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Consensus Control for Dynamic Systems on Lie Groups

DONG Runsha, GENG Zhiyong
The State Key Laboratory for Turbulence and Complex Systems,
Department of Mechanics and Aerospace Engineering, Peking University, Beijing 100871, P. R. China
E-mail: zygeng@pku.edu.cn

Abstract: This paper addresses the consensus problem of multi-agent systems with a complete communication topology consisting of second order dynamics defined on Lie groups. The specific results are illustrated on $SE(3)$ and $SE(2)$ (the special Euclidean groups of rigid body motions and each of which shares the geometric structure of a Lie group). The control algorithm proposed in the paper drive the multi-agent systems toward consensus of position as well as attitude simultaneously with specific motion patterns. Several numerical simulations are included to verify the theoretical results.

Key Words: Dynamic System, Lie Groups, Network Consensus

1 Introduction

The design and analysis of consensus control have received increasing attention in the last decade, due to the broad applications of consensus in various areas such that distributed sensor networks, vehicle formations, satellite synchronization, and so on [1–5]. The theoretical framework of consensus problems for multi-agent systems was introduced by Olfati-Saber and Murray in [6]. Further theoretical extensions based on their work generally involve two aspects, the dynamics of agents and the communication topologies of the network.

The problem of consensus control for agents with various dynamics evolving on vector space is well studied. Early seminal works [7] and [8] focus on the first and second order integrator dynamics, respectively. The works [9] and [10] have further broadened the scope of dynamics of agents and paid attention to the linear system and nonlinear system, where both of which evolve on the vector space. The recently literature [11] focuses on the rigid body defined by local coordinate in spatial frame.

It should be noted that for a vector space, the tangent space of which can be seen as the same vector space. However, in many applications, multiple agents cooperate with each other evolving on a nonlinear manifold, in which linear operation is no longer effective. So the extension of consensus control from on vector space (linear space) to on Lie groups (nonlinear manifold) is nontrivial.

Research about coordinated motion design on Lie groups is becoming more prevalent. The problem of stabilization of the relative equilibria (with Lie group structure) is formulated as a consensus one by using Lyapunov function and consensus estimator in [12]. The work [13] has shown that naive generalizations of Euclidean consensus algorithms fail to converge to the correct solution for consensus problems on SE(3). The paper [14] develops consensus algorithms for systems with simple (first order) dynamics on Lie groups. The paper [15] highlights the connection between consensus control on vector space and on nonlinear space, etc.

In this paper, the matrix Lie groups $SE(3)$ and $SE(2)$ are our major concerns, and they are the special Euclidean groups of rigid body motions in three-dimensional space and in two-dimensional space, respectively. In our recent paper[16], consensus based formation control law for rigid bodies with first order dynamics defined on Lie groups has been developed. We have noted that consensus control is one basic method for formation control. It is interesting to extend the study of consensus control from for first order to for second order dynamics on Lie groups before further research on formation control. However, it is not so easy.

Many results on consensus control algorithms are given based on the undirected graph [17] and the directed connected graph [1, 4, 6, 7], both of which have modeled the patterns of information exchange between agents. In this paper, we preliminarily consider a complete communication topology, since the complete communication topology is an extreme case contained in the directed connected communication topologies.

The rest of the present paper is organized as follows. In Section 2, the notations, some known and preliminary results on matrix Lie groups, and the basic concepts in graph theory are introduced. Section 3 establishes the problem of consensus control for multiple dynamic systems defined on Lie groups. Section 4 develops the consensus algorithm with complete communication topologies. And Section 5 shows several numerical simulations. The conclusion is drawn in the last section.

2 Background and Preliminaries

Let $\mathbb{C}^{n\times n}$ be the linear space of $n \times n$ complex matrices, and $\mathbb{R}^-$ the closed negative real axis $(-\infty, 0]$. $\mathrm{diag}(a_1, \cdots, a_n)$ denotes a diagonal matrix with diagonal elements $a_i, i = 1, \cdots, n$. The superscript T represents the transpose of a matrix. $\mathrm{tr}(\cdot)$ is the trace of a matrix. I is the identity matrix with compatible dimension. $\mathbf{0}$ is the matrix with each element being 0. $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number.

