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Abnormal Minimizers in Nonholonomic Mechanical Control Systems

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Abstract: In this paper, it is concerned with the problem of finding abnormal minimizers in nonholonomic mechanical control systems. Relations between nonholonomic mechanical and kinematic optimal control problems are established. For rank-two distributions, the abnormal minimizers in kinematic control systems are used to get abnormal minimizers for the length-minimizing problem in the corresponding nonholonomic mechanical control systems. Finally our ideas are illustrated by considering examples of nonholonomic mechanical systems.

Key Words: abnormal extremals, nonholonomic mechanical control systems, Pontryagain's Maximum Principle

1 Introduction

For years, it was thought that abnormal extremals could not be optimal, since they were isolated curves and variations were impossible. The idea wasn't changed until Montgomery [1] gave an example in 1994. He proved there exists a strict abnormal minimizer in the example. Later Liu and Sussmann [2] characterized abnormal minimizers for the length-minimizing problem in sub-Riemannian geometry with two-dimensional distributions. Their results imply abnormal minimizers are ubiquitous rather than exceptional.

For mechanical control systems, Maria Barbero Linan et al. [3] gave an example of nonholonomic mechanical control system and proved it has a strict abnormal minimizer.

In this paper, we focus on finding abnormal minimizers in nonholonomic mechanical control systems. To obtain abnormal minimizers in nonholonomic control systems, we first establish relations between nonholonomic and kinematic optimal solutions. Then we investigate the Hamilton's equations given by Pontryagain's Maximum principle for the nonholonomic mechanical control systems. Finally we build on previous work on abnormal extremals in rank-two distributions by Liu and Sussmann to get abnormal minimizers for the length-minimizing problem in nonholonomic mechanical control systems.

The paper is organized as follows: Preliminaries are provided in section 2. In section 3, we establish relations between nonholonomic and kinematic optimal control problems. After discussing the relations between the optimal solutions for both systems, we get into detailed study of the Hamilton's equations for the nonholonomic mechanical systems. Examples of nonholonomic mechanical system which have abnormal minimizers are studied in section 4. Conclusions are given in section 5.

Throughout this paper, we suppose that the manifolds are real, second countable and C^∞, and the maps are C^∞. All the used vector field distributions are supposed to have constant rank. As a reference for differential geometry, notation employed and geometric concepts, see [4].

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2 Preliminaries

2.1 Nonholonomic Mechanical and Kinematic Control Systems

Let (Q, g) be a Riemannian manifold of dimension n, ∇ be the Levi-civita connection associated to the Riemannian metric g. Let TQ be the tangent bundle of Q. Consider D be a nonintegrable distribution in Q with rank k and F be a vector field of Q describing an external force. Let $D^\perp$ be the orthogonal distribution to D according to the metric g. Then a nonholonomic mechanical control system is given by $\Sigma_D = (Q, g, F, D)$. A differentiable curve $\gamma : I \to Q$ is a solution of Σ_D for certain values of control functions $u^i \in C^\infty(R)$ if it satisfies the conditions

$$\nabla_{\dot\gamma}\dot\gamma = F\circ\gamma + \sum_{r=1}^{n-k}\mu^r Z_r\circ\gamma + \sum_{i=1}^{s}u^i Y_i\circ\gamma, \quad \dot\gamma \in D \tag{1}$$

where $Y_i(q) \in D(q)$, for $i = 1, \cdots, s$, span $\{Z_1(q), Z_2(q)\cdots Z_{n-k}(q)\} = D^\perp(q)$, and $u : I \to U \subseteq R^s$ being U an open set. The corresponding first order equation in TQ is given by the vector field

$$Y = S + F^V + \sum_{r=1}^{n-k}\mu^r Z_r^V + \sum_{i=1}^{s}u^i Y_i^V$$

where S is the geodesic spray and Y_i^V is the vertical lift of Y_i, analogously for F^V and Z_r^V.

