Finite Time Optimal Formation Control for Multiple Vehicles

Zhiyong, Geng
The State Key Laboratory for Turbulence and Complex Systems,
Department of Mechanics and Engineering Science, Peking University, Beijing 100871
E-mail: zygeng@pku.edu.cn

Abstract: The paper studies the problem of finite time formation control for multiple vehicles. The vehicle is modeled as the full actuated rigid body, both the formation time and the geometric structure of the formation are specified by the task. An open optimal control law is derived by using Pontryagins minimum principle, and is converted to the closed form by feeding the current state back under the assumption that the mode of communication between the vehicles is all-to-all. For the demonstration of the result, a numerical example of formation for planar vehicles is included.

Key Words: Formation Control, Consensus, Multiple Vehicles, Motion Planning

1 INTRODUCTION

The formation control for multiple vehicles is a cooperative control of a networked system with its nodes to be vehicles under some given communication topology. The researches on this mostly consider the dynamics of the vehicles evolving on Euclidean space, such as integrators , linear systems , Euler-Lagrange (EL) systems . However, when the vehicle is considered as a rigid body, its configuration space is a Euclidean group SE(3), which is a nonlinear manifold.

There are research works studying the problem under the framework of Lie group, for example, [1, 2] considered the formation of vehicles under the setting of Lie group, in which, the kinematics of the vehicle is molded by Frenet-Serret equations of motion, and presented the control law that leads to vehicle swarm. In order to keep the formation while moving in space, some researchers [3] studied the problem of coordinated motion on Lie groups by converting the problem of coordination to that of consensus on Lie algebra, and proposed the control law for the coordination. But they do not deal with the problem of achieving a specified formation which is quite important for some real applications. Recently, we studied the problem for multiple vehicles to achieve an arbitrary specified formation under the framework of Euclidean group, and proposed the asymptotic formation control laws for both kinematic model [4] and dynamic model [5]. We also considered the problem of finite time optimal formation control of multiple vehicles under the framework of Euclidean group, in which both the formation time and the geometric structure of the formation are specified by the task, and we proposed an optimal formation control law for the the kinematic model[6].

This paper is the continuation of [6], but the dynamic model of the vehicle is used, which is quite different from the kinematic model. In this paper, the formation time and the geometric structure of the formation are also arbitrarily specified by the task. We proposed a finite time optimal formation control law under the assumptions that the vehicles are full actuated with zero initial velocities and that the communications between the vehicles are all-to-all.

The paper is organized as follows. In section two, we introduce some notions and preliminary results which will be used. In section three, we focus mainly on the formulation of the problem. Section four is devoted to develop the main results of the paper. Section five includes an example and its numerical simulation results.

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2 PRELIMINARIES

In this section, we introduce some notions and preliminary results which will be used in the paper.

We use the Euclidian group

$$\text{SE}(N) = \left\{ \begin{bmatrix} R & p \\ 0_{1\times 3} & 1 \end{bmatrix} : R \in \text{SO(N)}, p \in \mathbb{R}^N \right\}, N = 2, 3,$$

to denote the configuration manifold of a rigid body. Where SO(N) is a $N\times N$ orthogonal matrix group with its element $R \in$ SO(N) satisfies $\det R = 1$. Each column of R is the base vector of the coordinate system fixed on the rigid body, thus R can be used to represent the attitude of the body, and $p \in \mathbb{R}^N$ is the position vector of the body.

Let TSE(N) denote the tangent bundle of SE(N) and T_gSE(N) be the tangent space of SE(N) at $g \in$SE(N). For the special case when g is the identity I, the tangent space T_ISE(N) that is denoted by $\mathfrak{se}(N)$ has the following structure,

$$\mathfrak{se}(N) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0_{1\times N} & 0 \end{bmatrix} : \hat{\omega} \in \mathfrak{so}(N), v \in \mathbb{R}^N \right\},$$

where $\mathfrak{so}(N) \subset \mathbb{R}^{N\times N}$ is the set of skew-symmetric matrices that represent the rotation velocities of the body, and $v \in \mathbb{R}^N$ is a real vector that represents the transition velocity of the body. It is obvious that $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$, and the outer product defined on $\mathbb{R}^3$, e.g., $\omega_1 \times \omega_2$ can be written as $\hat{\omega}_1\omega_2$ for $\omega_1, \omega_2 \in \mathbb{R}^3$, and $\hat{\omega}_1 \in \mathfrak{so}(3)$ is the isomorphic image of $\omega_1 \in \mathbb{R}^3$. It is obvious that $\mathfrak{se}(3)$, as a linear space, is isomorphic to $\mathbb{R}^6$. For this fact, we define an isomorphic map $\wedge : \mathbb{R}^6 \to \mathfrak{se}(3)$, such that

$$\wedge : \eta = \begin{bmatrix} \omega \\ v \end{bmatrix} \to \begin{bmatrix} \hat{\omega} & v \\ 0_{1\times 3} & 0 \end{bmatrix} = \hat{\eta},\ \omega, v \in \mathbb{R}^3$$

which is denoted by $\wedge(\eta) = \hat{\eta}$. The inverse of the map $\wedge$ is denote by $\vee : \mathfrak{se}(3) \to \mathbb{R}^6$. For $\mathfrak{se}(2)$, it is isomorphic to $\mathbb{R}^3$, the similar isomorphic map and its inverse can be defined, it is unnecessary to go into details. The meaning of the notion " $\hat{}$ " used for vectors in $\mathfrak{so}(3)$, $\mathfrak{se}(3)$ and $\mathfrak{se}(2)$ can be distinguished from the context.

Once defining Lie bracket on $\mathfrak{se}(N)$ by $[\hat{x}, \hat{y}] \triangleq \hat{x}\hat{y} - \hat{y}\hat{x},\ \hat{x}, \hat{y} \in \mathfrak{se}(N)$, $\mathfrak{se}(N)$ is a Lie algebra corresponding to SE(N) . For a given $\hat{x} \in \mathfrak{se}(N)$, it defines a linear map $\mathrm{ad}_{\hat{x}} : \mathfrak{se}(N) \to \mathfrak{se}(N)$, such that $\mathrm{ad}_{\hat{x}}(\hat{y}) = [\hat{x}, \hat{y}], \hat{y} \in \mathfrak{se}(N)$. In the following, we shall only discuss for the case of $N = 3$, but the results will also hold for the case of $N = 2$. For the Lie bracket defined on $\mathfrak{se}(3)$, we have the following lemma.