2.1 Mathematical Formulas

This paper focuses on the matrix Lie group $G \subset SE(3)$ ($SE(2)$ can be seen as a proper subgroup of $SE(3)$) whose Lie algebra is denoted by $\mathfrak{g} \subset \mathfrak{se}(3)$ (for $SE(2)$ case, $\mathfrak{g} \subset \mathfrak{se}(2)$). In the following, we only refer to a matrix Lie group $G \subset SE(3)$ and its Lie algebra $\mathfrak{g} \subset \mathfrak{se}(3)$ unless further indicated. For a comprehensive introduction in the contexts of $SE(2)$ and $SE(3)$, we refer the reader to [18, 19].

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Let $\exp : \mathfrak{g} \to G$ be the exponential map. It should be noted that 'exp' is a local diffeomorphism from the neighborhood of zero element of $\mathfrak{g}$ to the neighborhood of the identity element I of G. Hence, there exists the inverse $\exp^{-1} = \log : G \to \mathfrak{g}$. For the case of that G is a matrix group, the corresponding Lie algebra $\mathfrak{g}$ is a matrix linear space and on which a binary operation called Lie bracket is defined as $[A, B] = AB - BA$, where $[A, B] \in \mathfrak{g}$ for $A, B \in \mathfrak{g}$. For all $g \in G$ and all $A, B \in \mathfrak{g}$, the adjoint map Ad_g and the matrix commutator ad_A are defined as $\mathrm{Ad}_g(A) = gAg^{-1}, \quad \mathrm{ad}_A(B) = [A, B] = AB - BA$.

Lemma 1. *([20] Theorem 1.31) Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$. There is a unique logarithm X of A, and all of whose eigenvalues lie in the strip $\{z : -\pi < \mathrm{Im}(z) < \pi\}$. We refer to X as the principal logarithm of A and write*

$$X = \log(A).$$

Remark 1. *For example, matrix $g_s = \mathrm{diag}(-1, -1, 1) \in SE(2)$ represents the configuration at the origin in a planar reference frame with the attitude angle of π or $-\pi$ and it has two eigenvalues on $\mathbb{R}^-$. Then, we may identify one attitude angle (π or $-\pi$) for g_s. In this way, it avoids singularity which is excluded by assuming $\mathrm{tr}(g_s) \neq 1$ ([18] Lemma 2).*

Lemma 2. *([20] Theorem 11.1) For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$,*

$$\log(A) = \int_0^1 (A - I)[t(A - I) + I]^{-1} dt.$$

Corollary 1. *Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$. If $B \in \mathbb{C}^{n\times n}$ is invertible such that BAB^{-1} has no eigenvalues on $\mathbb{R}^-$, then,*

$$B(\log(A))B^{-1} = \log(BAB^{-1}).$$

Lemma 3. *([20] Theorem 11.2) For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ and $\alpha \in [-1, 1]$, we have $\log(A^\alpha) = \alpha \log(A)$. In particular, $\log(A^{-1}) = -\log(A)$ and $2\log(A^{1/2}) = \log(A)$.*

Lemma 4. *(Baker Campbell Hausdorff, [21]) For the two non-commutative operator X and Y, if Z is defined as $\exp Z = \exp X \circ \exp Y$, then Z can be rewritten as*

$$Z = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] + \tfrac{1}{12}[Y, [Y, X]] + \cdots$$

where $[\cdot, \cdot]$ is the Lie bracket.

Lemma 5. *(Differential of exponential, [18] Theorem 2) Let $g(t)$ be a smooth curve on G, $X(t) = \log(g(t))$ be the exponetial coordinates of $g(t)$, $\hat{\xi}^l = g^{-1}\dot{g}$ be the body velocity and $\hat{\xi}^r = \dot{g}g^{-1}$ be the spatial velocity. Then we can relate $\dot{X}$ and $\hat{\xi}^l$, $\hat{\xi}^r$ through:*

$$\dot{X} = \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{B}_n}{n!} \mathrm{ad}_X^n(\hat{\xi}^l), \quad \dot{X} = \sum_{n=0}^{\infty} \frac{\mathrm{B}_n}{n!} \mathrm{ad}_X^n(\hat{\xi}^r),$$

where $\{\mathrm{B}_n\}$ are the Bernoulli numbers. In the following, let $\mathcal{B}_X \triangleq \sum_{n=0}^{\infty} (\mathrm{B}_n/n!)\mathrm{ad}_X^n$ and $\mathcal{B}_{-X} \triangleq \sum_{n=0}^{\infty} ((-1)^n \mathrm{B}_n/n!)\mathrm{ad}_X^n$.

For more contexts of differential of exponential, we refer the reader to [22].

2.2 Graph Theory

In this subsection, we introduce the basic concepts in algebraic graph theory (we refer the reader to [23]) that will be used in the paper.