The coefficients $\mu^r \in C^\infty(TQ), r = 1, \cdots n-k$ are called Lagrange multipliers. They are determined by the condition $\dot\gamma \in D$ as follows: Suppose $\omega^1, \cdots \omega^{n-k} \in \Omega^1(Q)$, linearly independent at every point, satisfy $\omega^i(v_q) = 0, i = 1, \cdots n-k$ if and only if $v_q \in D$. Define $\hat\omega^i \in C^\infty(TQ)$ such that $\hat\omega^i(v_q) = \omega_q(v_q)$. Then we have

$$L_Y\hat\omega^i|_D = L_S\hat\omega^i + L_{F^V}\hat\omega^i + \sum_{r=1}^{n-k}\mu^r L_{Z_r^V}\hat\omega^i + \sum_{i=1}^{s}u^i L_{Y_i^V}\hat\omega^i|_D.$$

$Y_i(q) \in D(q)$ gives $L_{Y_i^V}\hat\omega^i|_D = 0$. So the above equations become

$$L_S\hat\omega^i + L_{F^V}\hat\omega^i + \sum_{r=1}^{n-k}\mu^r L_{Z_r^V}\hat\omega^i|_D = 0, \quad i = 1, \cdots n-k$$

$\mu^r|_D$ is determined by solving the linear equation above.

For the kinematic control system Σ_K associated to (1), a differentiable curve $\gamma : I \to Q$ is its solution if there exists certain values of the control functions $v^i \in C^\infty(R)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^{s} v^i(t) Y_i(\gamma(t)) \tag{2}$$

where $v : I \to V \subseteq R^s$ being V an open set.

The system (1) and (2) are equivalent if every solution of (1) is also a solution of (2), and vice versa.

Let $D_c \subseteq D$ be the distribution generated by the control vector fields. If rank D_c=rank D, Σ_D is called full actuated, otherwise we have underactuated systems. In the sequel, all the noholonomic mechanical control systems are referred to be full actuated.

Let the set of control values $U = V = R^k$, we have:

Theorem 2.1 ([5]) *Every full actuated nonholonomic mechanical control system Σ_D is equivalent to the associated kinematic control system Σ_K.*

Remark 2.2 *To establish equivalence, we choose $U = V = R^k$, because the same trajectory may have different control functions for the two systems. If we put bounds on the set of control values, equivalence may not hold. This fact is also an obstruction for linking time optimal control problems for these two systems.*

2.2 Pontryagin's Maximum Principle

The understanding of Pontryagin's Maximum Principle never vanishes since it was published in English in 1962 [6]. As a reference, see [7-14]. Here we follow the geometric approach in [15].

Let Q be a smooth n-dimensional manifold and $U \subseteq R^m$ be a bounded subset. Let X be a vector field along the projection $\pi : Q \times U \to Q$. If (x^i) are local coordinates on Q, the local expression of the vector field is $X = f^i \frac{\partial}{\partial x^i}$, where f^i are functions defined on an open set of $Q \times U$. Given $F : Q \times U \to R$, consider the function $S[\gamma, u] = \int_I F(\gamma, u)\, dt$ defined on curve (γ, u) with a compact interval as domain.

Now, we define the equivalent extended optimal control problem on $\hat{Q} = R \times Q$ with projection $\hat{\pi} : \hat{Q} \times U \to \hat{Q}$. Let $\hat{X}$ be the vector field along the projection $\hat{\pi} : \hat{Q} \times U \to \hat{Q}$ given by $\hat{X}(x^0, x, u) = F(x, u) \frac{\partial}{\partial x^0}\Big|_{(x^0,x,u)} + X(x, u)$ where x^0 is the natural coordinate on R. Given x_0, x_f, find $(\hat{\gamma}, u) : I \to \hat{Q} \times U$, such that

1) $\hat{\gamma}$ satisfies the endpoint conditions $\hat{\gamma}(t_0) = (0, x_0)$, $\gamma(t_f) = x_f$;
2) $\dot{\hat{\gamma}}(t) = \hat{X}(\hat{\gamma}(t), u(t))$ almost everywhere $t \in I$;
3) $\gamma^0(t_f)$ is minimum over all curves satisfying 1 and 2.

