi}_j^b(t_f) - \tilde{\xi}_i^b(t_f) = 0,\ i, j = 1, \cdots, n$$

are equivalent to the formation condition (12) $\square$

Thus, we shall focus on the consensus control problem.

4 The Main Results

4.1 Co-state of Optimal Trajectories

The problem of finding $\hat{u}_i$, is an optimal control problem. In order to use Pontryagin's minimum principle (PMP) [25], the Hamiltonian for this problem can be constructed as follows

$$H = -\frac{1}{2}\sum_{k=1}^{n} G_I(\hat{u}_k, \hat{u}_k) + \sum_{i=1}^{n}\hat{p}_{g_i}^*(g_i\hat{\xi}_i^b) + \sum_{i=1}^{n}\hat{p}_{\xi_i}^*(\hat{u}_i), \tag{15}$$

where $\hat{p}_{g_i}^* \in T_{g_i}^*\mathrm{SE}(3)$, and $\hat{p}_{\xi_i}^* \in \mathfrak{se}^*(s)$ are the co-states (Lagrangian multipliers). By using Lemma 2 and Remark 1, the corresponding Hamiltonian system can be written as

$$\begin{aligned} &\dot{g}_i = \frac{\partial H}{\partial \hat{p}_{g_i}} = g_i\hat{\xi}_i^b, && \dot{\hat{\xi}}_i^b = \frac{\partial H}{\partial \hat{p}_{\xi_i}} = \hat{u}_i, \\ &\dot{\hat{p}}_{g_i}^* = -\frac{\partial H}{\partial g_i^T} = -\hat{\xi}_i^b\hat{p}_{g_i}^*, && \dot{\hat{p}}_{\xi_i}^* = -\frac{\partial H}{\partial (\hat{\xi}_i^b)^*} = -\hat{p}_{g_i}^* g_i. \end{aligned} \tag{16}$$

Let $\hat{p}_{\xi_i} = G_I^\sharp(\hat{p}_{\xi_i}^*)$, and $\hat{p}_i = G_I^\sharp(\hat{p}_{g_i}^* g_i)$, then we have the following lemma.

Lemma 7. *Suppose that $[\hat{p}_i, \hat{\xi}_i^b] = 0$, then*

$$\hat{p}_{\xi_i}(t) = \hat{p}_{\xi_i}(t_0) + \hat{p}_i(t - t_0),$$

where, both $\hat{p}_{\xi_i}(t_0) \in \mathfrak{se}(3)$ and $\hat{p}_i \in \mathfrak{se}(3)$ are constants.

Proof. By writing $\hat{\xi}_i^b = g_i^{-1}\dot{g}_i$ and substituting to the equation $\dot{\hat{p}}_{g_i}^* = -\hat{\xi}_i^b\hat{p}_{g_i}^*$ of (16), it follows that

$$\frac{d}{dt}(g_i\hat{p}_{g_i}^*) = g_i\dot{\hat{p}}_{g_i}^* + \dot{g}_i\hat{p}_{g_i}^* = 0.$$

This shows that $g_i\hat{p}_{g_i}^*$ does not vary with time, and can be denoted as a constant $(\hat{p}_i^s)^* = g_i\hat{p}_{g_i}^*$. Let $\hat{p}_i^* = \hat{p}_{g_i}^* g_i$, then

$\hat{p}_i^* = g_i^{-1}(\hat{p}_i^s)^* g_i = \mathrm{Ad}_{g_i}^*(\hat{p}_i^s)^*$, and by sharp map $G_I^\sharp$, it follows that

$$\hat{p}_i = \mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s,$$

where $\hat{p}_i^s$ as the dual of $(\hat{p}_i^s)^*$ is constant. By differentiating the equality with respect to time, it follows that

$$\begin{aligned}\dot{\hat{p}}_i &= -g_i^{-1}\dot{g}_i g_i^{-1}\hat{p}_i^s g_i + g_i^{-1}\hat{p}_i^s \dot{g}_i \\ &= -\hat{\xi}_i^b \mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s + \mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s \hat{\xi}_i^b \\ &= [\hat{p}_i, \hat{\xi}_i^b] = 0.\end{aligned}$$

This implies that $\hat{p}_i$ is constant, and so is the $\hat{p}_i^*$. Substituting this result into the last equation of the Hamiltonian system (16) and integrating from t_0 to t, then we get

$$\hat{p}_{\xi_i}^*(t) = \hat{p}_{\xi_i}^*(t_0) - \hat{p}_i^*(t - t_0),$$

where $\hat{p}_{\xi_i}^*(t_0)$ is the initial value of $\hat{p}_{\xi_i}^*$. After the transform by sharp map, the result follows. ☐

Remark 2. *The condition of lemma 7 will make restrictions on the kind of the problem to be dealt with, we shall show that when the initial velocity is zero, the condition is automatically satisfied.*

4.2 Solution of Optimal Control

In this subsection, we studies the optimal control. According to PMP, the optimal control $\hat{u}_i$ must satisfies the necessary condition that

$$\frac{\partial H}{\partial \hat{u}_i} = -\hat{u}_i + \hat{p}_{\xi_i} = 0,\ i = 1, \cdots, n.$$

Since these equations have the unique solutions, the above condition is also sufficient. Recall Lemma 7, the optimal control can be written as

$$\hat{u}_i^{op} = \hat{p}_{\xi_i}(t_0) - \hat{p}_i(t - t_0). \tag{17}$$

For this control law, the constant $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ must be determined by boundary conditions (12). First, we give several results related to the transversality condition corresponding to (12) in the meaning of consensus.