The communication topology of N agents defined on Lie groups is modeled as a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$, where $\mathcal{V} = \{1, 2, ..., i, ..., N\}$ is a set of N integers with the number i meaning the i-th node (vertex) which represents the i-th agent, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is an edge set in which each edge is denoted by a pair of vertices (i, j) representing agent i can get the information from agent j. For a directed graph, $(i, j) \in \mathcal{E}$ does not necessarily mean $(j, i) \in \mathcal{E}$.

A directed graph is complete if there is an edge from every node to every other node. Let $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ be the normalized adjacency matrix of $\mathcal{G}$. For the complete graph, it follows that $a_{ii} = 0$ and $a_{ij} = 1/(N - 1)$, where $i, j \in \mathcal{V}$ and $i \neq j$.

3 Problem Formulation

3.1 Dynamic Systems on Lie Groups

On $SE(3)$ and $\mathfrak{se}(3)$, we represent a group element $g = (R, p) \in SO(3) \times \mathbb{R}^3$ and a velocity $\hat{\xi}^l = (\hat{\omega}, v) \in \mathfrak{so}(3) \times \mathbb{R}^3$ using homogeneous coordinates,

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \hat{\xi}^l = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix},$$

where $SO(3)$ is the rigid rotation group ($\mathfrak{so}(3)$ is the corresponding Lie algebra), $p \in \mathbb{R}^3$ represents the position of the rigid body in the spatial frame; $\omega \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ represent the angular velocity and linear velocity of the rigid body in the body frame, respectively. The operator $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined so that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$. For the angular velocity vector $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T \in \mathbb{R}^3$ in the body frame, $\hat{\omega} \in \mathfrak{so}(3)$ is given by

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Consider the problem of consensus control for N agents labeled $1, 2, \cdots, N$ on $SE(3)$. We assume that the i-th agent in the network is full actuated (where one actuator is available for each degree of freedom in the system) and described by the following equation:

$$\begin{aligned} \dot{g}_i &= g_i \hat{\xi}_i^l, \\ \dot{\hat{\xi}}_i^l &= u_i, \end{aligned} \tag{1}$$

where $i \in \mathcal{V}$, $g_i \in SE(3)$ is the configuration of the i-th agent, $\hat{\xi}_i^l \in \mathfrak{se}(3)$ represents the velocity seen from the body frame, and u_i is the control input.

Agents are said to achieve consensus if the following equations are satisfied,

$$g_i(t) = g_j(t), \ as \ t \to \infty, \tag{2}$$

$$\hat{\xi}_i^l(t) = \hat{\xi}_j^l(t), \ as \ t \to \infty, \tag{3}$$

where $i, j \in \mathcal{V}$ and $i \neq j$. Therefore, consensus means agents will be overlapping both in the position and the attitude angle, and will finally move at the same velocity.

For the dual version of systems (1), the agents can be described by the following equation,

$$\begin{aligned}\dot{g}_i &= \hat{\xi}_i^r g_i,\\ \dot{\hat{\xi}}_i^r &= \tilde{u}_i,\end{aligned} \tag{4}$$

where $i \in \mathcal{V}$, $g_i \in SE(3)$ is the same configuration of agent i as in (1), $\tilde{u}_i$ is the control input in another form and $\hat{\xi}_i^r \in \mathfrak{se}(3)$ represents the velocity seen from the spatial frame.

In the following, we only consider the design of u_i in the networked systems (1). However, it should be noted that for agent i there exists $\mathrm{Ad}_{g_i^{-1}}\hat{\xi}_i^r = \hat{\xi}_i^l$.

3.2 Relative Dynamic Systems on Lie Algebra

For the networked systems (1), consensus is achieved when (2) and (3) are fulfilled. To solve the problem of consensus control for dynamic systems on Lie group, we pay attention to the relative dynamic systems on the corresponding Lie algebra.

We define that $g_{ij}(t) = g_i(t)^{-1}g_j(t)$, which can be interpreted as the relative configuration of agent j with respect to agent i. Let $x_i = \log(g_i)$ and $x_{ij} = \log(g_{ij})$ be the exponential coordinates of the configuration and the relative configuration, respectively.

Note that the time derivative of the relative configuration g_{ij} is as follows,

$$\dot{g}_{ij} = -g_i^{-1}\dot{g}_i g_i^{-1} g_j + g_i^{-1}\dot{g}_j = g_{ij}(\hat{\xi}_j^l - \mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}_i^l). \tag{5}$$

Let $\hat{\xi}_{ij}^l \triangleq \hat{\xi}_j^l - \mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}_i^l$. Then,

$$\dot{g}_{ij} = g_{ij}\hat{\xi}_{ij}^l. \tag{6}$$

where $\hat{\xi}_{ij}^l$ can be interpreted as the relative velocity of agent j with respect to agent i.