Let $T^*\hat{Q}$ be the cotangent bundle with its natural symplectic structure denoted by ω. For each $u \in U$, $H^u : T^*\hat{Q} \to R$ is the Hamiltonian function defined by $H^u(\hat{x}, \hat{p}) = H(\hat{x}, \hat{p}, u) = \left\langle \hat{p}, \hat{X}(\hat{x}, u) \right\rangle = p_0 F(x, u) + \sum_{i=1}^{n} p_i f^i(x, u)$. The Hamiltonian vector field associated with H is a vector field along the projection $\hat{\pi}_1 : T^*\hat{Q} \times U \to T^*\hat{Q}$ given by $\hat{X}^{T^*}$, the cotangent lift of $\hat{X}$.

Define the Hamiltonian Problem as follows: Given the extended optimal control problem, find$(\hat{\sigma}, u) : I \to T^*\hat{Q} \times U$ such that

1) if $\hat{\gamma} = \pi_{\hat{Q}} \circ \hat{\sigma}, \gamma = \hat{\pi}_2 \circ \hat{\gamma}$ where $\pi_{\hat{Q}} : T^*\hat{Q} \to \hat{Q}$ and $\hat{\pi}_2 : \hat{Q} \to Q$, then $\hat{\gamma}(t_0) = (0, x_0)$, $\gamma(t_f) = x_f$;
2) $\dot{\hat{\sigma}}(t) = \hat{X}^{T^*}(\hat{\sigma}(t), u(t))$ almost everywhere $t \in I$.

Locally $(\hat{\sigma}, u)$ satisfies Hamilton's equations of the system $(T^*\hat{Q}, \omega, H^u)$:

$$\dot{x}^0 = \frac{\partial H^u}{\partial p_0} = F, \qquad \dot{p}_0 = -\frac{\partial H^u}{\partial x^0} = 0$$

$$\dot{x}^i = \frac{\partial H^u}{\partial p_i} = f^i, \qquad \dot{p}_i = -\frac{\partial H^u}{\partial x^i} = -p_0 \frac{\partial F}{\partial x^i} - p_j \frac{\partial f^j}{\partial x^i}$$

Theorem 2.3 ([15] Pontryagin's Maximum Principle) *Let $(\hat{\gamma}, u)$ be a solution of the extended optimal control problem. Then there exists $(\hat{\sigma}, u) : I \to T^*\hat{Q} \times U$, with fiber momenta $\hat{\lambda}(t) \in T^*_{\hat{\gamma}(t)}\hat{Q}$ such that:*

1) $(\hat{\sigma}, u)$is a solution of the Hamiltonian Problem;
2) a) $H(\hat{\sigma}(t), u(t)) = \max_{\tilde{u} \in U} H(\hat{\sigma}(t), \tilde{u}(t))$ almost everywhere $t \in I$;
b) $\max_{\tilde{u} \in U} H(\hat{\sigma}(t), \tilde{u}(t)) = const$ almost everywhere $t \in I$;
c) $\hat{\lambda}(t) = (\lambda_0, \lambda(t)) \neq 0$ for each $t \in I$.

Definition 2.4 ([2]) *A curve $\gamma : I \to Q$ is called*

1) an extremal if there exists $(\hat{\sigma}, u) : I \to T^*\hat{Q} \times U$ which satisfies the necessary conditions of Pontryagin's Maximum Principle and $\gamma = \pi_Q \circ \hat{\sigma}$ where $\pi_Q : T^*\hat{Q} \to Q$;
2) a normal extremal if it is an extremal with $\lambda_0 = -1$;
3) an abnormal extremal it is an extremal with $\lambda_0 = 0$;
4) a strict normal extremal if it is not an abnormal extremal but it is normal;
5) a strict abnormal extremal if it is not a normal extremal but it is abnormal.

3 Optimal Control Problem for Nonholonomic Mechanical Control Systems

3.1 Relations Between Nonholonomic and Kinematic Optimal Solutions

Let us consider the following two optimal control problems.
Given $x_0, x_f \in Q$,
(a) For the nonholonomic mechanical control system Σ_D with cost function $F : TQ \times U \to R$, $\gamma : I \to Q$ is called the optimal solution for Σ_D if it satisfies:

1) $\gamma(t_0) = x_0, \gamma(t_f) = x_f$;
2) $\gamma(t)$ is a solution of Σ_D;
3) $\gamma(t)$ minimizes $\int_I F(\dot{\gamma}(t), u(t))dt$ among all the curves satisfy 1 and 2.