Lemma 1. *Let $\hat{\eta}, \hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{se}(3)$, if $[\hat{\eta}, \hat{\xi}_i] = 0, i = 1, 2$, then $[\hat{\xi}_1, \hat{\xi}_2] = 0$.*

Proof. From the definition it is easy and is Omitted. □

The dual of TSE(3), T_gSE(3), and $\mathfrak{se}(3)$ will be denoted by T^*SE(3), T_g^*SE(3) and $\mathfrak{se}^*(3)$ respectively.

For each $\hat{x} \in \mathfrak{se}(3)$, it defines a left invariant vector field $\hat{x}_L : \mathrm{SE}(3) \to T\mathrm{SE}(3)$, such that $\hat{x}_L(q) = q\hat{x} \in T_q\mathrm{SE}(3)$, for $q \in \mathrm{SE}(3)$. We shall denote $\hat{x}_L(q)$ by $\hat{x}_q$ in the sequel. From the definition of the left invariant vector field, for any given $q \in \mathrm{SE}(3)$, it defines a left action map denoted simply by $q : \mathfrak{se}(3) \to T_q\mathrm{SE}(3)$ with abuse of notion.

For linear spaces T_qSE(3) and $\mathfrak{se}(3)$, defining inner products on the spaces,

$$G_q(\hat{x}_q, \hat{y}_q) \triangleq G_I(\hat{x}, \hat{y}) \triangleq x^T y, \tag{1}$$

then T_qSE(3) and $\mathfrak{se}(3)$ are inner product spaces, and SE(3) will be a Riemannian manifold with a Riemannian metric induced by the inner product.

Definition 1. *Let V be linear space, $B : V \times V \to \mathbb{R}$ be a bilinear map. The linear map $B^\flat : V \to V^*$ is called the flat map of B, if $B^\flat(u), u \in V$ satisfies*

$$(B^\flat(u))(v) = B(v, u), \forall v \in V,$$

and denote the image of the flat map $B^\flat(u)$ by $u^* \in V^*$. And if the flat map $B^\flat$ is invertible, then the inverse is denoted by $B^\sharp : V^* \to V$ which is called the sharp map of B, such that

$$B(v, B^\sharp(u^*)) = u^*(v), \forall v \in V.$$

By this definition and the above notions, we denote $\hat{x}_q^* = G_q^\flat(\hat{x}_q)$ and $\hat{x}^* = G_I^\flat(\hat{x})$, and it is ready to show that $\hat{x}_q^* = (q^{-1})^*(\hat{x}^*),\ \hat{x}^* = \vee^*(x)$, and $\hat{x}_q^*(\hat{y}_q) = \hat{x}^*(\hat{y}) = x^T y$.

By using the inner product of matrices, we have

$$\hat{x}^*(\hat{y}) = \left\langle [\hat{x}^*]^T, \hat{y} \right\rangle_{\mathbb{R}^{4\times 4}} = \mathrm{tr}(\hat{x}^*\hat{y}), \tag{2}$$

and

$$\hat{x}_q^*(\hat{y_q}) = \left\langle [\hat{x}_q^*]^T, \hat{y}_q \right\rangle_{\mathbb{R}^{4\times 4}} = \mathrm{tr}(\hat{x}_q^*\hat{y}_q), \tag{3}$$

where $\hat{x}^* = \mathrm{diag}\left(\frac{1}{2}I_3, 1\right)\hat{x}^T$, $\hat{x}_q^* = \mathrm{diag}\left(\frac{1}{2}I_3, 1\right)\hat{x}^T q^{-1}$.

Let $\{e_i\}$, $\{\hat{e}_i\}$, and $\{\hat{e}_{i,q}\}$ denote the orthonormal basis of $\mathbb{R}^6$, $\mathfrak{se}(3)$ and T_qSE(3) respectively, which satisfy $\hat{e}_{i,q} = q(\hat{e}_i) = q(\wedge(e_i)), i = 1, \cdots, 6$. And the dual basis of $(\mathbb{R}^6)^*$, $\mathfrak{se}^*(3)$ and T_q^*SE(3) will be $e_i^T = G_{\mathbb{R}^6}^\flat(e_i),\ \hat{e}_i^* = G_I^\flat(\hat{e}_i),\ \hat{e}_{i,q}^* = G_q^\flat(\hat{e}_{i,q})$.

Definition 2. *Let V be a finite dimensional linear space with a basis $\{\varepsilon_i\}$, $f \in C^1(V, \mathbb{R})$ be a function defined on V, then the derivative of f with respect $x = \sum_i \varepsilon_i x^i \in V$ is defined by*

$$\frac{\partial f}{\partial x} = \sum_i \varepsilon_i \frac{\partial f}{\partial x^i} \in V.$$

For a given $q \in \mathrm{SE}(3)$, an adjoint map $\mathrm{Ad}_q : \mathfrak{se}(3) \to \mathfrak{se}(3)$ is defined as $\mathrm{Ad}_q(\hat{x}) \triangleq q\hat{x}q^{-1}$, for $\hat{x} \in \mathfrak{se}(3)$, and $\mathrm{Ad}_q(\hat{x})$ is denote by $\hat{x}^s$. Although both $\hat{x}$ and $\mathrm{Ad}_q(\hat{x})$ are in $\mathfrak{se}(3)$, we shall still distinguish the image space $\mathrm{Ad}_q(\mathfrak{se}(3))$ of the adjoint map from its domain $\mathfrak{se}(3)$. By the similar way, we can define inner product on $\mathrm{Ad}_q(\mathfrak{se}(3))$,

$$G_{\mathrm{Ad}_q}(\hat{x}^s, \hat{y}^s) \triangleq x^T y,\ \hat{x}^s, \hat{y}^s \in \mathrm{Ad}_q(\mathfrak{se}(3)),$$