Note that $x_{ij} = \log(g_{ij}) \in \mathfrak{g}$. From Lemma 5, we have

$$\dot{x}_{ij} = \mathcal{B}_{-x_{ij}}\hat{\xi}_{ij}^l. \tag{7}$$

Before moving on further, we introduce some properties of the exponential coordinate of the relative configuration x_{ij} and the relative velocity $\hat{\xi}_{ij}^l$.

Properties 1. *For i, j, k represent the i-th, j-th, k-th agent in the networked systems ($i, j, k \in \mathcal{V}$), the exponential coordinate of the relative configuration has the following properties.*

(A) From Lemma 3, it follows that

$$x_{ji} = -x_{ij}.$$

(B) From Corollary 1 of Lemma 2, it follows that

$$\mathrm{Ad}_{g_{ij}^{-1}}x_{ij} = g_{ij}^{-1}(\log g_{ij})g_{ij} = x_{ij}.$$

(C) From Lemma 4, it follows that

$$\mathrm{Ad}_{g_{ij}^{-1}}x_{ki} = x_{ji} + x_{kj} + \text{higher order terms}, k \neq i.$$

It should be noted that the "higher order terms" in (C) of Properties 1 is composed of Lie brackets (with respect to the exponential coordinate of the relative configuration). In the following, we denote these "higher order terms" of relative configuration by $\mathfrak{h}-\mathfrak{o}-\mathfrak{t}$.

Properties 2. *For i, j, k represent the i-th, j-th, k-th agent in the networked systems ($i, j, k \in \mathcal{V}$), the relative velocity has the following properties.*

(D) From Corollary 1 of Lemma 2, it follows that

$$\hat{\xi}_{ij}^l = -\mathrm{Ad}_{g_{ji}}\hat{\xi}_{ji}^l, i \neq j \text{ and } \hat{\xi}_{ij}^l = 0, i = j,$$

(E)

$$\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}_{ij}^r = \hat{\xi}_{ij}^l.$$

(F)

$$\frac{d}{dt}\left(\mathrm{Ad}_{g_{ji}}\right)\hat{\xi}_k^l = \mathrm{Ad}_{g_{ji}}[\hat{\xi}_{ji}^l, \hat{\xi}_k^l].$$

(G)

$$\mathrm{Ad}_{g_{ik}}[\hat{\xi}_k^l, \hat{\xi}_{ik}^l] = -[\hat{\xi}_i^l, \hat{\xi}_{ki}^l].$$

(H) From (G) of Properties 2, one also has

$$\mathrm{Ad}_{g_{ij}^{-1}}[\hat{\xi}_i^l, \hat{\xi}_{ki}^l] = -\mathrm{Ad}_{g_{kj}^{-1}}[\hat{\xi}_k^l, \hat{\xi}_{ik}^l].$$

(I)

$$\hat{\xi}_{ik}^r = -\hat{\xi}_{ki}^l \text{ and } \mathrm{Ad}_{g_{ij}}\hat{\xi}_{ij}^l = -\hat{\xi}_{ji}^l.$$

(J) From (I) of Properties 2, one also has

$$\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}_{ik}^r = \hat{\xi}_{ij}^l - \hat{\xi}_{kj}^l.$$

Based on Properties 1 and 2, and in addition from Lemma 5 and the definition of $\hat{\xi}_{ij}^l$, one has the following relative dynamic systems,

$$\begin{aligned}\dot{x}_{ij} &= \mathcal{B}_{-x_{ij}}\hat{\xi}_{ij}^l,\\ \dot{\hat{\xi}}_{ij}^l &= u_j - \mathrm{Ad}_{g_{ij}^{-1}}u_i - [\hat{\xi}_j^l, \hat{\xi}_{ij}^l],\end{aligned} \tag{8}$$

where $i, j \in \mathcal{V}$ and $i \neq j$. In the following, when we refer to consensus is achieved we mean that the state of (8) satisfy $\lim_{t\to\infty} g_{ij}(t) = I$, which will be equivalent to

$$\lim_{t\to\infty} x_{ij}(t) = \mathbf{0} \text{ and } \lim_{t\to\infty}\hat{\xi}_{ij}^l(t) = \mathbf{0}.$$

For stability analysis of the relative dynamic system (8) on Lie groups, the following lemma will play an important role.