(b) For the kinematic control system Σ_K associated to Σ_D with cost function $G : Q \times V \to R$, $\gamma : I \to Q$ is called the optimal solution for Σ_K if it satisfies:

1) $\gamma(t_0) = x_0, \gamma(t_f) = x_f$;
2) $\gamma(t)$ is a solution of Σ_K;
3) $\gamma(t)$ minimizes $\int_I G(\gamma(t), v(t))dt$ among all the curves satisfy 1 and 2.

Proposition 3.1 *Let $U = V = R^k$ and $f : TQ \to R$. Let $G(\gamma(t), v) = f(\sum_{i=1}^{k} v^i Y_i(\gamma(t)))$ and $F = f$. If $\gamma(t)$ is the optimal solution for Σ_K, then it is also the optimal solution for Σ_D and vice versa.*

Proof. Let $\gamma(t)$ be the optimal solution for Σ_K. Suppose $\tilde{\gamma}(t)$ is a curve satisfying 1 and 2 for Σ_D, according to Theorem 2.1, $\tilde{\gamma}(t)$ satisfies 1 and 2 for Σ_K. So there exists $\tilde{v} : I \to R^k$ such that $\dot{\tilde{\gamma}}(t) = \sum_{i=1}^{s} \tilde{v}^i(t) Y_i(\tilde{\gamma}(t))$. Since (γ, v) is the optimal solution for Σ_K, we have

$$
\begin{aligned}
\int_I F(\dot{\gamma}(t))dt &= \int_I f(\sum_{i=1}^{k} v^i(t)Y_i(\gamma(t)))dt = \int_I G(\gamma(t), v(t))dt \\
&\leq \int_I G(\tilde{\gamma}(t), \tilde{v}(t))dt = \int_I f(\sum_{i=1}^{k} \tilde{v}^i(t)Y_i(\tilde{\gamma}(t)))dt \\
&= \int_I F(\dot{\tilde{\gamma}}(t))dt
\end{aligned}
$$

for all $\tilde{\gamma}$ satisfy 1 and 2 for Σ_D. On the other hand, according to Theorem 2.1, $\gamma(t)$ satisfies 1 and 2 for Σ_D. So $\gamma(t)$ is the optimal solution for Σ_D with cost function $F = f$. The proof of the remaining part is similar.

Remark 3.2 *In [3], time optimal problems for both systems are considered. To ensure the existence of time optimal solution, there must be bounds on the control sets for both systems. For the kinematic control system, according to Filippov theorem [7], when V is compact, the optimal solution exists. But for the nonholonomic mechanical control system, even with bounds on control, we still can't ensure the existence of optimal solution. On the other hand, even if there exists time optimal solution for some nonholonomic control system with bounds on control, according to remark 2.2, Theorem 2.1 may not hold for the two systems, so does time optimal equivalence.*

3.2 Abnormal Minimizers in Nonholonomic Mechanical Control Systems

Given a nonholonomic mechanical control system Σ_D with cost function $f : TQ \to R$, then Σ_D is a control system on the subbundle D of TQ. To apply Pontryagin's Maximum Principle, we use the local coordinate systems $(x^1, \cdots, x^n, y^1, \cdots, y^k, y^k+1, \cdots, y^n)$ on TQ given by the vector fields $Y_i, i = 1, \cdots k$ and $Z_j, j = 1, \cdots n-k$.

First we compute the Lagrange multipliers $\mu^r \in C^\infty(TQ), r = 1, \cdots n-k$. According to 2.1, let $\omega^i(x^1, \cdots, x^n, y^1, \cdots, y^k, y^k+1, \cdots, y^n) = y^{i+k}$, then we have

$$\mu^i = -F^{i+k} - (-\Gamma^{i+k}_{mn} y^m y^n), \quad i = 1, \cdots, n-k,$$

where $F = \sum_{i=1}^{k} F^i Y_i + \sum_{j=1}^{n-k} F^{j+k} Z_j$ and $\nabla_{X_i} X_j = \Gamma^k_{ij} X_k$ with $X_i = Y_i, i = 1, \cdots, k, X_j = Z_{j-k}, j = k+1, \cdots, n$. The system Σ_D has the local expression

$$
\begin{aligned}
\dot{x}^i &= \sum_{j=1}^{k} y^j Y^i_j + \sum_{l=k+1}^{n} y^l Z^i_{l-k}, \\
\dot{y}^i &= -\Gamma^i_{lm} y^l y^m + u^i + F^i, \quad i = 1, \cdots, k, \\
\dot{y}^j &= -\Gamma^j_{lm} y^l y^m + \mu^{j-k} + F^j = 0, \quad j = k+1, \cdots, n.
\end{aligned}
$$