and denote the image of $\hat{x}^s$ under the flat map of G_{Ad_q} by $(\hat{x}^s)^* = G_{\mathrm{Ad}_q}^\flat(\hat{x}^s)$, such that $(\hat{x}^s)^*(\hat{y}^s) = G_{\mathrm{Ad}_q}(\hat{x}^s, \hat{y}^s),\ \forall \hat{y}^s \in \mathrm{Ad}_q(\mathfrak{se}(3))$. Since $(\hat{x}^s)^*(\hat{y}^s) = \hat{x}^*(\hat{y}) = \hat{x}^*(\mathrm{Ad}_{q^{-1}}((\hat{y}^s)) = \mathrm{Ad}_{q^{-1}}^*\hat{x}^*(\hat{y}^s)$ holds for all $\hat{y}^s \in \mathrm{Ad}_q(\mathfrak{se}(3))$, this leads to $G_{\mathrm{Ad}_q}^\flat(\mathrm{Ad}_q\hat{x}) = (\hat{x}^s)^* = \mathrm{Ad}_{q^{-1}}^*\hat{x}^*$. And it is easy to show that $\mathrm{Ad}_{q^{-1}}^*\hat{x}^* = q\hat{x}^*q^{-1}$.

Let $\exp : \mathfrak{se}(N) \longrightarrow \mathrm{SE}(N)$ be exponential map. We can define its inverse for the element $g \in \mathrm{SE}(N)$ that has no negative real eigenvalues, and denote it by $\log(g)$ (see [8]).

3 PROBLEM FORMULATION

Consider n vehicles with the dynamics given by

$$\begin{cases} \dot{g}_i = g_i\hat{\xi}_i^b, & g_i(t_0) = g_i^0 \\ \dot{\hat{\xi}}_i^b = \hat{u}_i, & \hat{\xi}_i^b(t_0) = \hat{\xi}_i^{b,0} \end{cases},\ i = 1, \cdots, n, \tag{4}$$

where $g_i \in$SE(3) is the configuration of the i-th vehicle, $\hat{\xi}_i^b \in \mathfrak{se}(3)$ is the i-th velocity seen form the body fixed corroborate, and $\hat{u}_i \in \mathfrak{se}(3)$ is the i-th control input. The kinematics $\dot{g}_i = g_i\hat{\xi}_i^b$ can also be written as $\dot{g}_i = \hat{\xi}_i^s g_i$, where $\hat{\xi}_i^s = \mathrm{Ad}_g\hat{\xi}_i^b$ is the i-th velocity seen from spacial coordinate.

Now the problem is for given relative configurations $\bar{g}_{ij} \triangleq g_i^{-1}g_j, i, j = 1, \cdots, n$, determined by the formation task, and initial states $(g_i^0, \hat{\xi}_i^{b,0}), i = 1, \cdots, n$, to find the control inputs $\hat{u}_i, i = 1, \cdots, n$, such that

$$\begin{gathered} g_i^{-1}(t_f)g_j(t_f) = \bar{g}_{ij},\ \hat{\xi}_j^b(t_f) - \mathrm{Ad}_{\bar{g}_{ji}}\hat{\xi}_i^b(t_f) = 0, \\ i, j = 1, \cdots, n, \end{gathered} \tag{5}$$

where $t_f > t_0$ is the final time which is given, meanwhile, to minimize the following cost function,

$$J(\hat{u}_1, \cdots, \hat{u}_n) = \int_{t_0}^{t_f} \frac{1}{2}\sum_{k=1}^n G_I(\hat{u}_k, \hat{u}_k)dt. \tag{6}$$

For $\bar{g}_{ij} = I, i, j = 1, \cdots, n$, this implies that $\hat{\xi}_j^b(t_f) = \hat{\xi}_i^b(t_f), i, j = 1, \cdots, n$, the corresponding problem is consensus control. For the relation between the problems of formation and consensus, we have the following lemma.

Lemma 2. *Let $\tilde{g}_i = g_i\bar{g}_{i1}$, $\tilde{\xi}_i^b = Ad_{\bar{g}_{1i}}\hat{\xi}_i^b$, and $\tilde{u}_i = Ad_{\bar{g}_{1i}}\hat{u}_i$, for $i = 1, \cdots, n$, then the systems (4) achieve formation under the control $\hat{u}_i, i = 1, \cdots, n$, if and only the following systems achieve consensus under the control $\tilde{u}_i, i = 1, \cdots, n$.*

$$\begin{cases} \dot{\tilde{g}}_i = \tilde{g}_i\tilde{\xi}_i^b, & \tilde{g}_i(t_0) = \tilde{g}_i^0 \\ \dot{\tilde{\xi}}_i^b = \tilde{u}_i, & \tilde{\xi}_i^b(t_0) = \tilde{\xi}_i^{b,0} \end{cases}, \ i = 1, \cdots, n. \tag{7}$$

Proof. By noting that $\dot{\tilde{g}}_i = \tilde{g}_i\mathrm{Ad}_{\bar{g}_{1i}}\hat{\xi}_i^b$, it follows that $\dot{\tilde{g}}_i = \tilde{g}_i\tilde{\xi}_i^b$ and $\dot{\tilde{\xi}}_i^b = \mathrm{Ad}_{\bar{g}_{1i}}\hat{u}_i = \tilde{u}_i$. Besides, by utilizing $\bar{g}_{ij} = \bar{g}_{i1}\bar{g}_{1j}$, $g_i^{-1}(t_f)g_j(t_f) = \bar{g}_{ij} = \bar{g}_{i1}\bar{g}_{1j}$, and $\tilde{\xi}_j^b - \tilde{\xi}_i^b = \mathrm{Ad}_{\bar{g}_{1j}}\hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{1i}}\hat{\xi}_i^b = \mathrm{Ad}_{\bar{g}_{1j}}(\hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{ji}}\hat{\xi}_i^b)$. it is easy to obtain that the consensus conditions $\tilde{g}_i^{-1}(t_f)\tilde{g}_j(t_f) = I$, $\tilde{\xi}_j^b(t_f) - \tilde{\xi}_i^b(t_f) = 0$, $i, j = 1, \cdots, n$, are equivalent to the formation condition (5) □

Thus, we shall focus on the consensus control problem.