Lemma 8. *If* $\hat{\xi}_\alpha^b(t_f) = \hat{\xi}_\beta^b(t_f), \alpha, \beta = 1, \cdots, n$, *then*

$$\sum_{k=1}^n \hat{p}_{\xi_k}(t_f) = 0.$$

Proof. From the boundary conditions (12), let

$$h_{\alpha\beta}(\hat{\xi}_\alpha^b(t_f), \hat{\xi}_\beta^b(t_f)) = \hat{\xi}_\alpha^b(t_f) - \hat{\xi}_\beta^b(t_f), \alpha, \beta = 1, \cdots, n,$$

and the (i, j)-th entry of $h_{\alpha\beta}$ is denoted by $h_{\alpha\beta}^{ij}$. Then according to PMP, the transversality condition corresponding to $h_{\alpha\beta} = 0$ for fixed β can be written as

$$\begin{aligned}[\hat{p}_{\xi_k}^*(t_f)]^{pq} &= \sum_\alpha \sum_{i,j} \Gamma_{\alpha\beta}^{ij} \frac{\partial h_{\alpha\beta}^{ij}}{\partial[\hat{\xi}_k^b]^{pq}}(t_f) \\ &= \sum_\alpha \mathrm{tr}\left(\Gamma_{\alpha\beta}^T \frac{\partial(\hat{\xi}_\alpha^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_k^b]^{pq}}\right)(t_f)\end{aligned}$$

$$k = 1, \cdots, n,\ p, q = 1, 2, 3, 4.$$

where the superscript 'pq' represents the (p, q)-th entry of the corresponding matrix, and $\Gamma_{\alpha\beta} = [\Gamma_{\alpha\beta}^{ij}]$ is the parameter matrix to be determined.

For the case $k \neq \beta$, the above equality will be

$$\begin{aligned}[\hat{p}_{\xi_k}^*(t_f)]^{pq} &= \mathrm{tr}\left(\Gamma_{k\beta}^T \frac{\partial(\hat{\xi}_k^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_k^b]^{pq}}\right)(t_f) \\ &= \mathrm{tr}\left(\Gamma_{k\beta}^T, E_{pq}\right) = \Gamma_{k\beta}^{pq}\end{aligned},$$

this gives that $\hat{p}_{\xi_k}^*(t_f) = \Gamma_{k\beta},\ k \neq \beta$. For the case $k = \beta$, we have

$$\begin{aligned}[\hat{p}_{\xi_\beta}^*(t_f)]^{pq} &= \sum_{\alpha\neq\beta} \mathrm{tr}\left(\Gamma_{\alpha\beta}^T \frac{\partial(\hat{\xi}_\alpha^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_\beta^b]^{pq}}\right)(t_f) \\ &= -\sum_{\alpha\neq\beta} \mathrm{tr}\left(\Gamma_{\alpha\beta}^T, E_{pq}\right)(t_f) = -\sum_{\alpha\neq\beta} \Gamma_{\alpha\beta}^{pq}\end{aligned},$$

this leads to that $\hat{p}_{\xi_\beta}^*(t_f) = -\sum_{\alpha\neq\beta} \Gamma_{\alpha\beta}$. Combining the two cases, we get

$$\sum_{k=1}^n \hat{p}_{\xi_k}^*(t_f) = 0.$$

After taking its sharp image, the result follows. ☐

Lemma 9. *If* $g_\alpha^{-1}(t_f) g_\beta(t_f) = I, \alpha, \beta = 1, \cdots, n$, *then*

$$\sum_{k=1}^n \hat{p}_{g_k}(t_f) = 0.$$

Proof. Let

$$f_{\alpha\beta}(g_\alpha(t_f), g_\beta(t_f)) = g_\alpha^{-1}(t_f) g_\beta(t_f) - I,\ \alpha, \beta = 1, \cdots, n,$$

and the (i, j)-th entry of $f_{\alpha\beta}$ is denoted by $f_{\alpha\beta}^{ij}$. By the same argument as in lemma 8, the transversality condition corresponding to $f_{\alpha\beta} = 0$ for fixed β can be written as

$$\begin{aligned}[\hat{p}_{g_k}^*(t_f)]^{pq} &= \sum_{\alpha,\alpha\neq\beta} \sum_{i,j} \Lambda_{\alpha\beta}^{ij} \frac{\partial f_{\alpha\beta}^{ij}}{\partial g_k^{pq}}(t_f) \\ &= \sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T \frac{\partial g_\alpha^{-1} g_\beta}{\partial g_k^{pq}}\right)(t_f)\end{aligned}$$

$$k = 1, \cdots, n,\ p, q = 1, 2, 3, 4.$$

where the superscript 'pq' represents the (p, q)-th entry of the corresponding matrix, and $\Lambda_{\alpha\beta} = [\Lambda_{\alpha\beta}^{ij}]$ is the parameter matrix to be determined.