For the solutions of Σ_D lies in the distribution D, we have $y^j = 0, j = k+1, \cdots, n$. Restrict Σ_D to the subbundle D, We have the Hamiltonian function

$$H = \left\langle p, \sum_{i=1}^{k} y^i Y_i \right\rangle + \sum_{i=1}^{k} \left\langle q_i, -\sum_{l,m=1}^{k} \Gamma^i_{lm} y^l y^m + u^i + F^i \right\rangle + p_0 f.$$

And Hamilton's equations are

$$
\begin{aligned}
\dot{x}^0 &= f, \\
\dot{x}^i &= \sum_{j=1}^{k} y^j Y^i_j. \\
\dot{y}^i &= -\Gamma^i_{lm} y^l y^m + u^i + F^i, \\
\dot{p}_0 &= 0, \\
\dot{p}_i &= -\left\langle p, \sum_{j=1}^{k} y^j \frac{\partial Y_j}{\partial x^i} \right\rangle - \sum_{j=1}^{k} \left\langle q_j, -\sum_{l,m=1}^{k} \frac{\partial \Gamma^j_{lm}}{\partial x^i} y^l y^m + \frac{\partial F^j}{\partial x^i} \right\rangle - p_0 \frac{\partial f}{\partial x^i}, \\
\dot{q}_i &= -\langle p, Y_i \rangle - \sum_{l=1}^{k} \left\langle q_l, -\Gamma^l_{im} y^m - \Gamma^l_{mi} y^m \right\rangle - p_0 \frac{\partial f}{\partial y^i}.
\end{aligned} \tag{3}
$$

where p_i are the momenta of the state and q_i are the corresponding momenta to the velocities.

Since the cost function f doesn't contain the control terms, then the condition 2(a) in Theorem 2.3 gives

$$q_i = 0, \quad i = 1, \cdots, k.$$

This yields

$$\langle p, Y_i \rangle + p_0 \frac{\partial f}{\partial y^i} = 0, i = 1, \cdots, k. \tag{4}$$

Then (3) becomes

$$
\begin{aligned}
\dot{x}^0 &= f \\
\dot{x}^i &= \sum_{j=1}^{k} y^j Y^i_j \\
\dot{y}^i &= -\Gamma^i_{lm} y^l y^m + u^i + F^i, i = 1, \cdots, k \\
\dot{p}_0 &= 0 \\
\dot{p}_i &= -\left\langle p, \sum_{j=1}^{k} y^j \frac{\partial Y_j}{\partial x^i} \right\rangle - p_0 \frac{\partial f}{\partial x^i} \\
q_i &= 0 \\
(p, p_0) &\neq 0
\end{aligned} \tag{5}
$$

For the corresponding kinematic control system with cost function $G = f$, we have Hamiltonian $H = \left\langle \hat{\lambda}, \sum_{s=1}^{k} v^s Y_s \right\rangle + p_0 G$ with local expression $H = \sum_{s=1}^{k} v^s \hat{\lambda}_i Y^i_s + p_0 G$ and Hamilton's equations are

$$
\begin{aligned}
\dot{x}^0 &= G, & \dot{p}_0 &= 0 \\
\dot{x}^i &= v^s Y^i_s, & \dot{\hat{\lambda}}_i &= -\hat{\lambda}_j v^s \frac{\partial Y^j_s}{\partial x^i} - p_0 \frac{\partial G}{\partial x^i}
\end{aligned} \tag{6}
$$

with $(\hat{\lambda}, p_0) \neq 0$.