4 THE MAIN RESULTS

4.1 Co-State Of Optimal Trajectories

The problem of finding $\hat{u}_i$, is an optimal control problem. In order to use Pontryagin's minimum principle (PMP) [10], the Hamiltonian for this problem can be constructed as follows

$$H = -\frac{1}{2}\sum_{k=1}^{n} G_I(\hat{u}_k, \hat{u}_k) + \sum_{i=1}^{n}\hat{p}_{g_i}^*(g_i\hat{\xi}_i^b) + \sum_{i=1}^{n}\hat{p}_{\xi_i}^*(\hat{u}_i), \tag{8}$$

where $\hat{p}_{g_i}^* \in T_{g_i}^*\mathrm{SE}(3)$, and $\hat{p}_{\xi_i}^* \in \mathfrak{se}^*(s)$ are the co-states (Lagrangian multipliers). By Definition 2, the corresponding Hamiltonian system can be written as

$$\dot{g}_i = \frac{\partial H}{\partial \hat{p}_{g_i}} = g_i\hat{\xi}_i^b, \qquad \dot{\hat{\xi}}_i^b = \frac{\partial H}{\partial \hat{p}_{\xi_i}} = \hat{u}_i,$$

$$\dot{\hat{p}}_{g_i}^* = -\frac{\partial H}{\partial g_i^T} = -\hat{\xi}_i^b\hat{p}_{g_i}^*, \quad \dot{\hat{p}}_{\xi_i}^* = -\frac{\partial H}{\partial(\hat{\xi}_i^b)^*} = -\hat{p}_{g_i}^* g_i. \tag{9}$$

Let $\hat{p}_{\xi_i} = G_I^\sharp(\hat{p}_{\xi_i}^*)$, and $\hat{p}_i = G_I^\sharp(\hat{p}_{g_i}^* g_i)$, then we have the following lemma.

Lemma 3. *Suppose that $[\hat{p}_i, \hat{\xi}_i^b] = 0$, then*
$$\hat{p}_{\xi_i}(t) = \hat{p}_{\xi_i}(t_0) + \hat{p}_i(t - t_0),$$
where, both $\hat{p}_{\xi_i}(t_0) \in \mathfrak{se}(3)$ and $\hat{p}_i \in \mathfrak{se}(3)$ are constants.

Proof. By writing $\hat{\xi}_i^b = g_i^{-1}\dot{g}_i$ and substituting to the equation $\dot{\hat{p}}_{g_i}^* = -\hat{\xi}_i^b\hat{p}_{g_i}^*$ of (9), it follows that $\frac{d}{dt}(g_i\hat{p}_{g_i}^*) = g_i\dot{\hat{p}}_{g_i}^* + \dot{g}_i\hat{p}_{g_i}^* = 0$. This shows that $g_i\hat{p}_{g_i}^*$ does not vary with time, and can be denoted as a constant $(\hat{p}_i^s)^* = g_i\hat{p}_{g_i}^*$. Let $\hat{p}_i^* = \hat{p}_{g_i}^* g_i$, then $\hat{p}_i^* = g_i^{-1}(\hat{p}_i^s)^* g_i = \mathrm{Ad}_{g_i}^*(\hat{p}_i^s)^*$, and by sharp map $G_I^\sharp$, it follows that $\hat{p}_i = \mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s$, where $\hat{p}_i^s$ as the dual of $(\hat{p}_i^s)^*$ is constant. By differentiating the equality with respect to time, it follows that

$$\begin{aligned} \dot{\hat{p}}_i &= -g_i^{-1}\dot{g}_i g_i^{-1}\hat{p}_i^s g_i + g_i^{-1}\hat{p}_i^s\dot{g}_i \\ &= -\hat{\xi}_i^b\mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s + \mathrm{Ad}_{g_i^{-1}}\hat{p}_i^s\hat{\xi}_i^b = [\hat{p}_i, \hat{\xi}_i^b] = 0. \end{aligned}$$

This implies that $\hat{p}_i$ is constant, and so is the $\hat{p}_i^*$. Substituting this result into the last equation of the Hamiltonian system (9) and integrating from t_0 to t, then we get $\hat{p}_{\xi_i}^*(t) = \hat{p}_{\xi_i}^*(t_0) - \hat{p}_i^*(t - t_0)$, where $\hat{p}_{\xi_i}^*(t_0)$ is the initial value of $\hat{p}_{\xi_i}^*$. After the transform by sharp map, the result follows. □

Remark 1. *The condition of lemma 3 will make restrictions on the kind of the problem to be dealt with, we shall show that when the initial velocity is zero, the condition is automatically satisfied.*

4.2 Solution Of Optimal Control

In this subsection, we studies the optimal control. According to PMP, the optimal control $\hat{u}_i$ must satisfies the necessary condition that

$$\frac{\partial H}{\partial \hat{u}_i} = -\hat{u}_i + \hat{p}_{\xi_i} = 0, \ i = 1, \cdots, n.$$

Since these equations have the unique solutions, the above condition is also sufficient. Recall Lemma 3, the optimal control can be written as

$$\hat{u}_i^{op} = \hat{p}_{\xi_i}(t_0) - \hat{p}_i(t - t_0). \tag{10}$$

For this control law, the constant $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ must be determined by boundary conditions (5). First, we give several results related to the transversality condition corresponding to (5) in the meaning of consensus.

Lemma 4. *If $\hat{\xi}_\alpha^b(t_f) = \hat{\xi}_\beta^b(t_f), \alpha, \beta = 1, \cdots, n$, then*
$$\sum_{k=1}^{n}\hat{p}_{\xi_k}(t_f) = 0.$$