For the case $k \neq \beta$, the above equality will be

$$\begin{aligned}[\hat{p}_{g_k}^*(t_f)]^{pq} &= \mathrm{tr}\left(\Lambda_{k\beta}^T \frac{\partial g_k^{-1} g_\beta}{\partial g_k^{pq}}\right)(t_f) \\ &= -\mathrm{tr}\left(\Lambda_{k\beta}^T g_k^{-1} E_{pq} g_k^{-1} g_\beta\right)(t_f) \\ &= -(g_k^{-T} \Lambda_{k\beta}^T g_\beta^T g_k^{-T})^{pq}(t_f)\end{aligned}.$$

By noting that $g_\beta^T(t_f) g_k^{-T}(t_f) = I$, it follows that $\hat{p}_{g_k}^*(t_f) = -(g_k^{-T}\Lambda_{k\beta}^T)(t_f),\ k \neq \beta$. For the case $k = \beta$, we have

$$\begin{aligned}[\hat{p}_{g_\beta}^*(t_f)]^{pq} &= \sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T \frac{\partial g_\alpha^{-1} g_\beta}{\partial g_\beta^{pq}}\right)(t_f) \\ &= \sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T g_\alpha^{-1} E_{pq}\right)(t_f) \\ &= \sum_{\alpha,\alpha\neq\beta} \left(g_\alpha^{-T}\Lambda_{\alpha\beta}^T\right)^{pq}(t_f)\end{aligned},$$

this leads to that $\hat{p}^*_{g_\beta}(t_f) = \sum_{\alpha,\alpha\neq\beta}\left(g_\alpha^{-T}\Lambda^T_{\alpha\beta}\right)(t_f)$. Combining the two cases, we get

$$\sum_{k=1}^{n}\hat{p}^*_{g_k}(t_f) = 0.$$

After taking its sharp image, the result follows. □

Corollary 1.

1) $\sum_{k=1}^{n}\hat{p}_k = 0$.
2) $\sum_{k=1}^{n}\hat{p}_{\xi_k}(t_0) = 0$.

Proof. By noting that $g_i(t_f) = g_j(t_f), i, j = 1, \cdots, n$ and $\hat{p}_{g_k}(t_f) = g_k(t_f)\hat{p}_k$, the result of item 1) follows from lemma 9. From the lemma 7, by taking $t = t_f$, then

$$\hat{p}_{\xi_k}(t_f) = \hat{p}_{\xi_k}(t_0) - \hat{p}_k(t_f - t_0),$$

summing it up for index k, and using lemma 8 and item 1), then the result of item 2) follows. □

The optimal control (17) shows that the determination of the optimal control is boiled down to finding the initial co-states $\hat{p}_{\xi_i}(t_0)$ and parameter $\hat{p}_i$. For finding the initial co-states $\hat{p}_{\xi_i}(t_0)$ we give the following lemma.

Lemma 10.

$$\hat{p}_{\xi_i}(t_0) = \frac{1}{t_f - t_0}(\hat{\xi}^{b,0} - \hat{\xi}^{b,0}_i) + \frac{1}{2}(t_f - t_0)\hat{p}_i \qquad (18)$$

where $\hat{\xi}^{b,0} = \frac{1}{n}\sum_i \hat{\xi}^{b,0}_i$ is an arithmetic mean of initial velocities.

Proof. By substituting the optimal control (17) to the systems (11) and integrating the dynamics equation, we get

$$\hat{\xi}^b_k(t) = \hat{\xi}^{b,0}_k + \hat{p}_{\xi_k}(t_0)(t - t_0) - \frac{1}{2}\hat{p}_k(t - t_0)^2, \qquad (19)$$

then summing it up for index k, it follows from (19)that

$$\hat{\xi}^b(t) = \hat{\xi}^{b,0} + \frac{1}{n}\sum_{k=1}^{n}\hat{p}_{\xi_k}(t_0)(t - t_0) - \frac{1}{2n}\sum_{k=1}^{n}\hat{p}_k(t - t_0)^2.$$

where $\hat{\xi}^b(t) = \frac{1}{n}\sum_k \hat{\xi}^b_k(t)$. From corollary 1, it easy to get that

$$\hat{\xi}^b(t) = \hat{\xi}^{b,0}, \forall t \geq t_0.$$

Thus $\hat{\xi}^{b,f} \triangleq \hat{\xi}^b(t_f) = \hat{\xi}^{b,0}$, and the consensus condition implies that $\hat{\xi}^b_k(t_f) = \hat{\xi}^{b,f}$. By taking $t = t_f$ in (19) and using $\hat{\xi}^{b,0}$ to replace $\hat{\xi}^b_k(t_f)$, then $\hat{p}_{\xi_i}(t_0)$ can be solved. □

Remark 3. *From the proof of lemma 10, we know that the velocity under the optimal control has the form*

$$\hat{\xi}^b_i(t) = \frac{t - t_0}{t_f - t_0}\hat{\xi}^{b,0} - \frac{t - t_f}{t_f - t_0}\hat{\xi}^{b,0}_i - \frac{1}{2}(t - t_0)(t - t_f)\hat{p}_i, \qquad (20)$$

For this form of velocity, it is obvious that the condition of lemma 7, i.e., $[\hat{p}_i, \hat{\xi}^b_i(t)] = 0$, if and only $[\hat{p}_i, \hat{\xi}^{b,0}] = 0$ and $[\hat{p}_i, \hat{\xi}^{b,0}_i] = 0$. Generally, this dose not hold for arbitrary initial velocities $\hat{\xi}^{b,0}_i, i = 1, \cdots, n$. But there is an important case that the initial velocity $\hat{\xi}^{b,0}_i$ has the form $c\hat{p}_i$ for some scalar $c \in \mathbb{R}$, in this case $\hat{\xi}^{b,0} = 0$ by corollary 1, and $[\hat{p}_i, \hat{\xi}^{0,b}_i] = 0$.

Now the problem of determination of optimal control (17) is further reduced to finding parameter $\hat{p}_i$, which is not trivial, we need some preparatory work. In order to use the the final values of the configuration, we need to integrate the kinematic equation $\dot{g}_i = g_i\hat{\xi}^b_i$ by utilizing velocity (20). The following lemma gives the result.