Lemma 6. *([18] Theorem 6) For the second order fully actuated systems on $SE(3)$, let K_p and K_d be the positive-definite gains. Then, the control law*

$$u(g, \hat{\xi}^l) = -K_p\log(g) - K_d\hat{\xi}^l \tag{9}$$

locally exponentially stabilizes the state g at $I \in SE(3)$.

Furthermore, if scalar gains are employed ($K_p = k_pI_6$ and $K_d = k_dI_6$), then the control law in (9) exponentially stabilizes the state g at I from any initial condition $g(0) = (R(0), p(0))$ with $tr(R(0)) \neq -1$ and for all k_p and $\omega(0)$ (the initial angular velocity) such that

$$k_p > \frac{\|\omega(0)\|^2}{\pi^2 - \|R(0)\|_{SO(3)}^2}, \tag{10}$$

where $\|R\|_{SO(3)}$ is the distance between the element $R \in SO(3)$ and the identity $e_{SO(3)} = I \in SO(3)$ and given by the norm of the logarithmic function,

$$\|R\|_{SO(3)} = \langle\log(R), \log(R)\rangle^{1/2}.$$

Remark 2. *The relative equilibrium ([24], representing the time derivative of velocity equals to 0 for a single agent) of systems (8) on $SE(3)$ represents that all agents in the network are overlapping (after they have achieved consensus) and moving in uniform motion finally with the following patterns,*

- *in a straight line: moving in the same direction;*
- *in circular: drawing the same circle with the same radius in plane;*
- *in helical: drawing the same circular helix with the same radius, pitch, axis and axial direction of motion.*

The special case is that both agents are asymptotically static (staying at the same configuration and standing still).

4 Consensus Control on $SE(3)$

For the networked systems (1), we propose the consensus algorithm for agent i,

$$u_i = \sum_{k=1}^{N} a_{ik}\left(c_1 x_{ik} + c_2 \hat{\xi}^r_{ik} + [\hat{\xi}^l_i, \hat{\xi}^l_{ki}]\right), \tag{11}$$

where a_{ik} is the element of the normalized adjacent matrix, $c_1 > 0$ and $c_2 > 0$ are constant control gains. Then, we have the following main result.

Theorem 1. *If the communication topology between N agents is complete, then agents with dynamics (1) achieve consensus asymptotically under the control law (11) from any initial value with $g_{ij}(0) = (R_{ij}(0), p_{ij}(0)) \in SO(3) \times \mathbb{R}^3$ and $\mathrm{tr}(R_{ij}(0)) \neq -1$ for all c_1 and $\omega_{ij}(0)$ such that*

$$c_1 > \frac{N-1}{N} \cdot \frac{\|\omega_{ij}(0)\|^2}{\left(\pi^2 - \|R_{ij}(0)\|^2_{SO(3)}\right)},$$

where $\omega_{ij}(0)$ is defined by $\dot{g}_{ij}(0) = g_{ij}(0)\hat{\xi}^l_{ij}(0)$, $\hat{\xi}^l_{ij}(0) = (\hat{\omega}_{ij}(0), v_{ij}(0))$ and $(\hat{\cdot})^{-1} : \mathfrak{so}(3) \to \mathbb{R}^3$.

Proof. Based on the analysis above, we begin the proof from the relative dynamic systems (8). From equation (11), one has the control algorithm for agent j given by

$$u_j = \sum_{k=1}^{N} a_{jk}\left(c_1 x_{jk} + c_2 \hat{\xi}^r_{jk} + [\hat{\xi}^l_j, \hat{\xi}^l_{kj}]\right). \tag{12}$$

Substitute the control laws (11) and (12) into (8). It follows that

$$\begin{aligned}
\dot{\hat{\xi}}^l_{ij} &= \sum_{k=1}^{N} a_{jk}\left(c_1 x_{jk} + c_2 \hat{\xi}^r_{jk} + [\hat{\xi}^l_j, \hat{\xi}^l_{kj}]\right) \\
&\quad - \mathrm{Ad}_{g_{ij}^{-1}}\left(\sum_{k=1}^{N} a_{ik}\left(c_1 x_{ik} + c_2 \hat{\xi}^r_{ik} + [\hat{\xi}^l_i, \hat{\xi}^l_{ki}]\right)\right) - [\hat{\xi}^l_j, \hat{\xi}^l_{ij}] \\
&= c_1 \sum_{k=1}^{N}\left(a_{jk} x_{jk} - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}} x_{ik}\right) \\
&\quad + c_2 \sum_{k=1}^{N}\left(a_{jk}\hat{\xi}^r_{jk} - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^r_{ik}\right) \\
&\quad + \left(\sum_{k=1}^{N}\left(a_{jk}[\hat{\xi}^l_j, \hat{\xi}^l_{kj}] - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}}[\hat{\xi}^l_i, \hat{\xi}^l_{ki}]\right) - [\hat{\xi}^l_j, \hat{\xi}^l_{ij}]\right).
\end{aligned} \tag{13}$$