Proof. From the boundary conditions (5), let $h_{\alpha\beta}(\hat{\xi}_\alpha^b(t_f), \hat{\xi}_\beta^b(t_f)) = \hat{\xi}_\alpha^b(t_f) - \hat{\xi}_\beta^b(t_f), \alpha, \beta = 1, \cdots, n$, and the (i, j)-th entry of $h_{\alpha\beta}$ is denoted by $h_{\alpha\beta}^{ij}$. Then according to PMP, the transversality condition corresponding to $h_{\alpha\beta} = 0$ for fixed β can be written as

$$\begin{aligned} [\hat{p}_{\xi_k}^*(t_f)]^{pq} = &\sum_{\alpha}\sum_{i,j}\Gamma_{\alpha\beta}^{ij}\frac{\partial h_{\alpha\beta}^{ij}}{\partial[\hat{\xi}_k^b]^{pq}}(t_f) \\ = &\sum_{\alpha}\mathrm{tr}\left(\Gamma_{\alpha\beta}^T\frac{\partial(\hat{\xi}_\alpha^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_k^b]^{pq}}\right)(t_f) \\ &k = 1, \cdots, n, \ p, q = 1, 2, 3, 4. \end{aligned}$$

where the superscript 'pq' represents the (p, q)-th entry of the corresponding matrix, and $\Gamma_{\alpha\beta} = [\Gamma_{\alpha\beta}^{ij}]$ is the parameter matrix to be determined.
For the case $k \neq \beta$, the above equality will be

$$\begin{aligned} [\hat{p}_{\xi_k}^*(t_f)]^{pq} = &\ \mathrm{tr}\left(\Gamma_{k\beta}^T\frac{\partial(\hat{\xi}_k^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_k^b]^{pq}}\right)(t_f) \\ = &\ \mathrm{tr}\left(\Gamma_{k\beta}^T, E_{pq}\right) = \Gamma_{k\beta}^{pq} \end{aligned},$$

this gives that $\hat{p}_{\xi_k}^*(t_f) = \Gamma_{k\beta}, \ k \neq \beta$. For the case $k = \beta$, we have

$$\begin{aligned} [\hat{p}_{\xi_\beta}^*(t_f)]^{pq} = &\sum_{\alpha\neq\beta}\mathrm{tr}\left(\Gamma_{\alpha\beta}^T\frac{\partial(\hat{\xi}_\alpha^b - \hat{\xi}_\beta^b)}{\partial[\hat{\xi}_\beta^b]^{pq}}\right)(t_f) \\ = &-\sum_{\alpha\neq\beta}\mathrm{tr}\left(\Gamma_{\alpha\beta}^T, E_{pq}\right)(t_f) = -\sum_{\alpha\neq\beta}\Gamma_{\alpha\beta}^{pq} \end{aligned},$$

this leads to that $\hat{p}_{\xi_\beta}^*(t_f) = -\sum_{\alpha\neq\beta}\Gamma_{\alpha\beta}$. Combining the two cases, we get $\sum_{k=1}^{n}\hat{p}_{\xi_k}^*(t_f) = 0$. After taking its sharp image, the result follows. □

Lemma 5. *If $g_\alpha^{-1}(t_f)g_\beta(t_f) = I, \alpha, \beta = 1, \cdots, n$, then*

$$\sum_{k=1}^{n} \hat{p}_{g_k}(t_f) = 0.$$

Proof. Let $f_{\alpha\beta}(g_\alpha(t_f), g_\beta(t_f)) = g_\alpha^{-1}(t_f)g_\beta(t_f) - I, \alpha, \beta = 1, \cdots, n$, and the (i,j)-th entry of $f_{\alpha\beta}$ is denoted by $f_{\alpha\beta}^{ij}$. By the same argument as in lemma 4, the transversality condition corresponding to $f_{\alpha\beta} = 0$ for fixed β can be written as

$$\begin{aligned}
[\hat{p}^*_{g_k}(t_f)]^{pq} &= \sum_{\alpha,\alpha\neq\beta} \sum_{i,j} \Lambda_{\alpha\beta}^{ij} \frac{\partial f_{\alpha\beta}^{ij}}{\partial g_k^{pq}}(t_f) \\
&= \sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T \frac{\partial g_\alpha^{-1} g_\beta}{\partial g_k^{pq}}\right)(t_f) \\
& k = 1, \cdots, n,\ p, q = 1, 2, 3, 4.
\end{aligned}$$

where the superscript 'pq' represents the (p,q)-th entry of the corresponding matrix, and $\Lambda_{\alpha\beta} = [\Lambda_{\alpha\beta}^{ij}]$ is the parameter matrix to be determined.

For the case $k \neq \beta$, the above equality will be

$$\begin{aligned}
&[\hat{p}^*_{g_k}(t_f)]^{pq} = \mathrm{tr}\left(\Lambda_{k\beta}^T \frac{\partial g_k^{-1} g_\beta}{\partial g_k^{pq}}\right)(t_f) \\
= &-\mathrm{tr}\left(\Lambda_{k\beta}^T g_k^{-1} E_{pq} g_k^{-1} g_\beta\right)(t_f) - (g_k^{-T}\Lambda_{k\beta}^T g_\beta^T g_k^{-T})^{pq}(t_f)
\end{aligned}$$

By noting that $g_\beta^T(t_f) g_k^{-T}(t_f) = I$, it follows that $\hat{p}^*_{g_k}(t_f) = -(g_k^{-T}\Lambda_{k\beta}^T)(t_f)$, $k \neq \beta$. For the case $k = \beta$, we have

$$\begin{aligned}
&[\hat{p}^*_{g_\beta}(t_f)]^{pq} = \sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T \frac{\partial g_\alpha^{-1} g_\beta}{\partial g_\beta^{pq}}\right)(t_f) \\
= &\sum_{\alpha,\alpha\neq\beta} \mathrm{tr}\left(\Lambda_{\alpha\beta}^T g_\alpha^{-1} E_{pq}\right)(t_f) \sum_{\alpha,\alpha\neq\beta} \left(g_\alpha^{-T}\Lambda_{\alpha\beta}^T\right)^{pq}(t_f)
\end{aligned},$$

this leads to that $\hat{p}^*_{g_\beta}(t_f) = \sum_{\alpha,\alpha\neq\beta}\left(g_\alpha^{-T}\Lambda_{\alpha\beta}^T\right)(t_f)$. Combining the two cases, we get $\sum_{k=1}^n \hat{p}^*_{g_k}(t_f) = 0$. After taking its sharp image, the result follows. □

Corollary 1.

1) $\sum_{k=1}^n \hat{p}_k = 0$.
2) $\sum_{k=1}^n \hat{p}_{\xi_k}(t_0) = 0$.