Lemma 11. *Suppose that both $[\hat{p}_i, \hat{\xi}^{b,0}] = 0$ and $[\hat{p}_i, \hat{\xi}^{b,0}_i] = 0, i = 1, \cdots, n$. Let $x_i(t) = \log(g^{-1}(t_0)g_i(t))$, $i = 1, \cdots, n$. Then*

$$x_i(t_f) = (t_f - t_0)\hat{\xi}^{b,0}_i + \frac{(t_f - t_0)^2}{2n}\sum_{j=1}^{n}\hat{\xi}^{b,0}_{ij} + \frac{(t_f - t_0)^3}{12}\hat{p}_i \qquad (21)$$

where $\hat{\xi}^{b,0}_{ij} = \hat{\xi}^{b,0}_j - \hat{\xi}^{b,0}_i$ is the relative initial velocity of vehicle j with respect to vehicle i.

Proof. Let $x_i(t) = \log(g^{-1}(t_0)g_i(t))$, $i = 1, \cdots, n$, then according to lemma 5, we have

$$\dot{x}_i = \hat{\xi}^b_i + \sum_{k=1}^{\infty}\frac{\mathrm{B}_k}{k!}\mathrm{ad}^k_{-x_i}(\hat{\xi}^b_i), \ x_i(t_0) = 0. \qquad (22)$$

If we show that $x_i(t) = \int_{t_0}^{t}\hat{\xi}^b_i(\tau)d\tau$ solve this equation, then the result follows. From (20), $x_i(t)$ should has the form of $x_i(t) = a(t)\hat{\xi}^{b,0} + b(t)\hat{\xi}^{b,0}_i + c(t)\hat{p}_i$, where a, b and c are scalar functions of time. Thus, by lemma 1

$$\begin{aligned} \mathrm{ad}_{-x_i}(\hat{\xi}^b_i) &= -[x_i, \hat{\xi}^b_i] \\ &= -a(t)[\hat{\xi}^{b,0}, \hat{\xi}^b_i] - b(t)[\hat{\xi}^{b,0}_i, \hat{\xi}^b_i] - c(t)[\hat{p}_i, \hat{\xi}^b_i] \\ &= 0 \end{aligned}.$$

This implies that $\mathrm{ad}^k_{-x_i}(\hat{\xi}^b_i) = 0$, for $k \geq 1$, and thus equation (22) becomes $\dot{x}_i = \hat{\xi}^b_i$. By integrating this equation, the result follows. □

Now, we first determine the parameter $\hat{p}_i$ for the case of $n = 2$.

Lemma 12. *For the case that $n = 2$, and $\hat{\xi}^{b,0}_i = 0, i = 1, 2$,*

$$\hat{p}_1 = \frac{6}{(t_f - t_0)^3}x^0_{12}, \ \hat{p}_2 = \frac{6}{(t_f - t_0)^3}x^0_{21} \qquad (23)$$

where $x^0_{ij} = \log(g^0_{ij}) = \log(g_i^{-1}(t_0)g_j(t_0)), i, j = 1, 2$.

Proof. Let $x_i(t) = \log(g_i^{-1}(t_0)g_i(t))$. In the meaning of consensus, i.e., $g_1(t_f) = g_2(t_f)$, recall (21), it is easy to get

$$\exp(x_1(t_f)) \cdot \exp(-x_2(t_f)) = g^0_{12}.$$

By using Baker-Campbell-Hausdorff (BCH) formula (9), we have

$$\begin{aligned} x^0_{12} &= x_1(t_f) - x_2(t_f) - \frac{1}{2}[x_1(t_f), x_2(t_f)] + \cdots \\ &= \frac{(t_f - t_0)^3}{12}\hat{p}_1 - \frac{(t_f - t_0)^3}{12}\hat{p}_2 \\ &\quad -\frac{1}{2}[\frac{(t_f - t_0)^3}{12}\hat{p}_1, \frac{(t_f - t_0)^3}{12}\hat{p}_2] + \cdots \end{aligned}.$$

From corollary 1, we know that $\hat{p}_1 + \hat{p}_2 = 0$, this implies that all Lie brackets will vanish, and

$$x^0_{12} = \frac{(t_f - t_0)^3}{12}\hat{p}_1 - \frac{(t_f - t_0)^3}{12}\hat{p}_2 = \frac{(t_f - t_0)^3}{6}\hat{p}_1.$$

Thus,

$$\hat{p}_1 = \frac{6}{(t_f - t_0)^3} x_{12}^0,$$

and by noting $\hat{p}_1 = -\hat{p}_2$, $x_{21}^0 = -x_{12}^0$,

$$\hat{p}_2 = \frac{6}{(t_f - t_0)^3} x_{21}^0.$$

□

We now give the optimal control law for the case of $n = 2$.

Theorem 1. *For the case of $n = 2$ and $\hat{\xi}_i^{b,0} = 0, i = 1, 2$, the optimal consensus control is*

$$\hat{u}_i^{o,op} = \frac{3(t_f + t_0 - 2t)}{(t_f - t_0)^3} x_{ij}^0, \ i \neq j, \ i, j = 1, 2. \quad (24)$$

Proof. Simply by substituting $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ given by (18) and (23) into the control law (17), then the result follows. □

For the case of $n > 2$, we can not obtain the exact explicit optimal control law as that for the case of $n = 2$. In this case, we only can get the approximate explicit optimal control as shown in the following theorem.

Theorem 2. *For the case that $n > 2$, and $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$ the approximate optimal consensus control is*

$$\hat{u}_i^{o,op} \approx \frac{6(t_f + t_0 - 2t)}{n(t_f - t_0)^3} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij}^0, \ i, j = 1, \cdots, n. \quad (25)$$

where $x_{ij}^0 = \log(g_{ij}^0) = \log(g_i^{-1}(t_0) g_j(t_0))$.