Let $\mathrm{T}_1 = \sum_{k=1}^{N}\left(a_{jk} x_{jk} - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}} x_{ik}\right)$,

$$\begin{aligned}
\mathrm{T}_2 &= \sum_{k=1}^{N}\left(a_{jk}\hat{\xi}^r_{jk} - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^r_{ik}\right), \\
\mathrm{T}_3 &= \sum_{k=1}^{N}\left(a_{jk}[\hat{\xi}^l_j, \mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^l_{ki}] - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}}[\hat{\xi}^l_i, \hat{\xi}^l_{ki}]\right).
\end{aligned}$$

Then, we consider $\mathrm{T}_1, \mathrm{T}_2$ and T_3 in (13) respectively.

(T_1) From (C) of properties 1, it follows that

$$\mathrm{T}_1 = \sum_{k=1}^{N}(a_{jk} x_{jk} + a_{ik} x_{ji} + a_{ik} x_{kj}) + \mathfrak{h} - \mathfrak{o} - \mathfrak{t}.$$

From (A) of properties 1 and $\sum_{j=1}^{N} a_{ij} = 1, i, j \in \mathcal{V}$, one has

$$\mathrm{T}_1 = -(1 + a_{ji})x_{ij} + \sum_{\substack{k=1,\\ k\neq i}}^{N}(a_{jk} - a_{ik})x_{jk} + \mathfrak{h} - \mathfrak{o} - \mathfrak{t}. \tag{14}$$

(T_2) From equation (J) of properties 2, we have

$$\begin{aligned}
\mathrm{T}_2 &= \sum_{k=1}^{N}\left(a_{jk}\hat{\xi}^r_{jk} - a_{ik}\hat{\xi}^l_{ij} + a_{ik}\hat{\xi}^l_{kj}\right) \\
&= -\hat{\xi}^l_{ij} + a_{ji}\hat{\xi}^r_{ji} + \sum_{\substack{k=1,\\ k\neq i,j}}^{N}(a_{jk}\hat{\xi}^r_{jk} + a_{ik}\hat{\xi}^l_{kj}).
\end{aligned}$$

From equation (I) of properties 2, we can get

$$\mathrm{T}_2 = -(1 + a_{ji})\hat{\xi}^l_{ij} + \sum_{\substack{k=1,\\ k\neq i,j}}^{N}(a_{ik} - a_{jk})\hat{\xi}^l_{kj}. \tag{15}$$

(T_3) From equation (H), (J) of properties 2, it follows that

$$\mathrm{T}_3 = \sum_{k=1}^{N}(a_{jk}[\hat{\xi}^l_j, \mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^l_{ki}] - a_{ik}\mathrm{Ad}_{g_{ij}^{-1}}[\hat{\xi}^l_i, \hat{\xi}^l_{ki}]).$$

From $\mathrm{Ad}_g[\hat{\xi}^l_i, \hat{\xi}^l_j] = [\mathrm{Ad}_g\hat{\xi}^l_i, \mathrm{Ad}_g\hat{\xi}^l_j]$, we have

$$\mathrm{T}_3 = \mathrm{Ad}_{g_{ij}^{-1}}\sum_{k=1}^{N}([a_{jk}\mathrm{Ad}_{g_{ij}}\hat{\xi}^l_j - a_{ik}\hat{\xi}^l_i, \hat{\xi}^l_{ki}]). \tag{16}$$

Note that $a_{jk} = a_{ik} = \frac{1}{N-1}$ for a complete topology graph. From equation (A), (B) of properties 1 and (D), (E), (F) and (H) of properties 2, equation (13) (the summation of equations (14), (15) and (16) with constant gain) will be

$$\begin{aligned}
\dot{\hat{\xi}}^l_{ij} &= -c_1(1 + \tfrac{1}{N-1})x_{ij} + \mathfrak{h} - \mathfrak{o} - \mathfrak{t} \\
&\quad - c_2(1 + \tfrac{1}{N-1})\hat{\xi}^l_{ij} - \tfrac{1}{N-1}(\sum_{k=1}^{N}[\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^l_{ki}, \hat{\xi}^l_{ij}]).
\end{aligned} \tag{17}$$