Proof. By noting that $g_i(t_f) = g_j(t_f), i, j = 1, \cdots, n$ and $\hat{p}_{g_k}(t_f) = g_k(t_f)\hat{p}_k$, the result of item 1) follows from lemma 5. From the lemma 3, by taking $t = t_f$, then $\hat{p}_{\xi_k}(t_f) = \hat{p}_{\xi_k}(t_0) - \hat{p}_k(t_f - t_0)$, summing it up for index k, and using lemma 4 and item 1), then the result of item 2) follows. □

The optimal control (10) shows that the determination of the optimal control is boiled down to finding the initial costates $\hat{p}_{\xi_i}(t_0)$ and parameter $\hat{p}_i$. For finding the initial costates $\hat{p}_{\xi_i}(t_0)$ we give the following lemma.

Lemma 6.

$$\hat{p}_{\xi_i}(t_0) = \frac{1}{t_f - t_0}(\hat{\xi}^{b,0} - \hat{\xi}_i^{b,0}) + \frac{1}{2}(t_f - t_0)\hat{p}_i \qquad (11)$$

where $\hat{\xi}^{b,0} = \frac{1}{n}\sum_i \hat{\xi}_i^{b,0}$ is an arithmetic mean of initial velocities.

Proof. By substituting the optimal control (10) to the systems (4) and integrating the dynamics equation, we get

$$\hat{\xi}_k^b(t) = \hat{\xi}_k^{b,0} + \hat{p}_{\xi_k}(t_0)(t - t_0) - \frac{1}{2}\hat{p}_k(t - t_0)^2, \qquad (12)$$

then summing it up for index k, it follows from (12)that

$$\hat{\xi}^b(t) = \hat{\xi}^{b,0} + \frac{1}{n}\sum_{k=1}^n \hat{p}_{\xi_k}(t_0)(t - t_0) - \frac{1}{2n}\sum_{k=1}^n \hat{p}_k(t - t_0)^2.$$

where $\hat{\xi}^b(t) = \frac{1}{n}\sum_k \hat{\xi}_k^b(t)$. From corollary 1, it easy to get that $\hat{\xi}^b(t) = \hat{\xi}^{b,0}, \forall t \geq t_0$. Thus $\hat{\xi}^{b,f} \triangleq \hat{\xi}^b(t_f) = \hat{\xi}^{b,0}$, and the consensus condition implies that $\hat{\xi}_k^b(t_f) = \hat{\xi}^{b,f}$. By taking $t = t_f$ in (12) and using $\hat{\xi}^{b,0}$ to replace $\hat{\xi}_k^b(t_f)$, then $\hat{p}_{\xi_i}(t_0)$ can be solved. □

Remark 2. *From the proof of lemma 6, we know that the velocity under the optimal control has the form*

$$\hat{\xi}_i^b(t) = \frac{t - t_0}{t_f - t_0}\hat{\xi}^{b,0} - \frac{t - t_f}{t_f - t_0}\hat{\xi}_i^{b,0} - \frac{1}{2}(t - t_0)(t - t_f)\hat{p}_i, \qquad (13)$$

For this form of velocity, it is obvious that the condition of lemma 3, i.e., $[\hat{p}_i, \hat{\xi}_i^b(t)] = 0$, if and only $[\hat{p}_i, \hat{\xi}^{b,0}] = 0$ and $[\hat{p}_i, \hat{\xi}_i^{b,0}] = 0$. Generally, this dose not hold for arbitrary initial velocities $\hat{\xi}_i^{b,0}, i = 1, \cdots, n$. But there is an important case that the initial velocity $\hat{\xi}_i^{b,0}$ has the form $c\hat{p}_i$ for some scalar $c \in \mathbb{R}$, in this case $\hat{\xi}^{b,0} = 0$ by corollary 1, and $[\hat{p}_i, \hat{\xi}_i^{0,b}] = 0$.

Now the problem of determination of optimal control (10) is further reduced to finding parameter $\hat{p}_i$, which is not trivial, we need some preparatory work. In order to use the the final values of the configuration, we need to integrate the kinematic equation $\dot{g}_i = g_i\hat{\xi}_i^b$ by utilizing velocity (13). The following lemma gives the result.

Lemma 7. *Suppose that both $[\hat{p}_i, \hat{\xi}^{b,0}] = 0$ and $[\hat{p}_i, \hat{\xi}_i^{b,0}] = 0, i = 1, \cdots, n$. Let $x_i(t) = \log(g^{-1}(t_0)g_i(t))$, $i = 1, \cdots, n$. Then*

$$x_i(t_f) = (t_f - t_0)\hat{\xi}_i^{b,0} + \frac{(t_f - t_0)^2}{2n}\sum_{j=1}^n \hat{\xi}_{ij}^{b,0} + \frac{(t_f - t_0)^3}{12}\hat{p}_i \qquad (14)$$

where $\hat{\xi}_{ij}^{b,0} = \hat{\xi}_j^{b,0} - \hat{\xi}_i^{b,0}$ is the relative initial velocity of vehicle j with respect to vehicle i.

Proof. Let $x_i(t) = \log(g^{-1}(t_0)g_i(t))$, $i = 1, \cdots, n$, then according to the differential of exponential (see [9]), we have

$$\dot{x}_i = \hat{\xi}_i^b + \sum_{k=1}^\infty \frac{\mathrm{B}_k}{k!}\mathrm{ad}_{-x_i}^k(\hat{\xi}_i^b),\ x_i(t_0) = 0. \qquad (15)$$

where $\{\mathrm{B}_k\}$ are Bernoulli numbers. If we show that $x_i(t) = \int_{t_0}^t \hat{\xi}_i^b(\tau)d\tau$ solve this equation, then the result follows. From (13), $x_i(t)$ should has the form of $x_i(t) = a(t)\hat{\xi}^{b,0} + b(t)\hat{\xi}_i^{b,0} + c(t)\hat{p}_i$, where a, b and c are scalar functions of time. Thus, it can be varified by lemma 1 that $\mathrm{ad}_{-x_i}(\hat{\xi}_i^b) = 0$. This implies that $\mathrm{ad}_{-x_i}^k(\hat{\xi}_i^b) = 0$, for $k \geq 1$, and thus equation (15) becomes $\dot{x}_i = \hat{\xi}_i^b$. By integrating this equation, the result follows. □

Now, we first determine the parameter $\hat{p}_i$ for the case of $n = 2$.