Proof. Let $x_i(t) = \log(g_i^{-1}(t_0) g_i(t))$. Since $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$, it can be seen from (21) that

$$x_i(t_f) = \frac{(t_f - t_0)^3}{12} \hat{p}_i, \ i = 1, \cdots, n$$

In the meaning of consensus, i.e., $g_i(t_f) = g_j(t_f)$, it is easy to get

$$\exp(x_i(t_f)) \cdot \exp(-x_j(t_f)) = g_{ij}^0.$$

By using Baker-Campbell-Hausdorff (BCH) formula (9), we have

$$\begin{aligned} x_{ij}^0 &= x_i(t_f) - x_j(t_f) - \tfrac{1}{2}[x_i(t_f), x_j(t_f)] \\ &-\frac{1}{12}[x_i(t_f), [x_i(t_f), x_j(t_f)]] \\ &+\frac{1}{12}[x_j(t_f), [x_j(t_f), x_i(t_f)]] + \cdots, \end{aligned}$$

by summing it up for index j, and noting that $\sum_i x_i(t_f) = 0$, it follows that

$$\sum_{j=1}^{n} x_{ij}^0 = n x_i(t_f) + \frac{1}{12} \sum_{j=1}^{n} [x_j(t_f), [x_j(t_f), x_i(t_f)]] + \cdots, \quad i = 1, \cdots, n$$

This is a group of equations about parameters $\hat{p}_i, i = 1, \cdots, n$, it is generally imposable to solve $\hat{p}_i, i = 1, \cdots, n$, explicitly. For this problem, when the configurations approach consensus, the influence of the higher order Lie brackets will attenuate quickly, and can be omitted. Thus we get a group of approximate equations as follows,

$$\sum_{j=1}^{n} x_{ij}^0 \approx n x_i(t_f) = \frac{n(t_f - t_0)^3}{12} \hat{p}_i,$$

thus

$$\hat{p}_i \approx \frac{12}{n(t_f - t_0)^3} \sum_{j=1}^{n} x_{ij}^0.$$

Recall (18), we have $\hat{p}_{\xi_i}(t_0) = \frac{1}{2}(t_f - t_0)\hat{p}_i$, by substituting $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ into (17), then the result follows. □

Remark 4. *By substituting $\hat{p}_i, i = 1, \cdots, n$, into (21), the final configuration of $g_i, i = 1, 2$ will be*

$$g_i(t_f) = g_i^0 \exp\left(\frac{1}{n} \sum_{j=1}^{n} x_{ij}^0\right), \ i = 1, \cdots, n. \quad (26)$$

For $n = 2$, we have exactly that $g_1(t_f) = g_2(t_f)$.

Remark 5. *The control law given by (25) are open control that is determined by initial configuration. This control is sensitive to the perturbations in initial configurations and the disturbances in input control, which may cause failure in achieving the required consensus.*

In order to overcome this deficiency, it is suggested to use the closed optimal control, i.e., to take the current time and state as the initial time and state when constructing the optimal control [26]. In this case the control law (24) and (25) can not be used, since it is obtained under the assumption that the initial velocities are zeros, but the current velocities, when being taken as initial velocities, can not be zeros as shown in (20)

$$\hat{\xi}_i^b(t) = -\frac{1}{2}(t - t_0)(t - t_f)\hat{p}_i.$$

If we take the initial time as zero, and current time as t_0, the current velocities will be

$$\hat{\xi}_i^b(t_0) = -\frac{1}{2}(t_0)(t_0 - t_f)\hat{p}_i.$$

Obviously, this form of velocity has the property that $[\hat{p}_i, \hat{\xi}_i^b(t_0)] = 0$, and $\sum_i \hat{\xi}_i^b(t_0) = 0$ which satisfies the condition of lemma 11. See remark 3.

Theorem 3. *Suppose that $[\hat{p}_i, \hat{\xi}_i^b(t_0)] = 0, i = 1, \cdots, n$, and $\sum_i \hat{\xi}_i^b(t_0) = 0$, then the optimal control is*

$$\hat{u}_i^{op} \approx \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{6(t_f + t_0 - 2t)}{n(t_f - t_0)^3} x_{ij}(t_0) + \frac{2(2t_f + t_0 - 3t)}{n(t_f - t_0)^2} \hat{\xi}_{ij}^b(t_0) \right)$$
$$i = 1, \cdots, n. \quad (27)$$

Proof. Noting that in this case, we also have $\sum_i x_i(t_f) = 0$, by almost the same argument as that of theorem 2, then the result will follows. □

Theorem 4. *Suppose that the initial velocities satisfies $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$. For the required formation designated by relative configurations $\bar{g}_{ij}, i, j = 1, \cdots, n$, the* *optimal feedback formation control is*

$$\hat{u}_i^{f,op} \approx \sum_{\substack{j=1\\i\neq j}}^{n} \left(\frac{6}{n(t_f-t)^2} x_{ij}(t) + \frac{4}{n(t_f-t)} \hat{\xi}_{ij}^b(t) \right), \quad i = 1, \cdots, n. \tag{28}$$

where $x_{ij}(t) = \log(g_{ij}(t)\bar{g}_{ji})$, *and* $\hat{\xi}_{ij}^b(t) = \mathrm{Ad}_{\bar{g}_{ij}} \hat{\xi}_j^b(t) - \hat{\xi}_i^b(t)$.