And the condition 2(a) in Theorem 2.3 gives

$$\left\langle \hat{\lambda}, Y_s \right\rangle + p_0 \frac{\partial G}{\partial v^s} = 0, \quad s = 1, \cdots, k. \tag{7}$$

Comparing the above equations (4-7), we have:

Theorem 3.3 *Consider the optimal control problem (a) for a full-actuated nonholonomic mechanical control system Σ_D with the cost function $f : TQ \to R$, then $\gamma(t)$ is a normal (abnormal) extremal for (a) if and only if it is a normal (abnormal) extremal for the associated kinematic optimal control problem (b) with cost function $G = f$. Further, $\gamma(t)$ is a strict abnormal minimizer for (a) if and only if it is a strict abnormal minimizer for (b).*

Proof. Let $v^i = y^i, i = 1, \cdots, k$ and $\hat{\lambda} = p$, the result is obvious according to the equations (4-7) and Proposition 3.1.

The abnormal minimizers for length minimizing problem in kinematic control systems with two inputs have been discussed in [2]. With the above theorem, the results can be generalized to the case of full-actuated nonholonomic mechanical control systems naturally. Roughly speaking, for a generic full-actuated nonholonomic mechanical control system with two inputs, there always exist abnormal minimizers for the length minimizing problem.

4 Examples

In this section we give examples of nonholonomic mechanical control systems and study the abnormal minimizers for these systems.

Example 1 (Rolling Disk [10]): Consider a vertical unitary disk rolling on a horizontal plane. It can be thought as a simple model for a unicycle. The configuration space $Q = S^1 \times S^1 \times R^2$ is coordinated by (ϕ, θ, x, y) on $0 < \phi < 2\pi, 0 < \theta < 2\pi$ and the kinetic energy, associated to the Riemannian metric g on Q, is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2$$

where I is the moment of inertia about its center of mass in the direction of a line orthogonal to the disk, and J the moment of inertia about the vertical axis through the center of the disk. The nonholonomic constraints corresponding to rolling without slipping are: $\dot{x} = \dot{\theta}\cos\phi, \dot{y} = \dot{\theta}\sin\phi$. Then the constrained distribution D is generated by the orthonormal vector fields:

$$X_1 = \sqrt{\frac{2}{J}}\frac{\partial}{\partial\phi}, \quad X_2 = \sqrt{\frac{2}{m+I}}(\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta})$$

The control vector fields G_1, G_2 are related to torques on angles θ and ϕ where $i(G_1)g = d\theta, i(G_2)g = d\phi$. Then the dynamical equations are

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \sum_{j=1}^{2}\mu^j Z_j + \sum_{i=1}^{2}u^i G_i, \quad \dot{\gamma} \in D. \tag{8}$$

Since the component on $D^\perp$ is canceled by the Lagrange multipliers, we can take as the control forces their projections on D. For $\pi_D(G_i)$ generate the distribution D, then the system is a full-actuated nonholonomic mechanical control system with two inputs. The corresponding kinematic control system is given by

$$\dot{\gamma} = v^1 X_1 + v^2 X_2. \tag{9}$$

Consider the cost function $F = g(\dot{\gamma}, \dot{\gamma})$, which means minimizing the kinetic energy for the system (8). We treat the system (9). Let $X_3 = \frac{\partial}{\partial x}, X_4 = \frac{\partial}{\partial y}$, given a initial point $p_0 = (\phi_0, \theta_0, x_0, y_0)$, define the map Φ as

$$\Phi(x_1, x_2, x_3, x_4) = p_0 e^{x_4 X_4} e^{x_3 X_3} e^{x_2 X_2} e^{x_1 X_1}.$$

Since X_1, X_2, X_3, X_4 are linearly independent at p_0, Φ is well defined on a neighborhood of the origin in R^4, and maps some cube diffeomorphically onto a neighborhood U of p_0. The inverse map Φ^{-1} defines a chart such that p_0 becomes the point 0:

$$\Phi^{-1}(\phi, \theta, x, y) = (\sqrt{\frac{J}{2}}(\phi - \phi_0), \sqrt{\frac{m+I}{2}}(\theta - \theta_0),$$
$$x - x_0 - (\theta - \theta_0)\cos\phi_0, y - y_0 - (\theta - \theta_0)\sin\phi_0.$$