Lemma 8. *For the case that $n = 2$, and $\hat{\xi}_i^{b,0} = 0, i = 1, 2$,*

$$\hat{p}_1 = \frac{6}{(t_f - t_0)^3} x_{12}^0,\ \hat{p}_2 = \frac{6}{(t_f - t_0)^3} x_{21}^0 \tag{16}$$

where $x_{ij}^0 = \log(g_{ij}^0) = \log(g_i^{-1}(t_0) g_j(t_0)), i, j = 1, 2$.

Proof. Let $x_i(t) = \log(g_i^{-1}(t_0) g_i(t))$. In the meaning of consensus, i.e., $g_1(t_f) = g_2(t_f)$, recall (14), it is easy to get $\exp(x_1(t_f)) \cdot \exp(-x_2(t_f)) = g_{12}^0$. By using Baker-Campbell-Hausdorff (BCH) formula [7], we have

$$\begin{aligned} x_{12}^0 &= x_1(t_f) - x_2(t_f) - \tfrac{1}{2}[x_1(t_f), x_2(t_f)] + \cdots \\ &= \tfrac{(t_f - t_0)^3}{12}(\hat{p}_1 - \hat{p}_2) - \tfrac{(t_f - t_0)^6}{2 \times 12^2}[\hat{p}_1, \hat{p}_2] + \cdots \end{aligned}.$$

From corollary 1, $\hat{p}_1 + \hat{p}_2 = 0$, this implies that all Lie brackets will vanish, and $x_{12}^0 = \frac{(t_f - t_0)^3}{12}\hat{p}_1 - \frac{(t_f - t_0)^3}{12}\hat{p}_2 = \frac{(t_f - t_0)^3}{6}\hat{p}_1$. Thus, $\hat{p}_1 = \frac{6}{(t_f - t_0)^3} x_{12}^0$, and by noting $\hat{p}_1 = -\hat{p}_2$, $x_{21}^0 = -x_{12}^0$, we have $\hat{p}_2 = \frac{6}{(t_f - t_0)^3} x_{21}^0$. □

We now give the optimal control law for the case of $n = 2$.

Theorem 1. *For the case of $n = 2$ and $\hat{\xi}_i^{b,0} = 0, i = 1, 2$, the optimal consensus control is*

$$\hat{u}_i^{o,op} = \frac{3(t_f + t_0 - 2t)}{(t_f - t_0)^3} x_{ij}^0,\ i \neq j,\ i, j = 1, 2. \tag{17}$$

Proof. By substituting $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ given by (11) and (16) into the control law (10), the result follows. □

For the case of $n > 2$, we can not obtain the exact explicit optimal control law as that for the case of $n = 2$. In this case, we only can get the approximate explicit optimal control as shown in the following theorem.

Theorem 2. *For the case that $n > 2$, and $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$ the approximate optimal consensus control is*

$$\hat{u}_i^{o,op} \approx \frac{6(t_f + t_0 - 2t)}{n(t_f - t_0)^3} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij}^0,\ i, j = 1, \cdots, n. \tag{18}$$

where $x_{ij}^0 = \log(g_{ij}^0) = \log(g_i^{-1}(t_0) g_j(t_0))$.

Proof. Let $x_i(t) = \log(g_i^{-1}(t_0) g_i(t))$. Since $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$, it can be seen from (14) that $x_i(t_f) = \frac{(t_f - t_0)^3}{12}\hat{p}_i,\ i = 1, \cdots, n$. In the meaning of consensus, i.e., $g_i(t_f) = g_j(t_f)$, it is easy to get $\exp(x_i(t_f)) \cdot \exp(-x_j(t_f)) = g_{ij}^0$. By using Baker-Campbell-Hausdorff formula ([7]), we have

$$x_{ij}^0 = (x_i - x_j - \tfrac{1}{2}[x_i, x_j] + \tfrac{1}{12}[x_j, [x_j, x_i]] + \cdots)(t_f),$$

by summing it up for index j, and noting that $\sum_i x_i(t_f) = 0$, it follows that

$$\sum_{j=1}^{n} x_{ij}^0 = n x_i(t_f) + \frac{1}{12}\sum_{j=1}^{n}[x_j(t_f), [x_j(t_f), x_i(t_f)]] + \cdots,\quad i = 1, \cdots, n$$

This is a group of equations about unknowns $\hat{p}_i, i = 1, \cdots, n$, it is generally imposable to solve $\hat{p}_i, i = 1, \cdots, n$, explicitly. For this problem, when the configurations approach consensus, the influence of the higher order Lie brackets will attenuate quickly, and can be omitted. Thus we get a group of approximate equations as follows, $\sum_{j=1}^{n} x_{ij}^0 \approx n x_i(t_f) = \frac{n(t_f - t_0)^3}{12}\hat{p}_i$, thus $\hat{p}_i \approx \frac{12}{n(t_f - t_0)^3}\sum_{j=1}^{n} x_{ij}^0$. Recall (11), we have $\hat{p}_{\xi_i}(t_0) = \frac{1}{2}(t_f - t_0)\hat{p}_i$, by substituting $\hat{p}_{\xi_i}(t_0)$ and $\hat{p}_i$ into (10), then the result follows. □

Remark 3. *By substituting $\hat{p}_i, i = 1, \cdots, n$, into (14), the final configuration of $g_i, i = 1, 2$ will be*

$$g_i(t_f) = g_i^0 \exp\left(\frac{1}{n}\sum_{j=1}^{n} x_{ij}^0\right),\ i = 1, \cdots, n. \tag{19}$$

For $n = 2$, we have exactly that $g_1(t_f) = g_2(t_f)$.

In order to overcome the deficiency that open control is sensitive to perturbations, it is suggested to use the closed optimal control, i.e., to take the current time and state as the initial time and state when constructing the optimal control [11]. In this case the control law (17) and (18) can not be used, since it is obtained under the assumption that the initial velocities are zeros, but the current velocities, when being taken as initial velocities, can not be zeros as shown in (13), $\hat{\xi}_i^b(t) = -\frac{1}{2}(t - t_0)(t - t_f)\hat{p}_i$. If we take the initial time as zero, and current time as t_0, the current velocities will be $\hat{\xi}_i^b(t_0) = -\frac{1}{2}(t_0)(t_0 - t_f)\hat{p}_i$. Obviously, this form of velocity has the property that $[\hat{p}_i, \hat{\xi}_i^b(t_0)] = 0$, and $\sum_i \hat{\xi}_i^b(t_0) = 0$ which satisfies the condition of lemma 7. See remark 2.