Proof. Recalling lemma 6, by taking the current time t as the initial time t_0, and using $\tilde{u}_i^{f,op}$, $\tilde{x}_{ij} = \log \tilde{g}_i^{-1}\tilde{g}_j$, and $\tilde{\xi}_{ij}^b = \mathrm{Ad}_{\bar{g}_{1j}} \hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{1i}} \hat{\xi}_j^b$ to replace $\hat{u}_i^{op}$, x_{ij} and $\hat{\xi}_{ij}^b$ respectively in (27), we shall obtain

$$\tilde{u}_i^{f,op} \approx \sum_{\substack{j=1\\i\neq j}}^{n} \left(\frac{6}{n(t_f-t)^2} \tilde{x}_{ij}(t) + \frac{4}{n(t_f-t)} \tilde{\xi}_{ij}^b(t) \right), \quad i = 1, \cdots, n.$$

From lemma 6, the optimal feedback formation control can will be

$$\hat{u}_i^{f,op} = \mathrm{Ad}_{\bar{g}_{i1}} \tilde{u}_i^{f,op}.$$

Also, by lemma 4, we have

$$\begin{aligned} \mathrm{Ad}_{\bar{g}_{i1}} \tilde{x}_{ij} &= \bar{g}_{i1} \left(\log\left(\bar{g}_1^{-1} \bar{g}_j \right) \right) \bar{g}_{i1}^{-1} \\ &= \log\left(\bar{g}_{i1} \bar{g}_i^{-1} \bar{g}_j \bar{g}_{i1}^{-1} \right) \\ &= \log\left(g_i^{-1} g_j \bar{g}_{ji} \right) \end{aligned},$$

$$\begin{aligned} \mathrm{Ad}_{\bar{g}_{i1}} \tilde{\xi}_{ij}^b &= \mathrm{Ad}_{\bar{g}_{i1}} \left(\mathrm{Ad}_{\bar{g}_{1j}} \hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{1i}} \hat{\xi}_i^b \right) \\ &= \mathrm{Ad}_{\bar{g}_{ij}} \hat{\xi}_j^b - \hat{\xi}_i^b \end{aligned}.$$

This proves the theorem. □

Remark 6. *In theorem 4,*

1) *when* $n = 2$, *the optimal feedback control law is exact, no approximation is assumed in this case;*
2) *when* $\bar{g}_{ij} = I, i, j = 1, \cdots, n$, *the formation control law (28) will be reduced to the consensus control law.*

5 Simulations

Example 1. Consensus of two vehicles under open optimal control

Consider the consensus of two planar vehicles, the initial configurations are listed in Table 1.

Table 1: Initial Configurations of Agents

order number	$\theta(0)$	$x(0)$	$y(0)$
1	$\pi/2$	80	100
2	$-\pi/2$	-80	-100

$$g_1^0 = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 80 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 100 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g_2^0 = \begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) & -80 \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) & -100 \\ 0 & 0 & 1 \end{bmatrix},$$

it can be calculate that

$$x_{12}^0 = -x_{21}^0 = \log\left((g_1^0)^{-1} g_2^0 \right) = \begin{bmatrix} 0 & \pi & -251.3274 \\ -\pi & 0 & -100\pi \\ 0 & 0 & 0 \end{bmatrix},$$

and utilizing (26), the final configurations for consensus are

$$g_1(t_f) = g_2(t_f) = \begin{bmatrix} 1 & 0 & 100 \\ 0 & 1 & -80 \\ 0 & 0 & 1 \end{bmatrix}$$

If takeing $t_0 = 0$, $t_f = 10$, *then the optimal control will be*

$$u_1^{o,op} = -u_2^{o,op} = \frac{3(10-2t)}{1000} [-\pi \quad -251.3274 \quad -100\pi]^T,$$

the value of optimal cost is easy to to calculate from $u_i, i = 1, 2$ *by definition (13),*

$$J = \frac{1}{2} \int_0^{10} (u_1^T u_1 + u_2^T u_2) dt = 485.6414$$

The dynamical process of simulations under both open and feedback optimal controls (24) and (28) are same and are showed in Fig. 1 , Fig. 2 and Fig. 3.

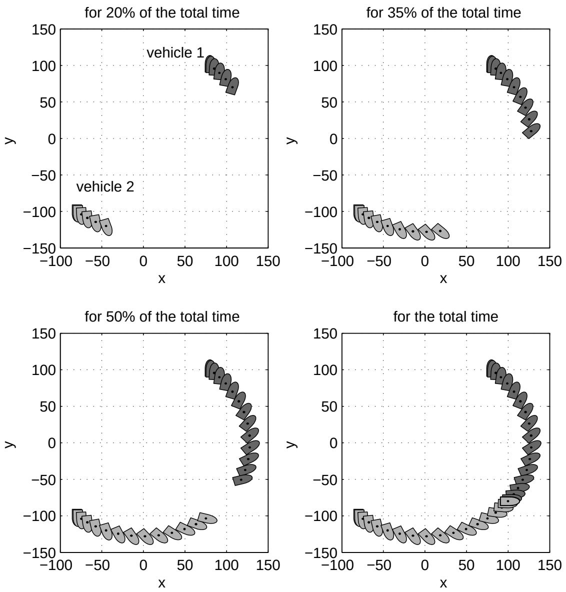

Fig. 1: The trajectories of 2 vehicles

Example 2. Consensus of two vehicles under open and feedback optimal controls with disturbances