Note that "$\mathfrak{h} - \mathfrak{o} - \mathfrak{t}$" and $\sum_{k=1}^{N}[\mathrm{Ad}_{g_{ij}^{-1}}\hat{\xi}^l_{ki}, \hat{\xi}^l_{ij}]$) in (17) are the higher order terms of the "exponential coordinates of the relative configuration" and the "relative velocity" accordingly. Since the impact of the higher order terms is less than the linear term, and the higher order terms will be asymptotically decreasing when agents are approaching consensus, it is reasonable to omit the higher order terms.

Approximatively, the relative dynamical systems (17) can be expressed by

$$\dot{\hat{\xi}}^{l}_{ij} = -c_1(1+\frac{1}{N-1})x_{ij} - c_2(1+\frac{1}{N-1})\hat{\xi}^{l}_{ij}. \quad (18)$$

Let $k_p = c_1(1+1/(N-1))$, $k_d = c_2(1+1/(N-1))$. Then, based on the control algorithm (11) and from lemma 6, it is easy to get that the control law (11) exponentially stabilizes the state x_{ij} and $\hat{\xi}^{l}_{ij}$ at $\mathbf{0}$ from any initial condition with $g_{ij}(0) = (R_{ij}(0), p_{ij}(0) \in SO(3) \times \mathbb{R}^3$ and $\mathrm{tr}(R_{ij}(0)) \neq -1$, for all c_1 and $\omega_{ij}(0)$ such that

$$c_1 > \frac{(N-1)}{N} \cdot \frac{\|\omega_{ij}(0)\|^2}{\left(\pi^2 - \|R_{ij}(0)\|^2_{SO(3)}\right)}, \quad (19)$$

where $i, j \in \mathcal{V}$. And equivalently, consensus is achieved asymptotically.

□

Remark 3. *For two agents ($N = 2$) with bidirectional communication, the higher order terms of the "exponential coordinates of the relative configuration" and the "relative velocity" in (17) do not exist. And then, the control algorithm (11) can accurately derive the relative dynamical system (18).*

5 Simulations

Without loss of generality and for simplicity, two numerical simulation examples are shown for 3 agents labeled 1,2,3 with dynamics described by (1) on $SE(2)$ in this section. Note that here group element $g_i = (R_i, p_i) \in SE(2)$, $\hat{\xi}^{l}_{i} = (\hat{\omega}, v) \in \mathfrak{se}(2)$ and

$$R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}, p_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix},$$
$$\hat{\omega}_i = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}, v_i = \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix}.$$

The communication topology is complete and shown in Fig.1. Note that these examples illustrate the relative equilibrium on $SE(2)$ (in uniform motions in circular and in a straight line) correspondingly.

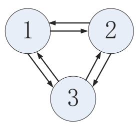

Fig. 1: The complete communication topology.

Example 1. The initial values of these agents are given in Table 1. where x_0, y_0 are the initial position with respect to horizontal coordinate x, vertical coordinate y respectively; θ_0 is the initial attitude angle;v_{x0}, v_{y0} are the initial velocity with respect to x, y coordinate ; and ω_0 is the initial angular velocity.

Then, with the control algorithm (11) (here let $c_1 = c_2 = 5$), the agents asymptotically achieve consensus and move in

Table 1: The initial values (Example 1)

node	x_0	y_0	θ_0	v_{x0}	v_{y0}	ω_0
1	0	10	$\pi/2$	0	0	0
2	30	43	0	0	0	0
3	30	0	$\pi/2$	10	0	-0.8

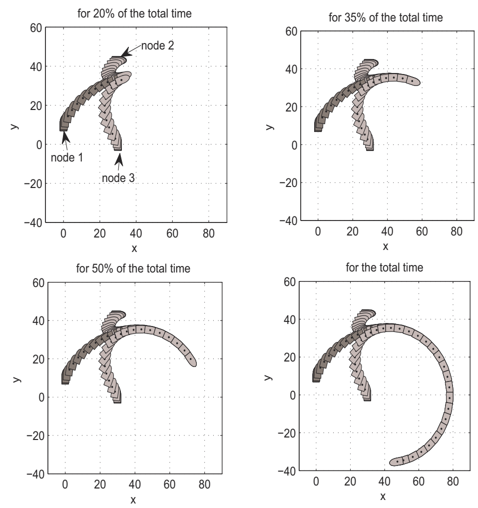

Fig. 2: Moving in circle uniformly after achieving consensus. The 4 subgraphs in sequence show agents' trajectories for 20%, 35%, 50% and total of the simulation time, respectively.