In this new chart,

$$X_1 = \frac{\partial}{\partial x_1},$$
$$X_2 = \frac{\partial}{\partial x_2} + \sqrt{\frac{2}{m+I}}[(-\cos\phi_0 + \cos(\sqrt{\frac{2}{J}}x_1 + \phi_0))\frac{\partial}{\partial x_3}$$
$$+(-\sin\phi_0 + \sin(\sqrt{\frac{2}{J}}x_1 + \phi_0))\frac{\partial}{\partial x_4}].$$

According to Lemma 3 in [2], when $\phi_0 \neq \pi$, the trajectory $\gamma(t) = (0, t, 0, 0)$ is an abnormal minimizer for t sufficiently small. In the original coordinate systems, we have the abnormal minimizer

$$(\phi(t), \theta(t), x(t), y(t)) = (\phi_0, \sqrt{\frac{2}{m+I}}t + \theta_0,$$
$$\sqrt{\frac{2}{m+I}}t\cos\phi_0 + x_0, \sqrt{\frac{2}{m+I}}t\sin\phi_0 + y_0) \tag{10}$$

which is also an abnormal minimizer for the nonholonomic mechanical control system (8) for t sufficiently small.

Just do a little modification to the above example, then we get the following example which has strict abnormal minimizer.

Example 2: Consider the two dimensional distribution D on R^4 generated by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1},$$
$$X_2 = (1 + x_1 x_2)\frac{\partial}{\partial x_2} + \sqrt{\frac{2}{m+I}}[(-\cos\phi_0 + \cos(\sqrt{\frac{2}{J}}x_1 + \phi_0))\frac{\partial}{\partial x_3}$$
$$+(-\sin\phi_0 + \sin(\sqrt{\frac{2}{J}}x_1 + \phi_0))\frac{\partial}{\partial x_4}].$$

Choose the Riemannian metric g such that X_1 and X_2 are the orthonormal basis for D. Suppose $\phi_0 \neq 0, \pi$. Then for the full-actuated nonholonomic mechanical control system given by the Riemannian metric and the constrained distribution D above, the trajectory $\gamma(t) = (0, t, 0, 0)$ (for t sufficiently small) is an abnormal minimizer for the cost function $F = g(\dot{\gamma}, \dot{\gamma})$ according to Theorem 3 above and Theorem 4 in [2]. If γ is a normal extremal for the mechanical system, it's also normal for the corresponding kinematic control system. Then there would have to be a solution $\lambda(t)$ of the adjoint equations such that $H = \lambda_1 v^1 + \lambda_2 v^2 + (v^1)^2 + (v^2)^2$ is minimized by $v^1 = 0, v^2 = 1$. This gives $\lambda_1 = 0, \lambda_2 = -2$. On the other hand, the adjoint equations gives:

$$\begin{aligned}
\dot{\lambda}_1 &= -\lambda_2 t + \frac{2}{\sqrt{J(m+I)}}\lambda_3 \sin\phi_0 - \frac{2}{\sqrt{J(m+I)}}\lambda_4 \cos\phi_0,\\
\dot{\lambda}_2 &= 0,\\
\dot{\lambda}_3 &= 0,\\
\dot{\lambda}_4 &= 0.
\end{aligned}$$

So we have

$$2t + \frac{2}{\sqrt{J(m+I)}}c_1 \sin\phi_0 - \frac{2}{\sqrt{J(m+I)}}c_2 \cos\phi_0 = 0.$$

This makes contradictions. That is, $\gamma(t)$ is a strict abnormal minimizer for the mechanical system above.

5 Conclusions

In this paper, abnormal minimizers for the noholonomic mechanical control systems are studied. After establishing the relations between the full-actuated nonholonomic mechanical control system and the corresponding kinematic control system, we study carefully about the Hamilton's equations given by Pontryagain's Maximum Principle for the noholonomic mechanical control systems. The properly chosen local coordinate systems make us see clearly about the structures for the Hamilton's equations, which yields Theorem 3.3, as the main result of the paper. Finally we study two examples of the noholonomic mechanical control systems which are proved to have abnormal minimizers. Future work will focus on finding abnormal minimizers for nonholonomic mechanical control systems with different cost functions, such as the force minimizing problem, for which the normal extremals have been investigated in [16].

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