Continuing the example 1 under the open optimal control (24) with disturbance

$$u_1(t) = u_1^{o,op}(t) + [0, \sin(t), -0.1\sin(t)]^T, \; u_2(t) = u_2^{o,op}(t), \quad 0 \leq t \leq 10,$$

$$u_1(t) = 0, \; u_2(t) = 0, \; t > 10.$$

The configurations at time $t_f = 10$ *are*

$$g_1(t_f) = \begin{bmatrix} 0.4937 & 0.8696 & 16.1905 \\ -0.8696 & 0.4937 & -39.6656 \\ 0 & 0 & 1 \end{bmatrix} .$$
$$g_2(t_f) = \begin{bmatrix} 1 & 0 & 100 \\ 0 & 1 & -80 \\ 0 & 0 & 1 \end{bmatrix}$$

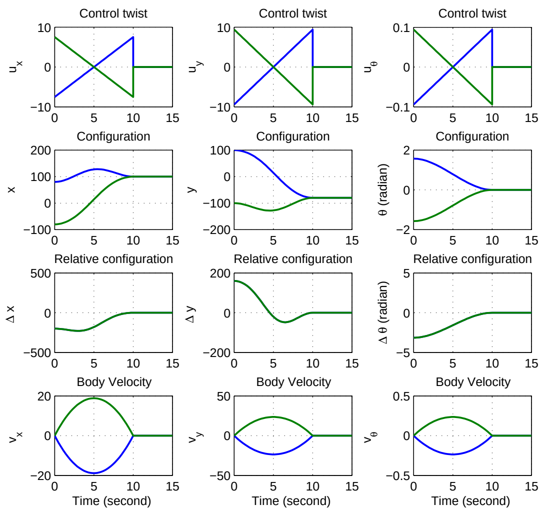

Fig. 2: The time behaviors of 2 vehicles during achieving consensus

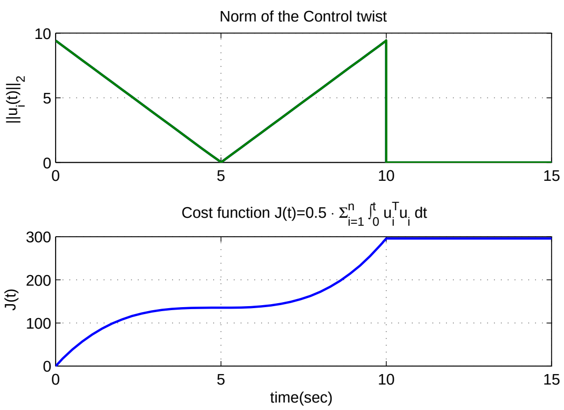

Fig. 3: The time behaviors of the norm and the cost of control

The system dose not achieve consensus at time $t_f = 10$, see figure 4 and figure 5.

Now, the system is driven by feedback optimal control with the same disturbance

$$u_1(t) = u_1^{f,op}(t) + [0, \sin(t), -0.1\sin(t)]^T,\ u_2(t) = u_2^{f,op}(t),\quad 0 \le t \le 10,$$

$$u_1(t) = 0,\ u_2(t) = 0,\ t > 10,$$

then the configurations at time $t_f = 10$ are

$$g_1(t_f) = g_2(t_f) = \begin{bmatrix} 0.5519 & 0.8339 & 96.7441 \\ -0.8339 & 0.5519 & -62.4848 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system dose achieves consensus at time $t_f = 10$, see figure 6 and figure 7.

Example 3. Formation of four vehicles under the feedback optimal control

Given four vehicles with the initial configurations shown in Table 2, now consider formation problem with the required formation given by Table 3. The control law to be utilized is the feedback optimal control law (28), the t_f is

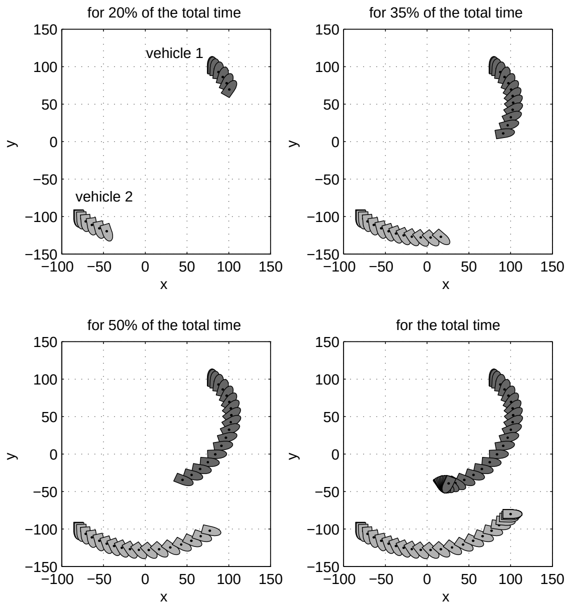

Fig. 4: The trajectories of 2 vehicles

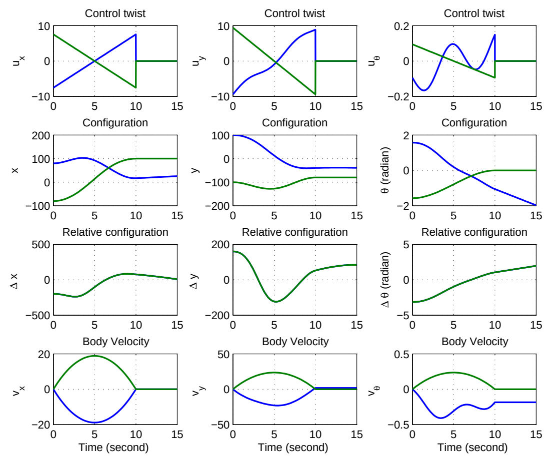

Fig. 5: The time behaviors of 2 vehicles during achieving consensus

taken to be 10 and the total simulation time is 15. The dynamical process of simulation is shown in Figure 8 and Figure 9. The result shows that the system archives the required formation at $t_f = 10$.