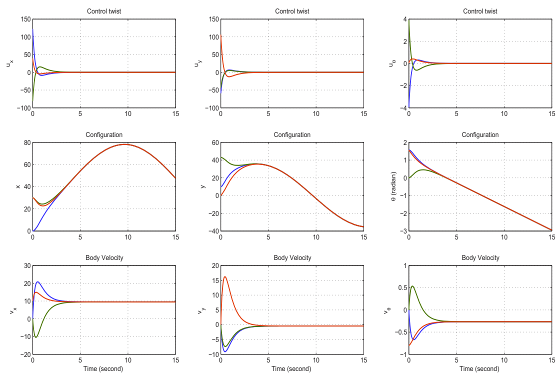

Fig. 3: States of agents (Example 1). From top to bottom: agents' control input, configuration and body velocity; from left to right: with respect to the x, y coordinate in a planar reference frame, and the attitude θ.

circle uniformly in plane, see Fig.2. The state values of these agents are shown in Fig.3.

Example 2. The initial values of three agents are given as in Table 2. Then, with the control algorithm (11) (here let

Table 2: The initial values (Example 2)

node	x_0	y_0	θ_0	v_{x0}	v_{y0}	ω_0
1	10	0	0	15	0	0
2	-15	10	$-\pi/4$	0	0	0
3	-20	-10	$\pi/4$	0	0	0

$c_1 = c_2 = 5$), the agents asymptotically achieve consensus and move in a straight line uniformly, see Fig.4. The state values of these agents are shown in Fig.5.

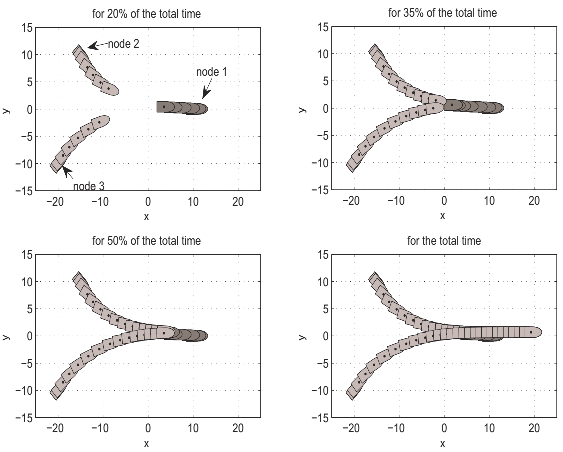

Fig. 4: Moving in a straight line uniformly after achieving consensus. The 4 subgraphs in sequence show agents' trajectories for 20%, 35%, 50% and total of the simulation time, respectively.

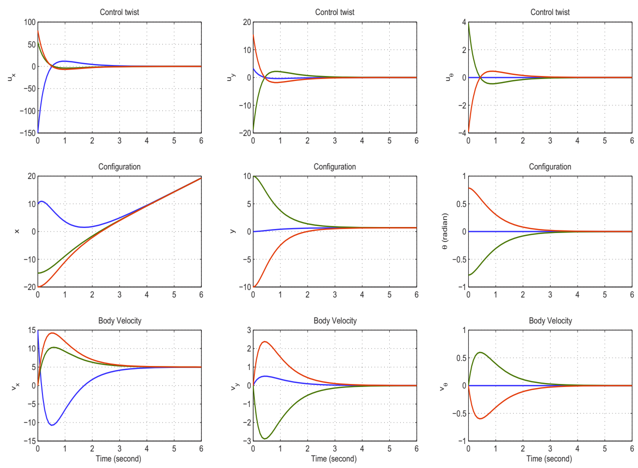

Fig. 5: States of agents (Example 2). From top to bottom: agents' control input, configuration and body velocity; from left to right: with respect to the x, y coordinate in a planar reference frame, and the attitude θ.

6 Conclusions

In this paper, we study the problem of consensus control for dynamical (second order) systems on Lie groups. The consensus algorithm is given for agents defined on $SE(3)$ and $SE(2)$ with the communication topology being complete. The numerical simulations are shown for consensus control on $SE(2)$. In the future, we plan to discuss multi-form communication patterns which leads to various behaviors more than consensus.

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