Theorem 3. *Suppose that $[\hat{p}_i, \hat{\xi}_i^b(t_0)] = 0, i = 1, \cdots, n$, and $\sum_i \hat{\xi}_i^b(t_0) = 0$, then the optimal control is*

$$\hat{u}_i^{op} \approx \sum_{\substack{j=1 \\ j \neq i}}^{n}\left(\frac{6(t_f + t_0 - 2t)}{n(t_f - t_0)^3} x_{ij}^0 + \frac{2(2t_f + t_0 - 3t)}{n(t_f - t_0)^2}\hat{\xi}_{ij}^{b,0}\right)\quad i = 1, \cdots, n. \tag{20}$$

Proof. Noting that in this case, we also have $\sum_i x_i(t_f) = 0$, by almost the same argument as that of theorem 2, then the result will follows. □

Theorem 4. *Suppose that the initial velocities satisfies $\hat{\xi}_i^{b,0} = 0, i = 1, \cdots, n$. For the required formation designated by relative configurations $\bar{g}_{ij}, i, j = 1, \cdots, n$, the optimal feedback formation control is*

$$\hat{u}_i^{f,op} \approx \sum_{\substack{j=1 \\ i \neq j}}^{n}\left(\frac{6}{n(t_f - t)^2} x_{ij}(t) + \frac{4}{n(t_f - t)}\hat{\xi}_{ij}^b(t)\right),\quad i = 1, \cdots, n. \tag{21}$$

where $x_{ij} = \log(g_{ij}\bar{g}_{ji})$, and $\hat{\xi}_{ij}^b = \mathrm{Ad}_{\bar{g}_{ij}}\hat{\xi}_j^b - \hat{\xi}_i^b$.

Proof. Recalling lemma 2, by taking the current time t as the initial time t_0, and using $\tilde{u}_i^{f,op}$, $\tilde{x}_{ij} = \log \tilde{g}_i^{-1}\tilde{g}_j$, and $\tilde{\xi}_{ij}^b = \mathrm{Ad}_{\bar{g}_{1j}}\hat{\xi}_j^b - \mathrm{Ad}_{\bar{g}_{1i}}\hat{\xi}_j^b$ to replace $\hat{u}_i^{op}$, x_{ij} and $\hat{\xi}_{ij}^b$ respectively in (20), we shall obtain

$$\tilde{u}_i^{f,op} \approx \sum_{\substack{j=1 \\ i \neq j}}^{n}\left(\frac{6}{n(t_f - t)^2}\tilde{x}_{ij}(t) + \frac{4}{n(t_f - t)}\tilde{\xi}_{ij}^b(t)\right),\quad i = 1, \cdots, n.$$

From lemma 2, the optimal feedback formation control can will be $\hat{u}_i^{f,op} = \mathrm{Ad}_{\bar{g}_{i1}} \tilde{u}_i^{f,op}$. Also, by the Corollary 1 of [4], we have $\mathrm{Ad}_{\bar{g}_{i1}} \tilde{x}_{ij} \log\left(g_i^{-1} g_j \bar{g}_{ji}\right)$. And $\mathrm{Ad}_{\bar{g}_{i1}} \tilde{\xi}_{ij}^b = \mathrm{Ad}_{\bar{g}_{ij}} \hat{\xi}_j^b - \hat{\xi}_i^b$. This proves the theorem. □

Remark 4. *In theorem 4,*

1) *when $n = 2$, the optimal feedback control law is exact, no approximation is assumed in this case;*
2) *when $\bar{g}_{ij} = I, i, j = 1, \cdots, n$, the formation control law (21) will be reduced to the consensus control law.*

5 SIMULATIONS

Given four vehicles with the initial configurations shown in Table 1, now consider formation problem with the required formation given by Table 2. The control law to be utilized is the feedback optimal control law (21), the t_f is taken to be 10 and the total simulation time is 15. The dynamical process of simulation is shown in Figure 1 and Figure 2. The result shows that the system archives the required formation at $t_f = 10$.

Table 1: Initial Configurations of Agents

order number	$\theta(0)$	$x(0)$	$y(0)$
1	0	100	100
2	$-\pi$	-100	-100
3	$\pi/2$	-100	100
4	$-\pi/2$	100	-100

Table 2: relative configurations at final time

relative index$(1i)$	$\theta_{1i}(t_f)$	$x_{1i}(t_f)$	$y_{1i}(t_f)$
(11)	0	0	0
(12)	0	-20	20
(13)	0	-20	-20
(14)	0	-40	0

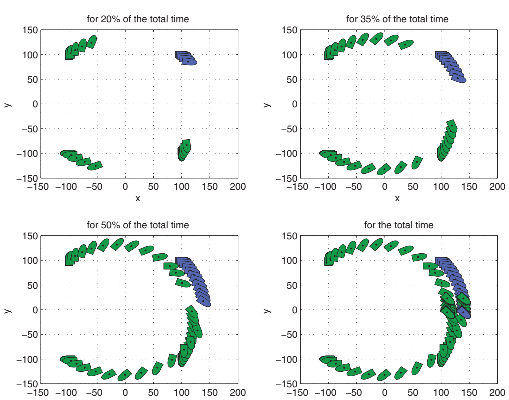

Figure 1: The trajectories of multiple vehicles with headings

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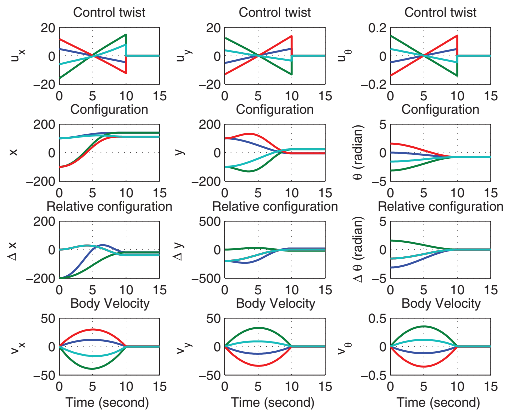

Figure 2: The time behaviors of vehicles during formation

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