Table 2: Initial Configurations of Agents

order number	$\theta(0)$	$x(0)$	$y(0)$
1	0	100	100
2	$-\pi$	-100	-100
3	$\pi/2$	-100	100
4	$-\pi/2$	100	-100

6 Conclusion

In this paper, we studied the finite time optimal formation control problems of multiple vehicles which are modeled by dynamics of rigid body evolving on the tangent bundle of

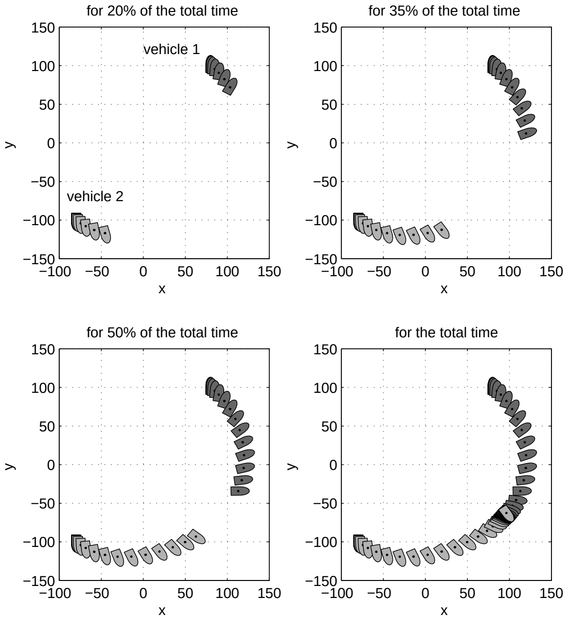

Fig. 6: The trajectories of 2 vehicles

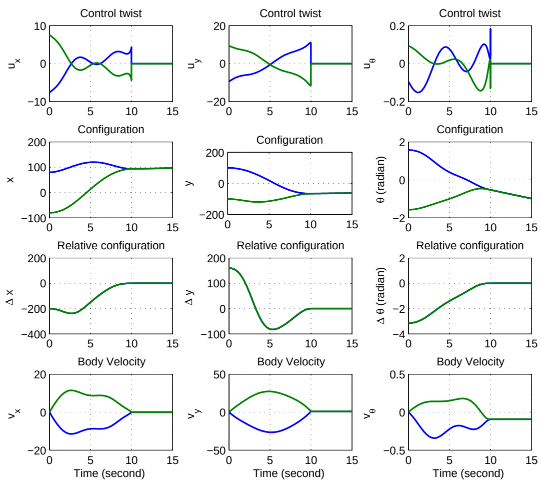

Fig. 7: The time behaviors of 2 vehicles during achieving consensus

Table 3: relative configurations at final time

relative index$(1i)$	$\theta_{1i}(t_f)$	$x_{1i}(t_f)$	$y_{1i}(t_f)$
(11)	0	0	0
(12)	0	-20	20
(13)	0	-20	-20
(14)	0	-40	0

Euclidean group SE(n), $n = 2, 3$. Both the formation time and the geometric structure of the formation are specified by the task. Under the assumption that the vehicles are all full actuated with zero initial velocities and that the communication between the vehicles is all-to-all, we derived a finite time formation control law, this control law is optimal when the number of formation vehicles is two and suboptimal when the number of formation vehicles is more than two.

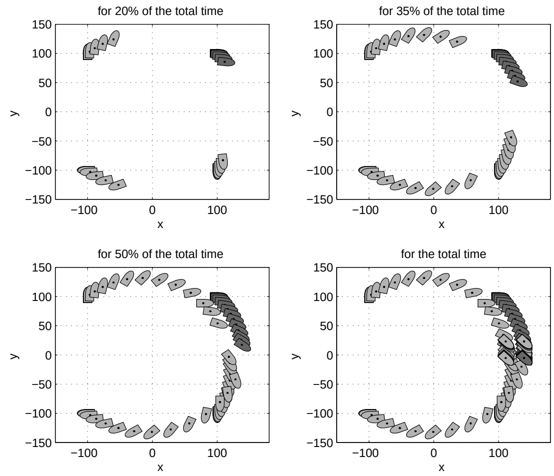

Fig. 8: The trajectories of multi vehicles

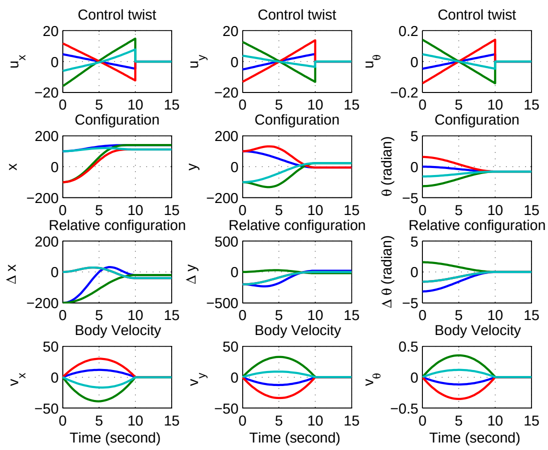

Fig. 9: The time behaviors of vehicles during formation

The further meaningful problems that worth being studied is the same formation control problem as above but with the following relaxed assumptions, 1) the initial velocities of vehicles are nonzero, 2) the vehicles are under actuated, 3) the communication topology of the vehicles are more general than that of all-to-all.

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